# Invariant Theory, Young Bitableaux, and Combinatorics

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## Contents



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The present document is an unofficial re-edition of the paper "Invariant Theory, Young Bitableaux, and Combinatorics" by J. Désarménien, Joseph P. S. Kung, Gian-Carlo Rota (originally published in: Advances in Mathematics, Volume 27, Issue 1, January 1978, pp. 63–92, doi:10.1016/0001-8708(78)90077-4).[1] This re-edition

*dedicated to Garrett Birkhoff

[1]See http://www.sciencedirect.com/science/article/pii/0001870878900774 for the publisher's webpage for the original article.

attempts to correct misprints[2] and add some comments (identifiable as footnotes, marked with "*Comment:*").

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I have tried to reproduce the original paper as faithfully as possibly while correcting errors and making use of the possibilities of contemporary LaTeX typography (e.g., numbering theorems, finishing proofs with □ signs). Nevertheless, errors and unintended discrepancies are likely to have happened. Please let me know if you find anything wrong![4]

# 1. Introduction

Since its emergence in the middle of the last century, invariant theory has oscillated between two clearly distinguishable poles. The first, and the one that was later to survive the temporary "death" of the field, is geometry. Invariants were identified with the invariants of surfaces. Their study, the aim of which was to give information about the solution of systems of polynomial equations, was to lead to the rise of commutative algebra. From this standpoint, projective invariants were eventually seen as poor relations of the richer algebraic invariants.

A casualty of this trend was the study of the projective generation of surfaces, a problem which was condemned by Cremona as "too difficult," and which has never quite recovered from the blow, despite the recent excitement over finite fields. In contrast, other heretical schools survived the *Fata Morgana* of algebra with the promise, not always fulfilled, that sooner or later they would be brought back into the commutative fold. Thus, the genial computations of the high school teacher Hermann Schubert were proclaimed a "problem" by Hilbert, who was articulating the general feeling at the time that enumerative geometry required a justification in terms of the dominant concepts of the day, namely, rings and fields.

Similarly, the mystical vision of Hermann Grassmann, another high school teacher, was only appreciated by other oddballs like Peano, Study, and several inevitable English gentleman-mathematicians. It took the advocacy of someone of the stature of Elie Cartan to get Grassmann's techniques accepted by a public by then avid for simplications, but reluctant to acknowledge embarrassing oversights; and then, only at the cost of putting them to a use for which they were not intended, though magically suited. The recognition that anti-commutativity is a sibling, with an equally noble genealogy, of commutativity is only now beginning, under the prod-

[2] All corrections I have made are identified in footnotes, marked with "*Correction:*".

[3] See https://web-beta.archive.org/web/20131022235944/http://www.elsevier.com/about/open-access/oa-and-elsevier/oa-license-policy for an archived version of this license on Elsevier's website.

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ding of the particle physicists, who with exquisite salesmanship have proclaimed it a law of nature.

The second pole of invariant theory was algorithmic. To be sure, all invariant theory is ultimately concerned with one problem. In crude, oversimplified, off-putting language, this problem is to generalize to tensors the eigenvalue theory of matrices, and all invariant theorists from Boole to Mumford have been, tacitly or otherwise, concerned with it. The algorithmic school, however, saw this problem as one of "explicit computation," an expression which was to smack of mathematical bad taste in the 1930s. In a century which prefers existence to construction, structure to algorithm, algebra to combinatorics, such a school could not thrive, and it did not, supported as it was more by the English and Italians than by the Germans and French. There were, however, weightier reasons for this defeat of the algorithmic school. Their most striking productions, the expansions that go under the names of Capelli, Clebsch, Gordan, and Young, were hopelessly tethered to characteristic zero, and seemed to belie the avowed combinatorial ideal of doing away with all numbers that are not integers, and preferably positive ones at that. To top it all, Igusa showed that, with the massive machinery of algebraic geometry, some of the results of classical invariant theory could be extended to fields of positive characteristic.

In this environment, the 1974 paper of Doubilet, Rota, and Stein [3], which for the first time succeeded in extending to arbitrary infinite fields, by constructive algorithmic methods, the two "fundamental theorems" of invariant theory, could only appear as an intrusion. To make things worse, the authors' sympathy for the nineteenth century went as far as to embrace matters of style, thus alienating many readers in a less romantic century. In 1976, de Concini and Procesi [1] charitably rewrote parts of that paper and developed some of the suggestions made therein, thus showing that the authors' claims were indeed well-founded.

In this paper we give a self-contained combinatorial presentation – the first one, to be sure – of vector invariant theory over an arbitrary infinite field. We begin by proving the *Straightening Formula*, which is probably one of the fundamental algorithms of multilinear algebra. This formula is the culmination of a trend of thought that can be traced back to Capelli, and was developed most notably by Alfred Young and the Scottish invariant theorists. Had it not been for the disrepute into which algorithmic methods had fallen in the thirties, the full proof of this formula would have appeared earlier than in 1974, and might have anticipated the current revival of classical invariant theory.

In comparison with other classical expansions, the straightening formula offers two advantages. First, it holds over the ring of integers. Second, it recognizes the crucial role played by the notion of a bitableau in obtaining a characteristic-free proof of the first fundamental theorem. In fact, we give two proofs of this result. Both of these proofs are based on new ideas, first presented in 1974. Even in characteristic zero, either of these proofs differs from any previously given, and is, in addition, much simpler as it only relies on elementary linear algebra and some combinatorics. The success of the notion of a bitableau also shows why previous attempts to prove the first fundamental theorem by expansion into *single* Young tableaux were bound to fail. Strangely, Alfred Young himself was the first to consider bitableaux in his study of the representations of the octahedral group, but it did not occur to him that they would be useful in the study of the projective group.

Less surprisingly, the straightening formula is also used to give a simple proof of the second fundamental theorem, in a version that has been proved by van der Waerden in characteristic zero. The present proof shows that the straightening formula is indeed the characteristic-free replacement of the Gordan-Capelli expansion.

The second fundamental theorem has lived in a limbo ever since Weyl's fumbling justification in "The Classical Groups" [4]. Some invariant theorists have taken the easy way out and claimed it as a result in algebraic geometry, stating certain facts about the coordinate rings of Grassmannians or flag manifolds. We believe on the contrary that the second fundamental theorem plays a crucial role in invariant theory which can perhaps be best understood by analogy with the predicate calculus. Here, two aspects have long been recognized as complementary: a syntactical aspect, where the subject is presented as a purely algebraic system subject to formal rules; and a semantical aspect, where the possible set-theoretic interpretations, or models, are classified. These two aspects are connected by the Gödel completeness theorem.

A corresponding situation obtains in invariant theory. Here, what we call the letter place algebra is the syntactic counterpart to the semantics of representing abstract brackets by actual inner products of vectors and covectors in a vector space. The second fundamental theorem is the invariant-theoretic analog of the Gödel completeness theorem. This suggests a host of questions on invariants which can be gleaned from analogous questions in the predicate calculus.

Other applications of the straightening formula, some of which were adumbrated in 1974, will be given elsewhere. We mention, as examples, a characteristic-free theory of symmetric functions, the study of polynomial identities in an associative algebra, the classification of transvectants, and connections with the algebra of second quantization. The present work is merely the first in what is hoped to be a far-reaching extension of the research program of projective invariant theory.

## 2. Young Tableaux

The fundamental combinatorial notion in this study is that of a *Young tableau*. Let $(\lambda) = \left(\lambda_1, \ldots, \lambda_p\right)$ be a *partition* of the integer $n$: that is, $(\lambda)$ is a finite sequence of positive integers such that

$$\lambda_1 + \cdots + \lambda_p = n$$

and

$$\lambda_1 \geq \cdots \geq \lambda_p > 0.$$

If $(\lambda)$ is a partition of $n$, then[5] its *shape*, also denoted by $(\lambda)$, is the set of integer points $(i, -j)$ in the plane, with $1 \leq j \leq p$ and $1 \leq i \leq \lambda_j$. The shape $(\lambda) = (\lambda_1, \ldots, \lambda_s)$ is said to be *longer* than the shape $(\mu) = (\mu_1, \ldots, \mu_t)$ if, considered as a finite sequence, $(\lambda)$ is greater than $(\mu)$ in the lexicographic order from left to right. Here, the definition of the lexicographic order is subject to the following caveat:[6] If a finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a proper prefix of a finite sequence $(\mu_1, \mu_2, \ldots, \mu_t)$ (that is, we have $t > s$ and $(\mu_1, \mu_2, \ldots, \mu_s) = (\lambda_1, \lambda_2, \ldots, \lambda_s)$), then $(\lambda)$ is understood to be greater than $(\mu)$ (not smaller than $(\mu)$ as with the usual definition of lexicographic order).[7] Thus, for example, $(3, 1)$ is longer (i.e., greater) than $(3, 1, 2)$. [8]

A *Young tableau* of[9] the shape $(\lambda)$ with values in the set $E$ is an assignment of an element of $E$ to each point in the shape $(\lambda)$. For example, $T_1$ and $T_2$ are Young tableaux of shape $(\lambda) = (5, 4, 2, 2, 1, 1)$ with values in the integers:[10]

$$T_1 = \begin{array}{llllll} 3 & 2 & 4 & 4 & 7 & 8 \\ 1 & 2 & 3 & 5 & & \\ 2 & 1 & & & & \\ 6 & 2 & & & & \\ 4 & & & & & \end{array} , \qquad T_2 = \begin{array}{llllll} 1 & 2 & 4 & 5 & 7 & 8 \\ 1 & 2 & 4 & 6 & & \\ 2 & 3 & & & & \\ 2 & 4 & & & & \\ 3 & & & & & \end{array} .$$

[11] If $p$ and $q$ are two integers, then the *cell* $(p, q)$ shall mean the point $(q, -p)$ in the plane. Thus, if $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition, then the shape of $(\lambda)$ is the set of all cells $(p, q)$ with $1 \leq p \leq \ell$ and $1 \leq q \leq \lambda_p$. These cells are called the *cells of* $(\lambda)$.

Recall that a Young tableau of shape $(\lambda)$ with values in $E$ is an assignment of an element of $E$ to each point in the shape $(\lambda)$; in other words, a Young tableau of shape $(\lambda)$ with values in $E$ is an assignment of an element of $E$ to each cell of $(\lambda)$.

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[5] *Correction:* Added "then" to make the sentence unambiguous.

[6] *Correction:* Added this caveat. Without it, for example, the Second Fundamental Theorem of Invariant Theory (as stated in this paper) would be false, because the partition $(d, 1)$ would be strictly longer than $(d)$.

[7] Thus, the lexicographic order is defined as follows:

A finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is greater than a finite sequence $(\mu_1, \mu_2, \ldots, \mu_t)$ in the lexicographic order if and only if one of the following two statements holds:

- There exists a $k \in \{1, 2, \ldots, \min\{s, t\}\}$ such that $\lambda_k > \mu_k$ but each $i \in \{1, 2, \ldots, k-1\}$ satisfies $\lambda_i = \mu_i$.
- We have $t > s$ and each $i \in \{1, 2, \ldots, s\}$ satisfies $\lambda_i = \mu_i$.

[8] *Comment:* If this sounds counterintuitive to you, think of the word "longer" as referring to the *horizontal* length of the partition's Young diagram (not the length of the partition as a list of numbers).

[9] *Correction:* Replaced "on" by "of".

[10] *Correction:* I have corrected these two tableaux. The original versions were their transposes, and they would fail the claim that $T_2$ is standard.

[11] *Correction:* This paragraph and the next two have been added by me.

If $R$ is any tableau, and if $p$ and $q$ are two positive integers such that the cell $(p,q)$ (that is, the point $(q,-p)$) belongs to $(\lambda)$, then we let $R(p,q)$ denote the element of $\mathcal{U}$ assigned to this cell $(p,q)$. This element $R(p,q)$ is called the *entry* of $R$ in cell $(p,q)$. For a given $p \geq 1$, the *p-th row* of the tableau $R$ consists of the entries of $R$ in all cells of the form $(p,q)$ with $q \geq 1$ (arranged in the order of increasing $q$). For a given $q \geq 1$, the *q-th column* of the tableau $R$ consists of the entries of $R$ in all cells of the form $(p,q)$ with $p \geq 1$ (arranged in the order of increasing $p$). For example, the 2-nd row of the above tableau $T_1$ is 1235, whereas its 3-rd column is 43.

In this paper, $E$ is always a totally ordered set. A Young tableau is said to be *standard* if the entries in each row are increasing[12] from left to right, and the entries in each column are nondecreasing downward[13]. In our previous example, $T_2$ is standard but $T_1$ is not. This definition, though unconventional, is the natural one for dealing with bitableaux (which are introduced in the sequel).

A word on notation: $\mathfrak{S}_p$ denotes the symmetric group on $p$ symbols, and for a permutation $\sigma \in \mathfrak{S}_p$, its signature is denoted $\operatorname{sgn}(\sigma)$.

## 3. The Straightening Formula

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{U} = \{u_1, \ldots, u_k\}$ be two alphabets, and let $P$ be the algebra of polynomials over the field $K$ in the indeterminates $(x_i \mid u_j)$; this $K$-algebra[14] $P$ is called the *letter place algebra*. Suppose $\left(x_{i_1}, \ldots, x_{i_p}\right)$ and $\left(u_{i_1}, \ldots, u_{i_p}\right)$ are two finite sequences with the same length of letters from $\mathcal{X}$ and $\mathcal{U}$. Their inner product[15] $\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$ is the polynomial in $P$ defined by

$$\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) = \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}(\sigma) \left(x_{i_{\sigma 1}} \mid u_{j_1}\right) \cdots \left(x_{i_{\sigma p}} \mid u_{j_p}\right).$$

[12] *Comment:* The word "increasing" means "strictly increasing" throughout this paper.

[13] *Comment:* This notion of "standard" is not the one commonly used nowadays in combinatorics (although it seems to have had some popularity in invariant theory). What is called a "standard Young tableau" in this paper would probably be called "cosemistandard Young tableau" or "transpose semistandard Young tableau" (indeed, it is a Young tableau whose transpose is semistandard, in today's language). The currently popular use of the word "standard" is different: It stands for a Young tableau $T$ with the following properties:

- The entries of $T$ are $1, 2, \ldots, n$ (for a fixed $n \in \mathbb{N}$).
- The entries of $T$ are pairwise distinct.
- The entries in each row of $T$ are increasing from left to right.
- The entries in each column of $T$ are increasing downward.

[14] *Correction:* Added the words "this $K$-algebra".

[15] *Correction:* The original wrote "$\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1}, \cdots, u_{j_p}\right)$" instead of "$\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$". I have removed the commas, since they seem to be unintentional.

The inner product is an antisymmetric function in $x_i$ and in[16] $u_j$ [17]. Thus, we may suppose, up to a change in sign, that in any inner product, the indices of $x$ and $u$ are increasing. Moreover, an inner product is nonzero if and only if no letter is repeated.

We define a total order on the set $\mathcal{U}$ by setting $u_1 < u_2 < \cdots < u_k$. Similarly, we define a total order on the set $\mathcal{X}$ by setting $x_1 < x_2 < \cdots < x_n$. Thus, a tableau with entries from $\mathcal{X}$ or from $\mathcal{U}$ may be standard.[18]

The *content* of a monomial in $P$ is the pair of vectors

$$(\alpha, \beta) = ((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_k)),$$

where $\alpha_s$ (resp. $\beta_t$) is the total degree of the factors in the monomial of the form $\left(x_s \mid u_j\right)$, $1 \leq j \leq k$ (resp. $\left(x_i \mid u_t\right)$, $1 \leq i \leq n$). The monomials[19] of content $(\alpha, \beta)$ generate[20] a subspace of $P$, denoted by $P(\alpha, \beta)$. The elements of $P(\alpha, \beta)$ are homogeneous polynomials, in which each monomial has the same content; we say that a polynomial in $P(\alpha, \beta)$ has content $(\alpha, \beta)$. It is clear that the product of a polynomial of content $(\alpha, \beta)$ and[21] a polynomial of content $(\alpha', \beta')$ is a polynomial of content $(\alpha + \alpha', \beta + \beta')$. For example, the inner product $\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$ has content $(\alpha, \beta)$ where $\alpha_i$ (resp. $\beta_i$) is 1 if $x_i$ is in the sequence $x_{i_1}, \ldots, x_{i_p}$ (resp. $u_i$ is in the sequence $u_{j_1}, \ldots, u_{j_p}$) and 0 otherwise.

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[16] *Comment:* I have added this "in" in order to stress that it is antisymmetric in the $p$ variables $i_1, i_2, \ldots, i_p$ and in the $p$ variables $j_1, j_2, \ldots, j_p$ separately, but not in all the $2p$ variables taken together.

[17] *Comment:* Explicitly, this is saying the following:

- If two of the numbers $i_1, i_2, \ldots, i_p$ are equal, then $\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) = 0$. The same holds if two of the numbers $j_1, j_2, \ldots, j_p$ are equal.
- If we interchange two of the numbers $i_1, i_2, \ldots, i_p$, then the inner product $\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$ gets multiplied by $-1$. The same happens if we interchange two of the numbers $j_1, j_2, \ldots, j_p$.

Both of these properties follow from the fact that

$$\begin{aligned}\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) &= \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}(\sigma) \left(x_{i_{\sigma 1}} \mid u_{j_1}\right) \cdots \left(x_{i_{\sigma p}} \mid u_{j_p}\right) \\ &= \begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix}\end{aligned}$$

(and from the antisymmetry of the determinant).

[18] *Correction:* I added these three sentences to clarify the definition of a "standard" tableau with entries in $\mathcal{X}$ or $\mathcal{U}$.

[19] *Correction:* Replaced "monomial" by "monomials" here.

[20] *Correction:* Replaced "generates" by "generate" here.

[21] *Correction:* Removed a misleading comma before this "and".

A *bitableau* is a pair $[T, T']$ of Young tableaux of the same shape $(\lambda)$, where the tableau $T$ has entries from $\mathcal{X}$ and the tableau $T'$ has entries from $\mathcal{U}$. The *content* of the bitableau $[T, T']$ is the pair of vectors $(\alpha, \beta)$ where $\alpha_i$ (resp. $\beta_j$) is the number of occurrences of $x_i$ in $T$ (resp. $u_j$ in $T'$). With a bitableau $[T, T']$ of content $(\alpha, \beta)$, we associate the polynomial, denoted by $(T \mid T')$, obtained by taking the product of the inner products of each row of $T$ with the corresponding row in $T'$. The polynomial $(T \mid T')$, which is in $P(\alpha, \beta)$, is called the *bideterminant* of the bitableau $[T, T']$, or simply, the bideterminant $(T \mid T')$.

**Example 3.1.**

$$\left( \begin{array}{lll|lll} x_1 & x_2 & x_3 & u_1 & u_3 & u_4 \\ x_2 & x_3 & & u_1 & u_2 & \\ x_1 & & & u_3 & & \end{array} \right) = (x_1 x_2 x_3 \mid u_1 u_3 u_4)(x_2 x_3 \mid u_1 u_2)(x_1 \mid u_3).$$

As for inner products, the bideterminant $(T \mid T')$ is nonzero if and only if no letter is repeated in any row of $T$ or $T'$. Moreover, we can suppose, up to a change of sign, that the entries in each row of $T$ and $T'$ in the bideterminant are increasing.

A bitableau $[T, T']$ is *standard* if both $T$ and $T'$ are standard. For example, the bitableau

$$\left[ \begin{array}{lll} x_1 & x_2 & x_3 \\ x_1 & x_3 & \\ x_2 & & \end{array} , \begin{array}{lll} u_1 & u_2 & u_4 \\ u_1 & u_3 & \\ u_3 & & \end{array} \right]$$

is standard.

We can now state the main result of this section.[22]

**Theorem 3.2** (the straightening formula)**.** Suppose $[T, T']$ is a bitableau of shape $(\lambda)$ and content $(\alpha, \beta)$. Then, its bideterminant $(T \mid T')$ is a linear combination, with integer coefficients, of bideterminants of standard bitableaux of the same content and of the same or longer shape[23].

[22]*Correction:* In the following theorem and also in the Corollary that follows it, I have replaced "standard tableaux" by "standard bitableaux".

[23]*Comment:* Here is a less ambiguous way to state this: Its bideterminant $(T \mid T')$ can be written in the form $(T \mid T') = \sum\limits_{j \in J} \alpha_j \left( S_j \mid S'_j \right)$, where $J$ is a finite set, each $\alpha_j$ is an integer, and each $\left[ S_j, S'_j \right]$ is a bitableau having the same content as $[T, T']$ and with the following property: The shape of $\left[ S_j, S'_j \right]$ is longer or equal to the shape of $[T, T']$. (The *shape* of a bitableau $[T, T']$ is defined to be the shape of the tableau $T$ (or, equivalently, the shape of the tableau $T'$).)

In general, we often speak of the "shape" or the "content" of a bideterminant, when meaning the shape or the content of the underlying bitableau. For example, a bideterminant of shape $(\lambda)$ and content $(\alpha, \beta)$ means a bideterminant of a bitableau of shape $(\lambda)$ and content $(\alpha, \beta)$.

**Example 3.3.**

$$\begin{pmatrix} x_2 & \big| & u_1 \\ x_1 & \big| & u_2 \end{pmatrix} = \begin{pmatrix} x_1 & \big| & u_1 \\ x_2 & \big| & u_2 \end{pmatrix} - \left( \ x_1 \quad x_2 \ \middle| \ u_1 \quad u_2 \ \right).$$

**Corollary 3.4.** The vector space $P(\alpha, \beta)$ is generated by the bideterminants of standard bitableaux of content $(\alpha, \beta)$.

*Proof of Corollary 3.4.* We only need to observe that the monomial $\left(x_{i_1} \mid u_{j_1}\right) \cdots \left(x_{i_p} \mid u_{j_p}\right)$ is the bideterminant of the bitableau:

$$\begin{bmatrix} x_{i_1} & & u_{j_1} \\ \vdots & , & \vdots \\ x_{i_p} & & u_{j_p} \end{bmatrix}.$$

$\square$

To facilitate the proof of Theorem 3.2, we introduce the notion of a *shuffle product*. Let

$$(i_1, \ldots, i_p, l_1, \ldots, l_q)$$

be an increasing sequence of integers, and

$$\begin{aligned} A &= \left(x_{i_1} \cdots x_{i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s}\right), \\ B &= \left(x_{l_1} \cdots x_{l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t}\right) \end{aligned}$$

be two inner products. The *shuffle product $AB$ supported by the variables* $x_{i_1}, \ldots, x_{i_p}, x_{l_1}, \ldots, x_{l_q}$ is defined by

$$\begin{aligned} &\left(\dot{x}_{i_1} \cdots \dot{x}_{i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s}\right) \left(\dot{x}_{l_1} \cdots \dot{x}_{l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t}\right) \\ &= \sum_{\sigma}{}' \operatorname{sgn}(\sigma) \\ &\qquad \left(x_{\sigma i_1} \cdots x_{\sigma i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s}\right) \left(x_{\sigma l_1} \cdots x_{\sigma l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t}\right), \end{aligned}$$

where the summation is over all permutations $\sigma$ of the set $\{i_1, \ldots, i_p, l_1, \ldots, l_q\}$ for which $\sigma i_1 < \cdots < \sigma i_p$ and $\sigma l_1 < \cdots < \sigma l_q$. This restricted summation is indicated by the notation $\sum'$. Another notational device is: A *dot* over a letter indicates that the letter is in the support of the shuffle product[24]. The notion of a shuffle product supported by letters in $\mathcal{U}$ is similar.

[24]*Comment:* We say that a letter is "in the support" of the shuffle product if this letter is one of the variables on which the shuffle product is supported.

**Example 3.5.** The shuffle product $(x_1x_2x_3 \mid \mathbf{u})\,(x_3x_4x_1 \mid \mathbf{u}')$ supported by $x_1, x_2$ in the first term and $x_3, x_4$ in the second is given by

$$\begin{aligned}
&(\dot{x}_1\dot{x}_2x_3 \mid \mathbf{u})\left(\dot{x}_3\dot{x}_4x_1 \mid \mathbf{u}'\right) \\
&= (x_1x_2x_3 \mid \mathbf{u})\left(x_3x_4x_1 \mid \mathbf{u}'\right) - (x_1x_3x_3 \mid \mathbf{u})\left(x_2x_4x_1 \mid \mathbf{u}'\right) \\
&\qquad + (x_1x_4x_3 \mid \mathbf{u})\left(x_2x_3x_1 \mid \mathbf{u}'\right) + (x_2x_3x_3 \mid \mathbf{u})\left(x_1x_4x_1 \mid \mathbf{u}'\right) \\
&\qquad - (x_2x_4x_3 \mid \mathbf{u})\left(x_1x_3x_1 \mid \mathbf{u}'\right) + (x_3x_4x_3 \mid \mathbf{u})\left(x_1x_2x_1 \mid \mathbf{u}'\right).
\end{aligned}$$

Only two of the terms in the expansion are nonzero, and after an appropriate reordering, we have

$$\begin{aligned}
&(\dot{x}_1\dot{x}_2x_3 \mid \mathbf{u})\left(x_1\dot{x}_3\dot{x}_4 \mid \mathbf{u}'\right) \\
&= (x_1x_2x_3 \mid \mathbf{u})\left(x_1x_3x_4 \mid \mathbf{u}'\right) - (x_1x_3x_4 \mid \mathbf{u})\left(x_1x_2x_3 \mid \mathbf{u}'\right).
\end{aligned}$$

Now, observe that, by definition,

$$\left(x_{i_1}\cdots x_{i_p} \mid u_{j_1}\cdots u_{j_p}\right) = \begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix}. \qquad (1)$$

We can expand the determinant by the first column to obtain the identity

$$\left(x_{i_1}\cdots x_{i_p} \mid u_{j_1}\cdots u_{j_p}\right) = \left(\dot{x}_{i_1} \mid u_{j_1}\right)\left(\dot{x}_{i_2}\cdots \dot{x}_{i_p} \mid u_{j_2}\cdots u_{j_p}\right). \qquad (2)$$

Similarly, using Laplace's expansion, we see that each $s \in \{0, 1, \ldots, p\}$ satisfies

$$\left(x_{i_1}\cdots x_{i_p} \mid u_{j_1}\cdots u_{j_p}\right) = \left(\dot{x}_{i_1}\cdots \dot{x}_{i_s} \mid u_{j_1}\cdots u_{j_s}\right)\left(\dot{x}_{i_{s+1}}\cdots \dot{x}_{i_p} \mid u_{j_{s+1}}\cdots u_{j_p}\right)$$

[25]. These two identities are examples of the fact that, under certain assumptions,

[25]*Comment:* This latter formula will be referred to as "*Laplace's identity*" further on in this paper. It follows from the following apocryphal property of determinants (sometimes known as "Laplace expansion in the first $s$ columns"):

**Laplace expansion in the first $s$ columns.** Given $p \in \mathbb{N}$ and $s \in \{0, 1, \ldots, p\}$ and a $p \times p$-matrix $A = \left(a_{i,j}\right)_{1 \le i \le p,\ 1 \le j \le p}$, we have

$$|A| = \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \operatorname{sgn}(\sigma) \begin{vmatrix} a_{\sigma 1,1} & \cdots & a_{\sigma 1,s} \\ \vdots & \ddots & \vdots \\ a_{\sigma s,1} & \cdots & a_{\sigma s,s} \end{vmatrix} \cdot \begin{vmatrix} a_{\sigma(s+1),s+1} & \cdots & a_{\sigma(s+1),p} \\ \vdots & \ddots & \vdots \\ a_{\sigma p,s+1} & \cdots & a_{\sigma p,p} \end{vmatrix}. \qquad (3)$$

Let me outline two proofs of (3):

- *First proof:* Argue that the right hand side of (3) is an alternating multilinear function in the

the shuffle product of two inner products is equal to an inner product of length

rows of $A$ (the alternating property follows from showing that the right hand side vanishes whenever two consecutive rows of $A$ are equal) that takes the value 1 when $A = I_p$. But it is well-known that the only such function is $|A|$.

- *Second proof:* For any permutations $\alpha \in \mathfrak{S}_s$ and $\beta \in \mathfrak{S}_{p-s}$, we define a permutation $\alpha \oplus \beta \in \mathfrak{S}_p$ as the map

$$\{1, 2, \ldots, p\} \to \{1, 2, \ldots, p\}, \qquad i \mapsto \begin{cases} \alpha(i), & \text{if } i \leq s; \\ \beta(i-s)+s, & \text{if } i > s \end{cases}.$$

Notice that the map $\mathfrak{S}_s \times \mathfrak{S}_{p-s} \to \mathfrak{S}_p$, $(\alpha, \beta) \mapsto \alpha \oplus \beta$ is a group homomorphism. Now, it is not hard to show that each permutation $\tau \in \mathfrak{S}_p$ can be **uniquely** written in the form $\tau = \sigma \circ (\alpha \oplus \beta)$, where $\sigma$ is a permutation in $\mathfrak{S}_p$ satisfying $\sigma 1 < \sigma 2 < \cdots < \sigma s$ and $\sigma(s+1) <$

$\sigma(s+2) < \cdots < \sigma p$, and where $\alpha \in \mathfrak{S}_s$ and $\beta \in \mathfrak{S}_{p-s}$. Thus,

$$\begin{aligned}
&\sum_{\tau \in \mathfrak{S}_p} \operatorname{sgn}(\tau) \prod_{i=1}^{p} a_{\tau i,i} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \sum_{\alpha \in \mathfrak{S}_s} \sum_{\beta \in \mathfrak{S}_{p-s}} \underbrace{\operatorname{sgn}(\sigma \circ (\alpha \oplus \beta))}_{= \operatorname{sgn}(\sigma) \operatorname{sgn}(\alpha \oplus \beta)} \underbrace{\prod_{i=1}^{p} a_{(\sigma \circ (\alpha \oplus \beta)) i,i}}_{= \left(\prod_{i=1}^{s} a_{(\sigma \circ (\alpha \oplus \beta)) i,i}\right) \left(\prod_{i=s+1}^{p} a_{(\sigma \circ (\alpha \oplus \beta)) i,i}\right)} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \sum_{\alpha \in \mathfrak{S}_s} \sum_{\beta \in \mathfrak{S}_{p-s}} \operatorname{sgn}(\sigma) \underbrace{\operatorname{sgn}(\alpha \oplus \beta)}_{= \operatorname{sgn}(\alpha) \operatorname{sgn}(\beta)} \left( \prod_{i=1}^{s} \underbrace{a_{(\sigma \circ (\alpha \oplus \beta)) i,i}}_{\substack{= a_{\sigma(\alpha i),i} \\ \text{(since } (\alpha \oplus \beta) i = \alpha i)}} \right) \left( \prod_{i=s+1}^{p} \underbrace{a_{(\sigma \circ (\alpha \oplus \beta)) i,i}}_{\substack{= a_{\sigma(\beta(i-s)+s),i} \\ \text{(since } (\alpha \oplus \beta) i = \beta(i-s)+s)}} \right) \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \sum_{\alpha \in \mathfrak{S}_s} \sum_{\beta \in \mathfrak{S}_{p-s}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\alpha) \operatorname{sgn}(\beta) \left( \prod_{i=1}^{s} a_{\sigma(\alpha i),i} \right) \underbrace{\left( \prod_{i=s+1}^{p} a_{\sigma(\beta(i-s)+s),i} \right)}_{\substack{= \prod_{i=1}^{p-s} a_{\sigma(\beta(i)+s),i+s} \\ \text{(here, we have substituted } i+s \\ \text{for } i \text{ in the sum)}}} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \sum_{\alpha \in \mathfrak{S}_s} \sum_{\beta \in \mathfrak{S}_{p-s}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\alpha) \operatorname{sgn}(\beta) \left( \prod_{i=1}^{s} a_{\sigma(\alpha i),i} \right) \left( \prod_{i=1}^{p-s} a_{\sigma(\beta(i)+s),i+s} \right) \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \operatorname{sgn}(\sigma) \underbrace{\left( \sum_{\alpha \in \mathfrak{S}_s} \operatorname{sgn}(\alpha) \prod_{i=1}^{s} a_{\sigma(\alpha i),i} \right)}_{= \begin{vmatrix} a_{\sigma 1,1} & \cdots & a_{\sigma 1,s} \\ \vdots & \ddots & \vdots \\ a_{\sigma s,1} & \cdots & a_{\sigma s,s} \end{vmatrix}} \underbrace{\left( \sum_{\beta \in \mathfrak{S}_{p-s}} \operatorname{sgn}(\beta) \prod_{i=1}^{p-s} a_{\sigma(\beta(i)+s),i+s} \right)}_{= \begin{vmatrix} a_{\sigma(s+1),s+1} & \cdots & a_{\sigma(s+1),p} \\ \vdots & \ddots & \vdots \\ a_{\sigma p,s+1} & \cdots & a_{\sigma p,p} \end{vmatrix}} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \operatorname{sgn}(\sigma) \begin{vmatrix} a_{\sigma 1,1} & \cdots & a_{\sigma 1,s} \\ \vdots & \ddots & \vdots \\ a_{\sigma s,1} & \cdots & a_{\sigma s,s} \end{vmatrix} \cdot \begin{vmatrix} a_{\sigma(s+1),s+1} & \cdots & a_{\sigma(s+1),p} \\ \vdots & \ddots & \vdots \\ a_{\sigma p,s+1} & \cdots & a_{\sigma p,p} \end{vmatrix}.
\end{aligned}$$

Since the left hand side of this equality is precisely $|A|$ (by the definition of $|A|$), this rewrites as

$$|A| = \sum_{\substack{\sigma \in \mathfrak{S}_p; \\ \sigma 1 < \sigma 2 < \cdots < \sigma s; \\ \sigma(s+1) < \sigma(s+2) < \cdots < \sigma p}} \operatorname{sgn}(\sigma) \begin{vmatrix} a_{\sigma 1,1} & \cdots & a_{\sigma 1,s} \\ \vdots & \ddots & \vdots \\ a_{\sigma s,1} & \cdots & a_{\sigma s,s} \end{vmatrix} \cdot \begin{vmatrix} a_{\sigma(s+1),s+1} & \cdots & a_{\sigma(s+1),p} \\ \vdots & \ddots & \vdots \\ a_{\sigma p,s+1} & \cdots & a_{\sigma p,p} \end{vmatrix}.$$

This proves (3).

greater than that of each of the two original inner products.[26]

**Lemma 3.6.** Let $\left(x_{i_1}\cdots x_{i_p}x_{i_{p+1}}\cdots x_{i_s}\mid u_{j_1}\cdots u_{j_s}\right)$ and $\left(x_{l_1}\cdots x_{l_q}x_{l_{q+1}}\cdots x_{l_t}\mid u_{m_1}\cdots u_{m_t}\right)$ be two inner products satisfying:

$$\begin{aligned}
&i_1<\cdots<i_p<l_1<\cdots<l_q,\\
&j_1<\cdots<j_s,\\
&m_1<\cdots<m_t,\\
&s<p+q, \qquad \text{and} \qquad t<p+q.
\end{aligned}$$

Then the shuffle product

$$C=\left(\dot{x}_{i_1}\cdots\dot{x}_{i_p}x_{i_{p+1}}\cdots x_{i_s}\mid u_{j_1}\cdots u_{j_s}\right)\left(\dot{x}_{l_1}\cdots\dot{x}_{l_q}x_{l_{q+1}}\cdots x_{l_t}\mid u_{m_1}\cdots u_{m_t}\right)$$

is a linear combination, with integer coefficients, of bideterminants of bitableaux of shape strictly longer than each of the partitions $(s)$ and $(t)$ of the integers $s$ and $t$.

*Proof of Lemma 3.6.* The proof is a computation with four steps. First, expand the shuffle product $C$:

$$C=\sum_{\sigma}{}'\operatorname{sgn}(\sigma)$$
$$\left(x_{\sigma i_1}\cdots x_{\sigma i_p}x_{i_{p+1}}\cdots x_{i_s}\mid u_{j_1}\cdots u_{j_s}\right)\left(x_{\sigma l_1}\cdots x_{\sigma l_q}x_{l_{q+1}}\cdots x_{l_t}\mid u_{m_1}\cdots u_{m_t}\right).$$

Now apply Laplace's identity to the letters in $U$;

$$\begin{aligned}
C=\sum_{\sigma}{}'\operatorname{sgn}(\sigma)&\left(x_{\sigma i_1}\cdots x_{\sigma i_p}\mid \dot{u}_{j_1}\cdots\dot{u}_{j_p}\right)\left(x_{i_{p+1}}\cdots x_{i_s}\mid \dot{u}_{j_{p+1}}\cdots\dot{u}_{j_s}\right)\\
&\cdot\left(x_{\sigma l_1}\cdots x_{\sigma l_q}\mid \overline{u}_{m_1}\cdots\overline{u}_{m_q}\right)\left(x_{l_{q+1}}\cdots x_{l_t}\mid \overline{u}_{m_{q+1}}\cdots\overline{u}_{m_t}\right).
\end{aligned}$$

To distinguish between the two shuffle products, a bar instead of a dot is used in the second. We next group together the first and third factor:

$$\begin{aligned}
C=\sum_{\sigma,\tau,\mu}{}'\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\operatorname{sgn}(\mu)&\left[\left(x_{\sigma i_1}\cdots x_{\sigma i_p}\mid u_{\tau j_1}\cdots u_{\tau j_p}\right)\left(x_{\sigma l_1}\cdots x_{\sigma l_q}\mid u_{\mu m_1}\cdots u_{\mu m_q}\right)\right]\\
&\cdot\left(x_{i_{p+1}}\cdots x_{i_s}\mid u_{\tau j_{p+1}}\cdots u_{\tau j_s}\right)\left(x_{l_{q+1}}\cdots x_{l_t}\mid u_{\mu m_{q+1}}\cdots u_{\mu m_t}\right).
\end{aligned}$$

[26] *Correction:* Replaced "$\left(x_{i_1}\cdots x_{i_p}x_{i_{p+1}}\cdots x_{i_1}\mid u_{j_1}\cdots u_{j_s}\right)$" by "$\left(x_{i_1}\cdots x_{i_p}x_{i_{p+1}}\cdots x_{i_s}\mid u_{j_1}\cdots u_{j_s}\right)$" in the following lemma.

Finally, we apply Laplace's identity, this time on the letters $x_{i_1}, \ldots, x_{i_p}, x_{l_1}, \ldots, x_{l_q}$:

$$C = \sum_{\tau,\mu}{}' \operatorname{sgn}(\tau) \operatorname{sgn}(\mu) \left( x_{i_1} \cdots x_{i_p} x_{l_1} \cdots x_{l_q} \mid u_{\tau j_1} \cdots u_{\tau j_p} u_{\mu m_1} \cdots u_{\mu m_q} \right)$$
$$\cdot \left( x_{i_{p+1}} \cdots x_{i_s} \mid u_{\tau j_{p+1}} \cdots u_{\tau j_s} \right) \left( x_{l_{q+1}} \cdots x_{l_t} \mid u_{\mu m_{q+1}} \cdots u_{\mu m_t} \right).$$

Each term in this last expansion for $C$ is a bideterminant with three rows[27], with the first row of length $p + q > s, t$. This concludes the proof of the lemma. □

Since the summation in the shuffle product always includes the identity permutation, we can restate the previous lemma in the following equivalent form:[28]

**Lemma 3.7.** Let $\left( x_{i_1} \cdots x_{i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s} \right)$ and $\left( x_{l_1} \cdots x_{l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t} \right)$ be two inner products satisfying

$$\begin{aligned} &i_1 < \cdots < i_p < l_1 < \cdots < l_q, \\ &j_1 < \cdots < j_s, \\ &m_1 < \cdots < m_t, \\ &s < p+q, \qquad \text{and} \qquad t < p+q. \end{aligned}$$

Then,

$$\begin{aligned} &\left( x_{i_1} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s} \right) \left( x_{l_1} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t} \right) \\ &= - \sum_{\substack{\sigma, \\ \sigma \neq \mathrm{id}}}{}' \operatorname{sgn}(\sigma) \left( x_{\sigma i_1} \cdots x_{\sigma i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s} \right) \\ &\qquad \left( x_{\sigma l_1} \cdots x_{\sigma l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t} \right) \\ &\qquad + D, \end{aligned}$$

where the summation is over all the nonidentical permutations $\sigma$ of $\{i_1, \ldots, i_p, l_1, \ldots, l_q\}$ satisfying $\sigma i_1 < \cdots < \sigma i_p$ and $\sigma l_1 < \cdots < \sigma l_q$, and the term $D$ is a linear combination with integer coefficients of bideterminants of bitableaux of shape strictly longer than $(s)$ and $(t)$.

Remarking that all we have done remains valid if we exchange the roles of the alphabets $\mathcal{X}$ and $\mathcal{U}$, we are now ready to prove Theorem 3.2.

*Proof of Theorem 3.2.* We begin by defining a total order on bitableaux of the same shape. Let $[T, T']$ be a bitableau of shape[29] $(\lambda) = (\lambda_1, \ldots, \lambda_p)$. The entry in $T$ (resp.

[27] *Comment:* Some of these rows can be empty. (This happens when $p = s$ or $q = t$.)
[28] *Correction:* Added "of $\{i_1, \ldots, i_p, l_1, \ldots, l_q\}$" in the following lemma.
[29] *Correction:* Replaced "form" by "shape".

$T'$) on row $s$ and column $t$ is denoted by $x_{i(s,t)}$ (resp. $u_{j(s,t)}$). With the bitableau $[T, T']$, we associate the sequence

$$\left(i(1,1), \ldots, i(1,\lambda_1), \ i(2,1), \ldots, i(2,\lambda_2), \ \ldots, \ i(p,1), \ldots, i\left(p,\lambda_p\right), \right.$$
$$\left. j(1,1), \ldots, j(1,\lambda_1), \ j(2,1), \ldots, j(2,\lambda_2), \ \ldots, \ j(p,1), \ldots, j\left(p,\lambda_p\right)\right),$$

which is the sequence obtained by reading off the bitableau row by row[30]. The bitableaux are now ordered according to the lexicographic order on their associated sequences.

Fix a pair of vectors $(\alpha, \beta)$, and consider from now on solely the bitableaux of content $(\alpha, \beta)$. Now, suppose the theorem is false for at least one of these bitableaux.[31] Let $(\lambda)$ be the longest shape with a bitableau of content $(\alpha, \beta)$ not satisfying the theorem[32]. Among the bitableaux of shape $(\lambda)$ and content $(\alpha, \beta)$, let $[T, T']$ be the smallest bitableau not satisfying the theorem:

$$[T, T'] = \left[\begin{array}{cccc} x_{i(1,1)} & \cdots & \cdots & x_{i(1,\lambda_1)} \\ \vdots & & \vdots & \\ x_{i(p,1)} & \cdots & x_{i(p,\lambda_p)} & \end{array}, \begin{array}{cccc} u_{j(1,1)} & \cdots & \cdots & u_{j(1,\lambda_1)} \\ \vdots & & \vdots & \\ u_{j(p,1)} & \cdots & u_{j(p,\lambda_p)} & \end{array}\right].$$

Suppose that $[T_1, T_1']$ is obtained from $[T, T']$ by putting each row in increasing order. Then, $[T_1, T_1']$ has a lexicographically smaller associated sequence than $[T, T']$ (unless $[T_1, T_1'] = [T, T']$) [33]. But $(T_1 \mid T_1') = \pm (T \mid T')$, and hence, if $[T, T']$ is a counterexample, so is $[T_1, T_1']$. We conclude that all the rows in $[T, T']$ are in increasing order.

Clearly, $[T, T']$ is not standard; let us suppose that $T$ is nonstandard. Then, there exist integers $l$ and $m$, with[34] $1 \le l \le p$ and[35] $1 \le m \le \lambda_{l+1}$, such that $i(l, m) > i(l+1, m)$. That is, we have the following situation:[36]

$$\begin{array}{ccccccccc} x_{i(l,1)} & \cdots & x_{i(l,m-1)} & x_{i(l,m)} & x_{i(l,m+1)} & \cdots & \cdots & x_{i(l,\lambda_l)} \\ x_{i(l+1,1)} & \cdots & x_{i(l+1,m-1)} & x_{i(l+1,m)} & x_{i(l+1,m+1)} & \cdots & x_{i(l+1,\lambda_{l+1})} & \end{array}.$$

We call such a situation a *violation*.

In the bideterminant $(T \mid T')$, consider the shuffle product of the two inner products corresponding to rows $l$ and $l + 1$, which support the letters

[30] *Comment:* ... where we first read the tableau $T$ and then the tableau $T'$.

[31] *Correction:* I have added the preceding two sentences (replacing "Now, suppose the theorem is false.") and also added the words "of content $(\alpha, \beta)$" in the next two sentences. Otherwise, the following sentence would not make sense (there might not be a "longest shape").

[32] *Comment:* This is well-defined, because there are only finitely many shapes that afford bitableaux of content $(\alpha, \beta)$, and at least one of these bitableaux fails to satisfy the theorem.

[33] *Correction:* I have added the preceding parenthetical.

[34] *Correction:* Added the word "with". Also, replaced "$1 \le m \le \lambda_l$" by "$1 \le m \le \lambda_{l+1}$".

[35] *Correction:* Replaced the ambiguous comma by an "and".

[36] *Correction:* Fixed the size of the parentheses in the table below, and removed an unnecessary comma.

$x_{i(l+1,1)}, \ldots, x_{i(l+1,m)}, x_{i(l,m)}, \ldots, x_{i(l,\lambda_l)}$. Since the support contains $\lambda_l + 1$ letters, and the length of each of the inner products is at most $\lambda_l$, we can apply Lemma 3.7[37] to obtain

$$\left(T \mid T'\right) = - \sum_{\substack{\sigma, \\ \sigma \neq \mathrm{id}}}{}' \pm \left(T_\sigma \mid T'\right) + D, \tag{4}$$

[37] *Correction:* Replaced "the lemma" by "Lemma 3.7".

where $D$ is a linear combination of bideterminants of shape greater[38] than $(\lambda)$. [39]

[38] *Comment:* Here and in the following, "greater" means "strictly longer".

[39] *Comment:* Let us explain why this holds.

For each $k \in \{1, 2, \ldots, p\}$, let

$$r_k = \left( x_{i(k,1)} \cdots x_{i(k,\lambda_k)} \mid u_{j(k,1)} \cdots u_{j(k,\lambda_k)} \right).$$

Then, $(T \mid T') = r_1 r_2 \cdots r_p$ (by the definition of the bideterminant $(T \mid T')$).

Now, consider the inner products

$$r_{l+1} = \left( x_{i(l+1,1)} \cdots x_{i(l+1,m-1)} x_{i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})} \right)$$

and

$$\begin{aligned} r_l &= \left( x_{i(l,1)} \cdots x_{i(l,m-1)} x_{i(l,m)} x_{i(l,m+1)} \cdots x_{i(l,\lambda_l)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)} \right) \\ &= \left( x_{i(l,m)} x_{i(l,m+1)} \cdots x_{i(l,\lambda_l)} x_{i(l,1)} \cdots x_{i(l,m-1)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)} \right). \end{aligned}$$

These two inner products satisfy

$$i\left(l+1,1\right) < \cdots < i\left(l+1,m\right) < i\left(l,m\right) < \cdots < i\left(l,\lambda_l\right)$$

(since $i(l,m) > i(l+1,m)$ and since the rows of $T$ are increasing) and $j(l+1,1) < \cdots < j(l+1,\lambda_{l+1})$ and $j(l,1) < \cdots < j(l,\lambda_l)$ and $\lambda_{l+1} < \lambda_l + 1$ and $\lambda_l < \lambda_l + 1$. Hence, Lemma 3.7 can be applied to these two inner products, showing that

$$\begin{aligned} & r_{l+1} r_l \\ &= - \sum_{\substack{\sigma, \\ \sigma \neq \mathrm{id}}}{}' \operatorname{sgn}(\sigma) \left( x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})} \right) \\ &\qquad \left( x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} x_{i(l,1)} \cdots x_{i(l,m-1)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)} \right) \\ &\qquad + D', \end{aligned} \tag{5}$$

where the summation is over all the nonidentical permutations $\sigma$ of $\{i(l+1,1), \ldots, i(l+1,m), i(l,m), \ldots, i(l,\lambda_l)\}$ satisfying $\sigma i(l+1,1) < \cdots < \sigma i(l+1,m)$ and $\sigma i(l,m) < \cdots < \sigma i(l,\lambda_l)$, and the term $D'$ is a linear combination with integer coefficients of bideterminants of bitableaux of shape strictly longer than $(\lambda_l)$ and $(\lambda_{l+1})$.

Now, multiplying the equality (5) by $(r_1 r_2 \cdots r_{l-1}) (r_{l+2} r_{l+3} \cdots r_p)$, we obtain

$$\begin{aligned} & r_{l+1} r_l \left( r_1 r_2 \cdots r_{l-1} \right) \left( r_{l+2} r_{l+3} \cdots r_p \right) \\ &= - \sum_{\substack{\sigma, \\ \sigma \neq \mathrm{id}}}{}' \operatorname{sgn}(\sigma) \left( x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})} \right) \\ &\qquad \left( x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} x_{i(l,1)} \cdots x_{i(l,m-1)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)} \right) \\ &\qquad \left( r_1 r_2 \cdots r_{l-1} \right) \left( r_{l+2} r_{l+3} \cdots r_p \right) \\ &\qquad + D' \left( r_1 r_2 \cdots r_{l-1} \right) \left( r_{l+2} r_{l+3} \cdots r_p \right). \end{aligned}$$

Since $r_{l+1}r_l\left(r_1r_2\cdots r_{l-1}\right)\left(r_{l+2}r_{l+3}\cdots r_p\right)=r_1r_2\cdots r_p=\left(T\mid T'\right)$, this rewrites as

$$\begin{aligned}&\left(T\mid T'\right)\\&=-\sum_{\substack{\sigma,\\ \sigma\neq\mathrm{id}}}{}' \operatorname{sgn}(\sigma)\left(x_{\sigma i(l+1,1)}\cdots x_{\sigma i(l+1,m)}x_{i(l+1,m+1)}\cdots x_{i(l+1,\lambda_{l+1})}\mid u_{j(l+1,1)}\cdots u_{j(l+1,\lambda_{l+1})}\right)\\&\qquad\left(x_{\sigma i(l,m)}x_{\sigma i(l,m+1)}\cdots x_{\sigma i(l,\lambda_l)}x_{i(l,1)}\cdots x_{i(l,m-1)}\mid u_{j(l,1)}\cdots u_{j(l,\lambda_l)}\right)\\&\qquad\left(r_1r_2\cdots r_{l-1}\right)\left(r_{l+2}r_{l+3}\cdots r_p\right)\\&\qquad+D'\left(r_1r_2\cdots r_{l-1}\right)\left(r_{l+2}r_{l+3}\cdots r_p\right).\end{aligned}$$

Thus, in order to prove (4), we need to verify the following two claims:

*Claim 1:* Let $\sigma$ be a nonidentical permutation of $\{i(l+1,1),\ldots,i(l+1,m),i(l,m),\ldots,i(l,\lambda_l)\}$ satisfying $\sigma i(l+1,1)<\cdots<\sigma i(l+1,m)$ and $\sigma i(l,m)<\cdots<\sigma i(l,\lambda_l)$. Then,

$$\begin{aligned}&\left(x_{\sigma i(l+1,1)}\cdots x_{\sigma i(l+1,m)}x_{i(l+1,m+1)}\cdots x_{i(l+1,\lambda_{l+1})}\mid u_{j(l+1,1)}\cdots u_{j(l+1,\lambda_{l+1})}\right)\\&\qquad\left(x_{\sigma i(l,m)}x_{\sigma i(l,m+1)}\cdots x_{\sigma i(l,\lambda_l)}x_{i(l,1)}\cdots x_{i(l,m-1)}\mid u_{j(l,1)}\cdots u_{j(l,\lambda_l)}\right)\\&\qquad\left(r_1r_2\cdots r_{l-1}\right)\left(r_{l+2}r_{l+3}\cdots r_p\right)\\&=\pm\left(T_\sigma\mid T'\right)\end{aligned}$$

for some tableau $T_\sigma$ which differs from $T$ only in rows $l$ and $l+1$, and whose rows $l$ and $l+1$ are

$$\begin{array}{cccccccc}x_{i(l,1)}&\cdots&x_{i(l,m-1)}&x_{\sigma i(l,m)}&x_{\sigma i(l,m+1)}&\cdots&\cdots&x_{\sigma i(l,\lambda_l)}\\x_{\sigma i(l+1,1)}&\cdots&x_{\sigma i(l+1,m-1)}&x_{\sigma i(l+1,m)}&x_{i(l+1,m+1)}&\cdots&x_{i(l+1,\lambda_{l+1})}&\end{array}.$$

*Claim 2:* Let $[S,S']$ be a bitableau of shape strictly longer than $(\lambda_l)$ and $(\lambda_{l+1})$. Then,

$$\left(S\mid S'\right)\left(r_1r_2\cdots r_{l-1}\right)\left(r_{l+2}r_{l+3}\cdots r_p\right)$$

is a bideterminant of shape greater than $(\lambda)$.

[*Proof of Claim 1:* Let $T_\sigma$ be the tableau which differs from $T$ only in rows $l$ and $l+1$, and whose rows $l$ and $l+1$ are

$$\begin{array}{cccccccc}x_{i(l,1)}&\cdots&x_{i(l,m-1)}&x_{\sigma i(l,m)}&x_{\sigma i(l,m+1)}&\cdots&\cdots&x_{\sigma i(l,\lambda_l)}\\x_{\sigma i(l+1,1)}&\cdots&x_{\sigma i(l+1,m-1)}&x_{\sigma i(l+1,m)}&x_{i(l+1,m+1)}&\cdots&x_{i(l+1,\lambda_{l+1})}&\end{array}.$$

Then,

$$\begin{aligned}
&\left(T_\sigma \mid T'\right) \\
&= (r_1 r_2 \cdots r_{l-1}) \\
&\qquad \cdot \left(x_{i(l,1)} \cdots x_{i(l,m-1)} x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)}\right) \\
&\qquad \cdot \left(x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})}\right) \\
&\qquad \cdot (r_{l+2} r_{l+3} \cdots r_p) \\
&= \left(x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})}\right) \\
&\qquad \underbrace{\left(x_{i(l,1)} \cdots x_{i(l,m-1)} x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)}\right)}_{\substack{= \pm \left(x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} x_{i(l,1)} \cdots x_{i(l,m-1)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)}\right) \\ \text{(by the antisymmetry of the inner product)}}} \\
&\qquad (r_1 r_2 \cdots r_{l-1}) (r_{l+2} r_{l+3} \cdots r_p) \\
&= \pm \left(x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})}\right) \\
&\qquad \left(x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} x_{i(l,1)} \cdots x_{i(l,m-1)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)}\right) \\
&\qquad (r_1 r_2 \cdots r_{l-1}) (r_{l+2} r_{l+3} \cdots r_p) .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l+1,m)} x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \mid u_{j(l+1,1)} \cdots u_{j(l+1,\lambda_{l+1})}\right) \\
&\qquad \left(x_{\sigma i(l,m)} x_{\sigma i(l,m+1)} \cdots x_{\sigma i(l,\lambda_l)} x_{i(l,1)} \cdots x_{i(l,m-1)} \mid u_{j(l,1)} \cdots u_{j(l,\lambda_l)}\right) \\
&\qquad (r_1 r_2 \cdots r_{l-1}) (r_{l+2} r_{l+3} \cdots r_p) \\
&= \pm \left(T_\sigma \mid T'\right) .
\end{aligned}$$

This proves Claim 1.]

[*Proof of Claim 2:* A *composition* shall mean a finite sequence of positive integers. Note that each partition is a composition, but not each composition is a partition. If $(\mu)$ is a composition, then the notion of the "shape of $(\mu)$" is defined in the same way as it has been defined when $\mu$ is a partition. If $(\mu)$ is a composition, then an *almost-tableau* of shape $(\mu)$ with values in the set $E$ means an assignment of an element of $E$ to each point in the shape $(\mu)$.

Define a Young tableau $\widetilde{T}$ by the following procedure:

- *Step 1:* Let $T_\circ$ be the Young tableau $T$.
- *Step 2:* Replace the $l$-th and the $(l+1)$-st rows of the tableau $T_\circ$ by the rows of $S$. (Notice that $S$ may have fewer or more than 2 rows; thus, the number of rows of $T_\circ$ is not necessarily preserved in this step. Also, the resulting array $T_\circ$ is an almost-tableau, but not necessarily a Young tableau, because the lengths of its rows might no longer be in weakly decreasing order.)
- *Step 3:* Permute the rows of the almost-tableau $T_\circ$ in such a way that their lengths become weakly decreasing (i.e., it becomes a Young tableau once again). Denote the resulting Young tableau $T_\circ$ by $\widetilde{T}$.

By our choice of $(\lambda)$, the polynomial[40] $D$ is also a linear combination of bideterminants of standard tableaux of shape greater than $(\lambda)$.

Now, each tableau $T_\sigma$ differs from $T$ only in rows $l$ and $l+1$, which in $T_\sigma$ are:[41]

$$\begin{array}{cccccccc} x_{i(l,1)} & \cdots & x_{i(l,m-1)} & x_{\sigma i(l,m)} & x_{\sigma i(l,m+1)} & \cdots & \cdots & x_{\sigma i(l,\lambda_l)} \\ x_{\sigma i(l+1,1)} & \cdots & x_{\sigma i(l+1,m-1)} & x_{\sigma i(l+1,m)} & x_{i(l+1,m+1)} & \cdots & x_{i(l+1,\lambda_{l+1})} & \end{array}.$$

In the tableau $T$, however, we have the inequalities

$$\begin{array}{ccc} & i\left(l,m\right) & < i\left(l,m+1\right) < \cdots < i\left(l,\lambda_l\right) \\ & \vee & \\ i\left(l+1,1\right) < \cdots < i\left(l+1,m-1\right) < & i\left(l+1,m\right) & \end{array}.$$

For any nonidentical permutation $\sigma$ in the shuffle product, the index $\sigma i\left(l,m\right)$ must equal one of the indices $i\left(l+1,1\right),\ldots,i\left(l+1,m\right)$; in particular, we have[42] $\sigma i\left(l,m\right) < i\left(l,m\right)$. Thus, the tableau $T_\sigma$ has a lexicographically smaller associated sequence than $T$. By our choice of $[T,T']$, however, this shows that[43] each of the bitableaux[44] $[T_\sigma,T']$ satisfies the theorem, and hence, by substitution, we can write

Similarly, define a Young tableau $\widetilde{T'}$ (using the tableaux $T'$ and $S'$ instead of $T$ and $S$); in doing so, make sure to permute the rows in the same way as during the construction of $\widetilde{T}$.

The resulting bitableau $\left[\widetilde{T},\widetilde{T'}\right]$ has the bideterminant

$$\left(\widetilde{T} \mid \widetilde{T'}\right) = \left(S \mid S'\right) \left(r_1 r_2 \cdots r_{l-1}\right) \left(r_{l+2} r_{l+3} \cdots r_p\right)$$

(because its rows are precisely the rows of the bitableau $[T,T']$ except for its $l$-th and $(l+1)$-st rows, which have been replaced by the rows of the bitableau $[S,S']$).

Now, we shall show that the shape of $\widetilde{T}$ is greater than $(\lambda)$.

Indeed, the bitableau $[S,S']$ has shape strictly longer than $(\lambda_l)$. In other words, the tableau $S$ has shape strictly longer than $(\lambda_l)$. Thus, the first row of $S$ has length $> \lambda_l$. Thus, in Step 2 of our procedure by which we defined $\widetilde{T}$, the shape of the almost-tableau $T_\circ$ has increased in lexicographic order (because the $l$-th row, which used to have length $\lambda_l$, now has length $> \lambda_l$, while the rows above it have preserved their lengths). Furthermore, in Step 3 of the procedure, the shape of $T_\circ$ has either stayed unchanged or increased in lexicographic order (because when we sort a list of numbers in weakly decreasing order, this list either stays unchanged or increases in lexicographic order). Thus, throughout the procedure, the shape of $T_\circ$ has increased in lexicographic order. Therefore, the shape of the tableau obtained at the end of the procedure is greater than the shape of the tableau at its beginning. In other words, the shape of $\widetilde{T}$ is greater than the shape of $T$ (because the tableau obtained at the end of the procedure is $\widetilde{T}$, while the tableau at its beginning is $T$). In other words, the shape of $\widetilde{T}$ is greater than $(\lambda)$ (since the shape of $T$ is $(\lambda)$). Hence, $\left(\widetilde{T} \mid \widetilde{T'}\right)$ is a bideterminant of shape greater than $(\lambda)$.

Because of $\left(\widetilde{T} \mid \widetilde{T'}\right) = \left(S \mid S'\right)\left(r_1 r_2 \cdots r_{l-1}\right)\left(r_{l+2} r_{l+3} \cdots r_p\right)$, this rewrites as follows: $\left(S \mid S'\right)\left(r_1 r_2 \cdots r_{l-1}\right)\left(r_{l+2} r_{l+3} \cdots r_p\right)$ is a bideterminant of shape greater than $(\lambda)$. This proves Claim 2.]

[40] *Correction:* Added the words "the polynomial" in order to separate expressions.

[41] *Correction:* Added the words "which in $T_\sigma$ are".

[42] *Correction:* Added "we have" and replaced comma by semicolon for disambiguation.

[43] *Correction:* Added the words "this shows that" for clarity.

[44] *Correction:* Replaced "tableaux" by "bitableaux".

$(T \mid T')$ as a sum of bideterminants of standard bitableaux[45] of shape equal to or longer than $(\lambda)$. This contradicts our initial assumption.

It remains to observe that if $T$ were standard, then $T'$ would have to be nonstandard; the same reasoning can then be applied to $T'$ to yield a contradiction. This concludes the proof of the theorem. $\square$

The proof contains implicitly an algorithm for expressing any bitableau as a linear combination of standard bitableaux[46] by successive corrections of violations. This is inefficient for practical computations, as the number of bitableaux introduced during a correction is, in general, very large.

As an exercise, apply the algorithm to obtain the following identity (only the subscripts are shown):

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 2 \\ 1 & 4 & 1 & 3 \\ 2 & & 1 & \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \\ 2 & & 1 & \end{array}\right) - \left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 3 \\ 3 & & 1 & \end{array}\right) + \left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 4 & & 1 & \end{array}\right) - \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & & & 1 & & \\ 4 & & & 1 & & \end{array}\right).$$

## 4. The Basis Theorem

As we have seen, the standard bideterminants (i.e., the bideterminants of standard bitableaux[47]) of content $(\alpha, \beta)$ span[48] the vector space $P(\alpha, \beta)$. In this section, using the technique of set polarization operators, we show that, in fact, they form a basis.

We augment the alphabets $\mathcal{X}$ and $\mathcal{U}$ by adding new letters from the sets $\mathcal{S}$ and $\mathcal{T}$, respectively. The sets $\mathcal{S}$ and $\mathcal{T}$ are supposed finite, but large enough that the ensuing constructions can be performed. This enlarges the algebra $P$, even though the vector space $P(\alpha, \beta)$ remains unchanged.[49]

Thus, a bitableau can have entries from all of the four alphabets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{S}$ and $\mathcal{T}$ (more precisely, it consists of a tableau with entries from $\mathcal{X} \cup \mathcal{S}$ and a tableau with entries from $\mathcal{U} \cup \mathcal{T}$). A bitableau with entries from the alphabets $\mathcal{X}$ and $\mathcal{U}$ only will

[45] *Correction:* Replaced "tableaux" by "bitableaux".

[46] *Comment:* More precisely: "... for expressing the bideterminant of any bitableau as a linear combination of bideterminants of standard bitableaux".

[47] *Correction:* Replaced "tableaux" by "bitableaux".

[48] *Correction:* Replaced "spans" by "span".

[49] *Comment:* Here, we identify each pair of vectors $(\alpha, \beta) = ((\alpha_1, \alpha_2, \ldots, \alpha_n), (\beta_1, \beta_2, \ldots, \beta_k))$ with the pair of vectors

$$\left(\left(\alpha_1, \alpha_2, \ldots, \alpha_n, \underbrace{0, 0, \ldots, 0}_{|\mathcal{S}| \text{ many zeroes}}\right), \left(\beta_1, \beta_2, \ldots, \beta_k, \underbrace{0, 0, \ldots, 0}_{|\mathcal{T}| \text{ many zeroes}}\right)\right);$$

this allows us to speak of $P(\alpha, \beta)$ even after the alphabets $\mathcal{X}$ and $\mathcal{U}$ have been enlarged.

be called an $(\mathcal{X},\mathcal{U})$-*bitableau*. The $(\mathcal{X},\mathcal{U})$-*content* of a bitableau $[T,T']$ is defined as the pair of vectors $(\alpha,\beta)$ where $\alpha_i$ (resp. $\beta_j$) is the number of occurrences of $x_i$ in $T$ (resp. $u_j$ in $T'$). (Thus, the $(\mathcal{X},\mathcal{U})$-content differs from the usual content only in that it disregards the new letters from $\mathcal{S}$ and $\mathcal{T}$.) An $(\mathcal{X},\mathcal{U})$-*bideterminant* means a bideterminant of an $(\mathcal{X},\mathcal{U})$-bitableau. [50]

The $(\mathcal{X},\mathcal{U})$-*content* of a monomial in $P$ is the pair of vectors

$$(\alpha,\beta) = ((\alpha_1,\ldots,\alpha_n),(\beta_1,\ldots,\beta_k)),$$

where $\alpha_s$ (resp. $\beta_t$) is the total degree of the factors in the monomial of the form $(x_s \mid u)$ with $u \in \mathcal{U}\cup\mathcal{T}$ (resp. $(x \mid u_t)$ with $x \in \mathcal{X}\cup\mathcal{S}$). [51]

Let $x_i$, $u_i$, $s_j$, and $t_j$ be letters from the alphabets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{S}$, and $\mathcal{T}$. The *set polarization operators* $D^l(s_j,x_i)$ (for integers $l \geq 0$) [52] are defined as follows: Let

$$M = (x_{i_1} \mid u'_1)\cdots\left(x_{i_p} \mid u'_p\right)\cdot(s_{j_1} \mid v'_1)\cdots\left(s_{j_q} \mid v'_q\right)$$

be a monomial of $(\mathcal{X},\mathcal{U})$-content[53] $(\alpha,\beta)$, where $u'_1,\ldots,u'_p,v'_1,\ldots,v'_q$ are elements of $\mathcal{U}\cup\mathcal{T}$. Then,

(i) if $\alpha_i < l$, we set $D^l(s_j,x_i)\,M = 0$,

(ii) if $\alpha_i \geq l$, we set $D^l(s_j,x_i)\,M = \sum_r M_r$, where $M_1,\ldots,M_r,\ldots,M_{\binom{\alpha_i}{l}}$ are all the $\binom{\alpha_i}{l}$ distinct monomials obtained from $M$ by replacing each subset of $l$ letters $x_i$ by $l$ letters $s_j$. (In particular, each of the monomials $M_r$ contains the letter $x_i$ exactly[54] $(\alpha_i - l)$ times and the letter $s_j$ exactly[55] $l$ times.)[56]

The operator $D^l(s_j,x_i)$ is now extended to all of $P(\alpha,\beta)$ by linearity.

[50] *Correction:* I have added the preceding paragraph, since the original paper was cavalier about where these issues (i.e., in which contexts to disregard the new letters from $\mathcal{S}$ and $\mathcal{T}$, and in which context to count them in). I hope my correction is correct!

[51] *Correction:* I have added the preceding paragraph, since the original paper was cavalier about where these issues (i.e., in which contexts to disregard the new letters from $\mathcal{S}$ and $\mathcal{T}$, and in which context to count them in). I hope my correction is correct!

[52] *Correction:* Added "for integers $l \geq 0$".

[53] *Correction:* Replaced "content" by "$(\mathcal{X},\mathcal{U})$-content".

[54] *Correction:* Added the word "exactly" in order to separate two unrelated expressions.

[55] *Correction:* Added the word "exactly" in order to separate two unrelated expressions.

[56] *Comment:* Notice that case (i) should be regarded as a particular case of case (ii). Indeed, if $\alpha_i < l$, then $\binom{\alpha_i}{l} = 0$, and there is no subset of $l$ letters $x_i$ in $M$ (because $M$ has fewer than $l$ letters $x_i$ in total); therefore, if we apply the rule for defining $D^l(s_j,x_i)\,M$ in case (ii) to case (i), then we obtain $D^l(s_j,x_i)\,M = \sum_r M_r = (\text{empty sum}) = 0$, which is precisely how we defined $D^l(s_j,x_i)\,M$ in case (i). Therefore, strictly speaking, there was no need to treat case (i) separately.

The operator $D^0(s_j, x_i)$ [57] is the identity operator, and for $1 \leq l \leq \alpha_i$, the set polarization operator $D^l(s_j, x_i)$ maps a polynomial in $P(\alpha, \beta)$ to a polynomial lying outside $P(\alpha, \beta)$. [58]

For bideterminants, the set polarization operators act in the following simple way.[59]

**Lemma 4.1.** Let $(T \mid T')$ be a bideterminant of $(\mathcal{X}, \mathcal{U})$-content $(\alpha, \beta)$. Then:

(i) If $\alpha_i < l$, then $D^l(s_j, x_i)(T \mid T') = 0$.

(ii) If $\alpha_i \geq l$, then $D^l(s_j, x_i)(T \mid T') = \sum_r (T_r \mid T')$, where $T_1, \ldots, T_r, \ldots, T_{\binom{\alpha_i}{l}}$ are all the distinct $\binom{\alpha_i}{l}$ tableaux obtained from $T$ by replacing each subset of $l$ letters $x_i$ by $l$ letters $s_j$.

*Proof of Lemma 4.1.* Expand $(T \mid T')$ into a sum of monomials $M^t$ of $(\mathcal{X}, \mathcal{U})$-content[60] $(\alpha, \beta)$:

$$(T \mid T') = \sum_t M^t.$$

Now, if $\alpha_i < l$, then[61] $D^l(s_j, x_i) M^t = 0$ for all the monomials $M^t$. Hence (i).

Now, suppose that $\alpha_i \geq l$. Then,

$$D^l(s_j, x_i)(T \mid T') = \sum_t D^l(s_j, x_i) M^t = \sum_t \sum_r M^t_r,$$

[57] *Correction:* Replaced "$D^o$" by "$D^0$".

[58] *Comment:* It is not hard to prove that for a given choice of $x_i$ and $s_j$, the sequence $(D^0(s_j, x_i), D^1(s_j, x_i), D^2(s_j, x_i), \ldots)$ is a *divided-powers Hasse-Schmidt derivation* in the sense of [17, Definition 2.6]. Explicitly, this means that the following holds:

- We have $D^0(s_j, x_i) = \mathrm{id}$.
- We have $\left(D^l(s_j, x_i)\right)(ab) = \sum_{g=0}^{l} \left(D^g(s_j, x_i)\, a\right) \left(D^{l-g}(s_j, x_i)\, b\right)$ for all $a \in P$ and $b \in P$.
- We have $D^p(s_j, x_i) \cdot D^q(s_j, x_i) = \binom{p+q}{p} D^{p+q}(s_j, x_i)$ for all $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

As a consequence, the operator $D^1(s_j, x_i)$ is a derivation on $P$, and each $l \in \mathbb{N}$ satisfies $l! \cdot D^l(s_j, x_i) = \left(D^1(s_j, x_i)\right)^l$. Thus, if the ground field $K$ has characteristic 0, then the operators $D^l(s_j, x_i)$ can all be computed from $D^1(s_j, x_i)$.

Operators such as $D^l(s_j, x_i)$ are the hallmark of characteristic-free invariant theory (and, generally, tend to occur when studying polynomials over fields of arbitrary characteristic).

[59] *Correction:* In the following lemma, I replaced "content" by "$(\mathcal{X}, \mathcal{U})$-content". Also, added the words "then" to parts (i) and (ii) in order to separate formulas, and replaced run-on-sentence by three separate sentences.

[60] *Correction:* Replaced "content" by "$(\mathcal{X}, \mathcal{U})$-content".

[61] *Correction:* Added the word "then".

where $M_r^t$ (for a fixed $t$) [62] are the $\binom{\alpha_i}{l}$ monomials obtained from $M^t$ according to rule (ii) of the definition of $D^l\left(s_j, x_i\right)$ [63]. Interchanging the order of summation, we have

$$D^l\left(s_j, x_i\right)\left(T \mid T'\right) = \sum_r \sum_t M_r^t,$$

But[64]

$$\sum_t M_r^t = \left(T_r \mid T'\right),$$

where the same set of $l$ letters $x_i$ are replaced by $l$ letters $s_j$ on both sides of the equation. Hence,

$$D^l\left(s_j, x_i\right)\left(T \mid T'\right) = \sum_r \left(T_r \mid T'\right).$$

This proves the lemma. □

The set polarization operators $D^l\left(t_j, u_i\right)$ are defined in an analogous manner, and the analog of the previous lemma is true for these operators.

**Example 4.2.**

$$D^2\left(s_1, x_2\right)\left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ x_1 & x_2 & & T' \\ x_2 & & & \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} x_1 & s_1 & x_3 & \\ x_1 & s_1 & & T' \\ x_2 & & & \end{array}\right) + \left(\begin{array}{ccc|c} x_1 & s_1 & x_3 & \\ x_1 & x_2 & & T' \\ s_1 & & & \end{array}\right) + \left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ x_1 & s_1 & & T' \\ s_1 & & & \end{array}\right),$$

while

$$D^2\left(s_1, x_3\right)\left(\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ x_1 & x_2 & & T' \\ x_2 & & & \end{array}\right) = 0.$$

Note that since the alphabets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{S}$, and $\mathcal{T}$ are disjoint, the set polarization operators commute.

The set polarization operators are the building blocks of the *Capelli operator* $C\left(T, T'\right)$ [65], which is defined for each bitableau $\left[T, T'\right]$ of shape $(\lambda)$ as follows: Let $\alpha_i\left(q\right)$ (resp. $\beta_j\left(q\right)$) be the number of occurrences of $x_i$ [66] (resp. $u_j$) in the

[62] *Correction:* Added "(for a fixed $t$)".
[63] *Correction:* Added the words "of the definition of $D^l\left(s_j, x_i\right)$".
[64] *Correction:* Removed a useless comma at this point.
[65] *Correction:* Added "$C\left(T, T'\right)$".
[66] *Correction:* Replaced "$x_j$" by "$x_i$".

$q$th column of $T$ (resp. $T'$). The Capelli operator $C(T,T')$ [67] is defined by the following formula:[68]

$$C(T,T') = \prod_{1\leq q\leq\lambda_1}\left(\prod_{1\leq i\leq n} D^{\alpha_i(q)}(s_q,x_i)\right)\left(\prod_{1\leq j\leq k} D^{\beta_j(q)}(t_q,u_j)\right).$$

[69]

**Example 4.3.** Suppose

$$[T,T'] = \left[\begin{array}{ccccccc} x_1 & x_2 & x_3 & & u_1 & u_2 & u_3 \\ x_1 & x_2 & & , & u_1 & u_3 & \\ x_2 & & & & u_1 & & \end{array}\right].$$

Then

$$\begin{aligned} C(T,T') &= D^2(s_1,x_1)\,D^1(s_1,x_2)\,D^2(s_2,x_2)\,D^1(s_3,x_3) \\ &\quad\cdot D^3(t_1,u_1)\,D^1(t_2,u_2)\,D^1(t_2,u_3)\,D^1(t_3,u_3). \end{aligned}$$

We now impose a new total order on $(\mathcal{X},\mathcal{U})$-bitableaux[70] of the same shape. Associate with each $(\mathcal{X},\mathcal{U})$-bitableau[71] the sequence formed by reading off the indices down each column, successively, first in $T$ and then in $T'$. The bitableaux are then ordered according to the lexicographic order of their associated column sequences.[72]

**Example 4.4.** For the bitableau in the preceding example, the associated column sequence is

$$(1,1,2,2,2,3,1,1,1,2,3,3).$$

If the $(\mathcal{X},\mathcal{U})$-bitableau[73] $[T,T']$ is standard, the associated column sequence can be written

$$\left(1^{\alpha_1(1)}\cdots n^{\alpha_n(1)}2^{\alpha_2(2)}\cdots n^{\alpha_n(2)}\cdots n^{\alpha_n(\lambda_1)}1^{\beta_1(1)}\cdots k^{\beta_k(1)}\cdots k^{\beta_k(\lambda_1)}\right).$$

[67] *Correction:* Added "$C(T,T')$".

[68] *Correction:* In the following formula, I replaced "$D^{\beta_l(q)}(t_q,u_j)$" by "$D^{\beta_j(q)}(t_q,u_j)$".

[69] *Comment:* Note that this is a product of commuting operators (since the set polarization operators commute). Note also that $\lambda_1$ is the number of columns of the shape $(\lambda)$.

[70] *Correction:* Replaced "bitableaux" by "$(\mathcal{X},\mathcal{U})$-bitableaux".

[71] *Correction:* Replaced "bitableau" by "$(\mathcal{X},\mathcal{U})$-bitableau".

[72] *Comment:* This is indeed a total order on the set of all bitableaux of a given shape $(\lambda)$. This is because a $(\mathcal{X},\mathcal{U})$-bitableau **of a given shape** $(\lambda)$ is uniquely determined by its associated column sequence.

[73] *Correction:* Added the words "the $(\mathcal{X},\mathcal{U})$-bitableau".

We have used the fact that for the $(\mathcal{X},\mathcal{U})$-bitableau $[T,T']$ to be standard[74], we must have $\alpha_i(q) = \beta_i(q) = 0$ for $i < q$ [75].

**Theorem 4.5.** Let $[T,T']$ and $[V,V']$ be two standard $(\mathcal{X},\mathcal{U})$-bitableaux of shape $(\lambda)$ and $(\mu)$ with the same content. Then:

(i) We have $C(T,T')(T \mid T') \neq 0$.

(ii) If $(\mu)$ is longer than $(\lambda)$, then $C(T,T')(V \mid V') = 0$.

(iii) If $(\lambda) = (\mu)$, and if $[V,V']$ is greater than $[T,T']$ in the lexicographic order of their associated column sequences, then $C(T,T')(V \mid V') = 0$.

[76]

*Proof of Theorem 4.5.* (i) We calculate $C(T,T')(T \mid T')$ explicitly; we have[77]

$$(T \mid T') = \left( \begin{array}{r|c} \alpha_1(1) \text{ rows} \left\{ \begin{array}{cc} x_1 & \cdots \\ \vdots & \vdots \\ x_1 & \cdots \end{array} \right. & \\ \vdots & \\ \alpha_l(1) \text{ rows} \left\{ \begin{array}{cc} x_l & \cdots \\ \vdots & \vdots \\ x_l & \cdots \end{array} \right. & T' \\ \vdots & \\ \alpha_n(1) \text{ rows} \left\{ \begin{array}{cc} x_n & \cdots \\ \vdots & \vdots \\ x_n & \cdots \end{array} \right. & \end{array} \right).$$

Now, the letter $x_1$ can only be found in the first column of $T$. Hence, $D^{\alpha_1(1)}(s_1,x_1)(T \mid T')$ [78] consists of a single term, obtained by substituting $s_1$ for all the $x_1$ in $T$.

[74] *Correction:* Replaced "for the bideterminant $(T \mid T')$ to be nonzero" by "for the $(\mathcal{X},\mathcal{U})$-bitableau $[T,T']$ to be standard". (Indeed, the bideterminant $(T \mid T')$ being nonzero was not a requirement.)

[75] *Comment:* This is because the entries in each row must be strictly increasing.

[76] *Correction:* In this theorem, I have replaced "Let $(T \mid T')$ and $(V \mid V')$ be two standard bideterminants" by "Let $[T,T']$ and $[V,V']$ be two standard $(\mathcal{X},\mathcal{U})$-bitableaux" for precision. Also, I have made each of (i), (ii) and (iii) a separate sentence. Also, I have replaced "smaller than" by "smaller or equal to" in part (iii). Finally, I have rewritten part (iii) in contrapositive form, since this is how this part is actually being proved and used (and it is more natural to state it like this from a constructive point of view).

[77] *Correction:* In the formula below, replaced "$\alpha_n(a)$" by "$\alpha_n(1)$"; added more "$\vdots$"s and "$\cdots$"s; added the words "rows" for clarity.

[78] *Correction:* Replaced "$D^{\alpha_1(1)}(s_1,x_i)$" by "$D^{\alpha_1(1)}(s_1,x_1)$".

Assume that $\left(\prod_{1\leq i\leq l-1} D^{\alpha_i(1)}(s_1, x_i)\right)(T \mid T')$ [79] consists of a single term, obtained by substituting all the letters $x_i$, $1 \leq i \leq l-1$, in the first column of $T$ by the letter $s_1$. Then,

$$
\begin{aligned}
&\left(\prod_{1\leq i\leq l} D^{\alpha_i(1)}(s_1, x_i)\right)(T \mid T') \\
&= D^{\alpha_l(1)}(s_1, x_l)\left(\prod_{1\leq i\leq l-1} D^{\alpha_i(1)}(s_1, x_i)\right)(T \mid T') \\
&= D^{\alpha_1(1)}(s_1, x_l)\left(\begin{array}{r l c}
\sum\limits_{1\leq i\leq l-1} \alpha_i(1) \text{ rows} & \left\{\begin{array}{cc} s_1 & \cdots \\ \vdots & \vdots \\ s_1 & \cdots \end{array}\right. & \\
\alpha_l(1) \text{ rows} & \left\{\begin{array}{cc} x_l & \cdots \\ \vdots & \vdots \\ x_l & \cdots \end{array}\right. & \\
& \quad\vdots & T' \\
\alpha_n(1) \text{ rows} & \left\{\begin{array}{cc} x_n & \cdots \\ \vdots & \vdots \\ x_n & \cdots \end{array}\right. &
\end{array}\right).
\end{aligned}
$$

Since tableau $T$ is standard, any occurrence of $x_l$ in other than the first column must be in the first $\sum\limits_{1\leq i\leq l-1} \alpha_i(1)$ rows. If any of these $x_l$ are chosen for substitution during the polarization[80], the letter $s_1$ would be repeated within a row, and the resulting bideterminant would be zero. Hence, the only nonzero term in the above expression is the term obtained by substituting $s_1$ for all the $\alpha_l(1)$ letters $x_l$ in the first column of $T$. By induction, we have shown that the expression $\left(\prod_{1\leq i\leq n} D^{\alpha_1(1)}(s_1, x_i)\right)(T \mid T')$ consists of a single nonzero term, obtained by substituting $s_1$ for all the letters in the first column of $T$.

Repeating this argument for the other columns, we can easily see that $C(T, T')(T \mid T')$ is obtained by substituting $s_q$ (resp. $t_q$) for all the letters in the $q$th column of $T$

[79] *Correction:* Replaced "$\alpha_1(1)$" by "$\alpha_i(1)$". Added parentheses around the product. Similarly in the equalities below.

[80] *Comment:* "Polarization" here means the application of the operator $D^{\alpha_1(1)}(s_1, x_l)$.

(resp. $T'$); that is,[81]

$$C\left(T,T'\right)\left(T\mid T'\right)=\left(\begin{array}{cccc|cccc} s_1 & s_2 & \cdots & s_{\lambda_1} & t_1 & t_2 & \cdots & t_{\lambda_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_1 & s_2 & \cdots & s_{\lambda_1} & t_1 & t_2 & \cdots & t_{\lambda_1} \\ \vdots & \vdots & \iddots & & \vdots & \vdots & \iddots & \\ s_1 & s_2 & & & t_1 & t_2 & & \\ \vdots & & & & \vdots & & & \\ s_1 & & & & t_1 & & & \end{array}\right).$$

The two tableaux appearing on the right hand side will be denoted by $U$ and $U'$. Thus, $C\left(T,T'\right)\left(T\mid T'\right)=\left(U\mid U'\right)\neq 0$. [82]

(ii) The expression $C\left(T,T'\right)\left(V\mid V'\right)$ consists of a sum of bideterminants of the same shape $(\mu)$. If it is nonzero, then one of the bideterminants, say $\left(W\mid W'\right)$, is nonzero. The content of $\left(W\mid W'\right)$ is the same as that of $\left(U\mid U'\right)=C\left(T,T'\right)\left(T\mid T'\right)$ (since the set polarization operators transform the content of a polynomial in a predictable way (i.e., the content of the output is determined by the content of the input), and therefore so does the operator $C\left(T,T'\right)$) [83]; that is, for $1\leq l\leq\lambda_1$, the letters $s_l$ and $t_l$ occur in $\left(W\mid W'\right)$ exactly[84] $\widetilde{\lambda}_l$ times, where

$$\begin{aligned}\widetilde{\lambda}_l &= \sum_{1\leq i\leq n}\alpha_i\left(l\right)\\ &= \text{the height of the } l\text{th column.}\end{aligned}$$

We are required to show that $(\mu)$ is shorter or equal to[85] $(\lambda)$. If $(\mu)\neq(\lambda)$, let $m$ be the smallest integer such that $\lambda_m\neq\mu_m$ [86]. [87] We claim that: For $1\leq i\leq m-1$, the contents of the $i$th row in $U$ and in $W$ are identical.

[81] *Correction:* Added a few more dots and an extra $s_1$ to the bitableau below to hopefully make it clearer.

[82] *Correction:* Added the previous two sentences.

[83] *Correction:* Inserted the preceding parenthetical sentence.

[84] *Correction:* Added the word "exactly".

[85] *Correction:* Replaced "shorter than" by "shorter or equal to".

[86] *Comment:* Such an $m$ exists, for the following reason: The tableaux $T$ and $V$ have the same content. Thus, they have the same number of boxes. In other words, $(\lambda)$ and $(\mu)$ are partitions of one and the same integer. Hence, $(\lambda)$ cannot be a proper subsequence of $(\mu)$, and $(\mu)$ cannot be a proper subsequence of $(\lambda)$. Therefore, if $(\mu)\neq(\lambda)$, then there exists an integer $m$ satisfying $\lambda_m\neq\mu_m$.

[87] *Correction:* At this spot, I have removed the following two sentences:

"If $m=1$, it must be the case that $\mu_1<\lambda_1$, for the first row of $W$ contains $\mu_1$ *distinct* letters chosen from the set $\left\{s_1,\ldots,s_{\lambda_1}\right\}$.

Now, suppose that $m\geq 2$.".

In fact, these sentences are not wrong, but they are unnecessary, since the argument that follows does not require $m\geq 2$, and thus the $m=1$ case needs not be treated separately.

The proof is by strong induction[88] on $i$. [89] Namely,[90] assume that the proposition is true up to the $i$th row. The letters $s_l$, for $l \geq \lambda_i + 1$, have all been used in the first $i-1$ rows in $U$, hence in $W$ (since $U$ and $W$ have the same content)[91]. For the $i$th row in $W$, we thus[92] have to choose $\mu_i$ distinct letters from $\{s_1, \ldots, s_{\lambda_i}\}$. But $\lambda_i = \mu_i$; hence the contents of the $i$th row in $U$ and $W$ are identical. This proves the claim.[93]

Now, consider the $m$th row. As the contents of the[94] first $m-1$ rows are identical, the $m$th row in $W$ contains $\mu_m$ distinct letters from the set $\{s_1, \ldots, s_{\lambda_m}\}$ [95]. Since $\mu_m \neq \lambda_m$, we must have $\mu_m < \lambda_m$.

(iii) We can now suppose that $(\mu) = (\lambda)$. [96] Recall that the associated column sequence of $[T, T']$ [97] is

$$\left(1^{\alpha_1(1)} \ldots n^{\alpha_n(1)} \ldots n^{\alpha_n(\lambda_1)} 1^{\beta_1(1)} \ldots k^{\beta_k(1)} \ldots k^{\beta_k(\lambda_1)}\right).$$

We shall denote by $\gamma_i(q)$ (resp. $\delta_j(q)$) the number of occurrences of $x_i$ (resp. $u_j$) [98] in the $q$th column of $V$ (resp. $V'$). The associated column sequence of $[V, V']$ [99] is[100]

$$\left(1^{\gamma_1(1)} \ldots n^{\gamma_n(1)} \ldots n^{\gamma_n(\lambda_1)} 1^{\delta_1(1)} \ldots k^{\delta_k(1)} \ldots k^{\delta_k(\lambda_1)}\right).$$

Suppose now that $[T, T']$ and $[V, V']$ differ in the left tableau; the reasoning is similar if the only[101] difference lies in the right tableau.

Let $p$ be the first column where $T$ and $V$ differ, and in the $p$th column, let $x_l$ be

[88] *Correction:* Replaced "induction" by "strong induction".

[89] *Correction:* At this spot, I have removed the following sentence:

"If $i = 1$, then $\mu_1 = \lambda_1$; by our preceding observations, the first row in both $W$ and $U$ consists of the set $\{s_1, \ldots, s_{\lambda_1}\}$ arranged in some order.".

In fact, this sentence is useless here, since a strong induction does not contain an induction base.

[90] *Correction:* Replaced "Now" by "Namely,".

[91] *Correction:* Inserted the preceding parenthetical sentence.

[92] *Correction:* Added the word "thus" for clarity of the argument.

[93] *Correction:* Added the preceding sentence.

[94] *Correction:* Added "contents of the".

[95] *Comment:* This follows from noticing that the letters $s_l$, for $l \geq \lambda_m + 1$, have all been used in the first $m-1$ rows in $U$, hence in $W$ (since $U$ and $W$ have the same content), and thus are not available for the $m$th row any more.

[96] *Correction:* Removed "and $C(T, T')(V \mid V') \neq 0$" from this sentence, as it was an instance of useless proof by contradiction.

[97] *Correction:* Replaced "of $(T \mid T')$" by "of $[T, T']$".

[98] *Correction:* Replaced "(resp. $u_i$)" by "(resp. $u_j$)".

[99] *Correction:* Replaced "of $(V \mid V')$" by "of $[V, V']$".

[100] *Correction:* In the column sequence below, I replaced "$\delta_k(3)$" by "$\delta_k(1)$".

[101] *Correction:* Added the word "only" for clarity.

the smallest index that is different. That is, we have[102]

for $1 \leq i \leq n$ and $1 \leq q \leq p-1$, we have $\alpha_i(q) = \gamma_i(q)$;
for $1 \leq i \leq l-1$, we have $\alpha_i(p) = \gamma_i(p)$;
but $\alpha_l(p) \neq \gamma_l(p)$.

Now, assume that $[V, V']$ [103] is lexicographically *greater* than $[T, T']$ [104]; that is, $\alpha_l(p) > \gamma_l(p)$ [105]. Consider the action of the Capelli operator $C(T, T')$ on $(V \mid V')$. The polarizations $D^{\alpha_1(1)}(s_1, x_1), D^{\alpha_2(1)}(s_1, x_2), \ldots, D^{\alpha_{l-1}(p)}(s_p, x_{l-1})$ act on $(V \mid V')$ exactly as they do on $(T \mid T')$. At this instant[106], the expression[107]

$$\left(\prod_{1\leq i\leq l-1} D^{\alpha_i(p)}\left(s_p, x_i\right)\right) \cdot \left(\prod_{1\leq q\leq p-1}\left(\prod_{1\leq i\leq n} D^{\alpha_i(q)}\left(s_q, x_i\right)\right)\right)(V \mid V')$$

[102] *Correction:* Added the words "we have" in the following statements, and replaced "$\delta_i(p)$" by "$\gamma_i(p)$".

[103] *Correction:* Replaced "$(V \mid V')$" by "$[V, V']$".

[104] *Correction:* Replaced "$(T \mid T')$" by "$[T, T']$".

[105] *Comment:* Let me explain why the assumption (that $[V, V']$ is lexicographically *greater* than $[T, T']$) yields $\alpha_l(p) > \gamma_l(p)$.

Indeed, assume the contrary. Thus, $\alpha_l(p) \leq \gamma_l(p)$, so that $\alpha_l(p) < \gamma_l(p)$ (since $\alpha_l(p) \neq \gamma_l(p)$). Hence, the topmost $\sum_{1\leq i\leq l} \alpha_i(p)$ indices in the $p$th column of $T$ equal the corresponding indices in the $p$th column of $V$ (because for $1 \leq i \leq l-1$, we have $\alpha_i(p) = \gamma_i(p)$), but the $\left(\sum_{1\leq i\leq l} \alpha_i(p)\right) + 1$-st index from the top is greater in $T$ than in $V$ (in fact, the index in $T$ is $> p$ whereas the index in $V$ is $= p$). Therefore, the associated column sequence of $[T, T']$ is greater than the associated column sequence of $[V, V']$ (because the first $p-1$ columns of $T$ and of $V$ are equal). This contradicts the assumption that $[V, V']$ is lexicographically *greater* than $[T, T']$. This contradiction completes the proof.

[106] *Comment:* i.e., after these polarizations have been applied to $(V \mid V')$

[107] *Correction:* Replaced "$1 \leq \alpha \leq p-1$" by "$1 \leq q \leq p-1$" in the following equality. Also, added some parenthesis to disambiguate the products.

is a bideterminant of the form[108]

$$\left(\begin{array}{rl|c}
\sum\limits_{1\leq i\leq l-1}\alpha_i(p)\text{ rows} & \left\{\begin{array}{cccccc} s_1 & s_2 & \cdots & s_{p-1} & s_p & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ldots \\ s_1 & s_2 & \cdots & s_{p-1} & s_p & \cdots \end{array}\right. & \\
\gamma_l(p)\text{ rows} & \left\{\begin{array}{cccccc} s_1 & s_2 & \cdots & s_{p-1} & x_l & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ldots \\ s_1 & s_2 & \cdots & s_{p-1} & x_l & \cdots \end{array}\right. & \\
 & \begin{array}{cccccc} \vdots & \vdots & \ddots & \vdots & \vdots & \ldots \\ s_1 & s_2 & \cdots & s_{p-1} & & \\ \vdots & \vdots & \iddots & & & \\ s_1 & s_2 & & & & \\ \vdots & & & & & \\ s_1 & & & & & \end{array} & V'
\end{array}\right).$$

That is, the first $p-1$ columns are replaced by the appropriate letters $s_q$; in the $p$th column, the first $\sum\limits_{1\leq i\leq l-1}\alpha_i(p)$ letters are replaced by $s_p$, and the remainder of the tableau is unchanged.

Since $\alpha_l(p) > \gamma_l(p)$, and $V$ is standard, any choice of $\alpha_l(p)$ [109] letters $x_l$ must involve a letter $x_l$ lying in the first $\sum\limits_{1\leq i\leq l-1}\alpha_i(p)$ rows of $V$. The set polarization operator $D^{\alpha_l(p)}(s_p, x_l)$ [110] substitutes $s_p$ for this particular $x_l$. The resulting bideterminant is zero, since there are two letters $s_p$ in a single row. This proves $C(T,T')(V \mid V') = 0$. [111] □

With Theorem 4.5 proved, we can now proceed to the main result of this section.[112]

**Theorem 4.6.** The standard $(\mathcal{X},\mathcal{U})$-bideterminants of content $(\alpha,\beta)$ form a basis of the vector space $P(\alpha,\beta)$.

*Proof of Theorem 4.6.* By Corollary 3.4[113], the standard $(\mathcal{X},\mathcal{U})$-bideterminants[114] span the vector space $P(\alpha,\beta)$. Suppose we have a nontrivial linear relation between these

[108] *Correction:* Added "$\cdots$"s at the end of the rows of the left tableau.

[109] *Correction:* Replaced "$\gamma_l(p)$" by "$\alpha_l(p)$".

[110] *Correction:* Replaced "$D^{\alpha_i(p)}(s_p, x_l)$" by "$D^{\alpha_l(p)}(s_p, x_l)$".

[111] *Correction:* Replaced the original sentence "This contradicts the assumption; we have therefore proved that $(V \mid V')$ is lexicographically smaller than $(T \mid T')$" (which was imprecise and also a needless proof by contradiction) by "This proves $C(T,T')(V \mid V') = 0$".

[112] *Correction:* In the following theorem, I have replaced "bideterminant" by "$(\mathcal{X},\mathcal{U})$-bideterminants".

[113] *Correction:* Replaced "By Theorem 3.2" by "By Corollary 3.4".

[114] *Correction:* Replaced "bideterminants" by "$(\mathcal{X},\mathcal{U})$-bideterminants".

bideterminants. We can write this linear relation as follows:

$$a\left(T \mid T'\right) + A + B = 0,$$

where $a$ is a nonzero scalar in $K$, $(\lambda)$ is the shortest shape occurring in the linear relation, $[T, T']$ [115] is the bitableau of shape $(\lambda)$ with the lexicographically smallest associated *column* sequence, $A$ is the linear combination of the remaining bitableaux[116] of shape $(\lambda)$, and $B$ is the linear combination of the remainder of the bitableaux, which are necessarily of shape longer than $(\lambda)$.

Applying the Capelli operator $C\left(T, T'\right)$ to the relation we have, by Theorem 4.5,[117]

$$\begin{aligned} C\left(T, T'\right) A &= 0 \qquad &&(\text{by Theorem 4.5 (iii)}), \\ C\left(T, T'\right) B &= 0 \qquad &&(\text{by Theorem 4.5 (ii)}), \end{aligned}$$

but

$$C\left(T, T'\right)\left(T \mid T'\right) \neq 0 \qquad (\text{by Theorem 4.5 (i)}).$$

This implies $aC\left(T, T'\right)\left(T \mid T'\right) = 0$, which is a contradiction.[118] □

The proof of this theorem contains another algorithm for expressing any bideterminant as a linear combination of standard bideterminants[119]. Suppose

$$\left(T \mid T'\right) = \sum_i a_i \left(T_i \mid T_i'\right)$$

is the unique decomposition of $\left(T \mid T'\right)$ into standard bitableaux, written so that if $i < j$, then either $T_i$ is of shape shorter than $T_j$, or $T_i$ and $T_j$ [120] have the same

[115] *Correction:* Replaced "$(T \mid T')$" by "$[T, T']$".

[116] *Correction:* Replaced "tableaux" by "bitableaux".

[117] *Correction:* Here, I have added the comments "by Theorem 4.5 (iii)", "by Theorem 4.5 (ii)" and "by Theorem 4.5 (i)".

[118] *Comment:* Here is (what I believe to be) a clearer way to write up this argument:

Applying the map $C\left(T, T'\right)$ to the equality $a\left(T \mid T'\right) + A + B = 0$, we obtain

$$C\left(T, T'\right)\left(a\left(T \mid T'\right) + A + B\right) = C\left(T, T'\right) 0 = 0.$$

Thus,

$$\begin{aligned} 0 &= C\left(T, T'\right)\left(a\left(T \mid T'\right) + A + B\right) \\ &= aC\left(T, T'\right)\left(T \mid T'\right) + \underbrace{C\left(T, T'\right) A}_{=0} + \underbrace{C\left(T, T'\right) B}_{=0} \\ &= aC\left(T, T'\right)\left(T \mid T'\right). \end{aligned}$$

In other words, $aC\left(T, T'\right)\left(T \mid T'\right) = 0$. Since $C\left(T, T'\right)\left(T \mid T'\right) \neq 0$, this entails that $a = 0$. This contradicts $a \neq 0$.

[119] *Correction:* Replaced "any bitableau as a linear combination of standard bitableaux" by "any bideterminant as a linear combination of standard bideterminants".

[120] *Correction:* Replaced "or $T_j$" by "or $T_i$ and $T_j$".

shape, and $\left[T_i, T_i'\right]$ has a lexicographically smaller associated column sequence than $\left[T_j, T_j'\right]$ [121]. The coefficients $a_i$ are called the *straightening coefficients*. They can be computed by applying the Capelli operators $C\left(T_j, T_j'\right)$ to both sides of the linear relation; by Theorem 4.5, we obtain a triangular array of equations between bideterminants with entries from the alphabets $\mathcal{S}$ and $\mathcal{T}$. From this, we can extract a triangular system of linear equations for the coefficients $a_i$, which can then be solved.[122]

**Example 4.7.** Consider the $(\mathcal{X},\mathcal{U})$-bideterminant (where, for simplicity, all but the subscripts are suppressed):

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 2 \\ 1 & 4 & 1 & 3 \\ 2 & & 1 & \end{array}\right).$$

The standard bitableaux of the same or longer shape of the same content are

$$\left[\begin{array}{ccccc} 1 & 3 & & 1 & 2 \\ 2 & 4 & , & 1 & 3 \\ 2 & & & 1 & \end{array}\right], \qquad \left[\begin{array}{ccccc} 1 & 2 & & 1 & 2 \\ 2 & 4 & , & 1 & 3 \\ 3 & & & 1 & \end{array}\right], \qquad \left[\begin{array}{ccccc} 1 & 2 & & 1 & 2 \\ 2 & 3 & , & 1 & 3 \\ 4 & & & 1 & \end{array}\right],$$

$$\left[\begin{array}{ccccccc} 1 & 3 & 4 & & 1 & 2 & 3 \\ 2 & & & , & 1 & & \\ 2 & & & & 1 & & \end{array}\right], \qquad \left[\begin{array}{ccccccc} 1 & 2 & 4 & & 1 & 2 & 3 \\ 2 & & & , & 1 & & \\ 3 & & & & 1 & & \end{array}\right],$$

$$\left[\begin{array}{ccccccc} 1 & 2 & 3 & & 1 & 2 & 3 \\ 2 & & & , & 1 & & \\ 4 & & & & 1 & & \end{array}\right].$$

Let $a_1, \ldots, a_6$ be the corresponding straightening coefficients. We obtain, through the Capelli operators, the equations:

$$\begin{array}{ccccccccc} a_1 & & & & & & & = & 1 \\ & a_2 & & & & & & = & -1 \\ -a_1 & & +a_3 & & & & & = & 0 \\ & & & a_4 & & & & = & 0 \\ & & & & a_5 & & & = & 0 \\ & & a_3 & & & & +a_6 & = & 0 \end{array}$$

[121] *Correction:* Replaced "and $T_i$ has a lexicographically smaller associated column sequence than $T_j$" by "and $\left[T_i, T_i'\right]$ has a lexicographically smaller associated column sequence than $\left[T_j, T_j'\right]$".

[122] *Correction:* In the following example, I have replaced "bideterminant" by "$(\mathcal{X},\mathcal{U})$-bideterminant".

Therefore, we obtain

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 2 \\ 1 & 4 & 1 & 3 \\ 2 & & 1 & \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \\ 2 & & 1 & \end{array}\right) - \left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 3 \\ 3 & & 1 & \end{array}\right) + \left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 4 & & 1 & \end{array}\right) - \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & & & 1 & & \\ 4 & & & 1 & & \end{array}\right).$$

From Theorem 4.6, we obtain a basis of the whole polynomial ring in the $(x_i \mid u_j)$: [123]

**Corollary 4.8.** The standard $(\mathcal{X}, \mathcal{U})$-bideterminants form a basis of the algebra of polynomials over the field $K$ in the indeterminates $(x_i \mid u_j)$.

*Proof of Corollary 4.8.* This algebra (as a vector space) is the direct sum $\bigoplus_{(\alpha,\beta)} P(\alpha, \beta)$. Each of the addends has a basis consisting of the standard $(\mathcal{X}, \mathcal{U})$-bideterminants of content $(\alpha, \beta)$ (by Theorem 4.6). Thus, the whole algebra has a basis consisting of all the standard $(\mathcal{X}, \mathcal{U})$-bideterminants. □

# 5. Invariant Theory

## 5.1. The Second Fundamental Theorem

Classical invariant theory is concerned with the behavior of forms under the action of linear transformations. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ be a dual basis for the vector space $V_d$ (of dimension $d$). [124] A *form* $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ on $m$ vectors in $V_d$ is a polynomial in the $md$ scalar products of the $m$ vectors $\mathbf{x}_i$ with the $d$ covectors $\mathbf{u}_j$ [125] in the dual basis. More pedantically, consider the polynomial algebra $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$ in the indeterminates $x_{ir}, u_{jr}, s_{pr}, t_{qr}$, where $1 \leq i \leq n$, $1 \leq j \leq k$, $1 \leq p$, $1 \leq q$, and

[123] *Correction:* Added this sentence and the next corollary.

[124] *Comment:* I am not sure what "dual basis" means here; I suspect that it means "basis for the dual space" (i.e., a *dual basis* of a vector space $V$ means a basis of the dual space $V^*$ of $V$). However, this is not very important, because the vector space $V_d$ and its dual basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ are used only for motivation. A proper definition of the notion of forms shall be given two sentences later.

[125] *Correction:* Replaced "$\mathbf{u}_i$" by "$\mathbf{u}_j$".

$1 \leq r \leq d$. In this algebra, we distinguish the following polynomials:

$$\left\langle x_i \mid u_j \right\rangle = \sum_{1 \leq r \leq d} x_{ir} u_{jr},$$
$$\left\langle s_p \mid u_j \right\rangle = \sum_{1 \leq r \leq d} s_{pr} u_{jr},$$
$$\left\langle x_i \mid t_q \right\rangle = \sum_{1 \leq r \leq d} x_{ir} t_{qr},$$
$$\left\langle s_p \mid t_q \right\rangle = \sum_{1 \leq r \leq d} s_{pr} t_{qr}.$$

Let $\widehat{P}$ be the subalgebra of $K\left[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}\right]$ generated by these polynomials; $\widehat{P}$ is called the *algebra of forms*. Rigorously speaking, we define a *form* to be an element of $\widehat{P}$. [126]

There exists an algebra homomorphism[127] $\phi$ from $P$ (constructed as in the previous section from the alphabets $\mathcal{X}, \mathcal{U}, \mathcal{S}$, and $\mathcal{T}$) to $\widehat{P}$ defined by

$$\phi : \begin{array}{ccc} \left(x_i \mid u_j\right) & \mapsto & \left\langle x_i \mid u_j \right\rangle, \\ \left(s_p \mid u_j\right) & \mapsto & \left\langle s_p \mid u_j \right\rangle, \\ \left(x_i \mid t_q\right) & \mapsto & \left\langle x_i \mid t_q \right\rangle, \\ \left(s_p \mid t_q\right) & \mapsto & \left\langle s_p \mid t_q \right\rangle. \end{array}$$

Consider a monomial $m = \left(x_{i_1} \mid u_{j_1}\right) \cdots \left(x_{i_a} \mid u_{j_a}\right)$ in $P$. Its image in $\widehat{P}$ under $\phi$ is given by

$$\phi m = \sum_f x_{i_1 f_1} u_{j_1 f_1} \cdots x_{i_a f_a} u_{j_a f_a},$$

where the summation is over the set of all functions $f : i \mapsto f_i$ from $\{1, \ldots, a\}$ to $\{1, \ldots, d\}$. We shall use the simpler notation

$$\phi m = \sum_f m_f.$$

[128] The restriction of the homomorphism $\phi$ to $P\left(\alpha, \beta\right)$ is called the *Pascal homomorphism*.[129]

[126] *Correction:* Added the preceding sentence.

[127] *Correction:* Replaced "a homomorphism" by "an algebra homomorphism".

[128] *Comment:* Thus, $m_f$ denotes the monomial $x_{i_1 f_1} u_{j_1 f_1} \cdots x_{i_a f_a} u_{j_a f_a}$ whenever $f : i \mapsto f_i$ is a function from $\{1, \ldots, a\}$ to $\{1, \ldots, d\}$. Of course, this definition depends not just on the monomial $m$, but also on the order of the factors in the product $\left(x_{i_1} \mid u_{j_1}\right) \cdots \left(x_{i_a} \mid u_{j_a}\right)$; we can thus only use it when this order is fixed.

[129] *Correction:* In the following theorem, I have replaced "bideterminants" by "$(\mathcal{X},\mathcal{U})$-bideterminants".

**Theorem 5.1** (the second fundamental theorem of invariant theory)**.** The kernel of the Pascal homomorphism is the subspace of $P(\alpha, \beta)$ spanned by the standard $(\mathcal{X}, \mathcal{U})$-bideterminants of shape strictly longer than $(d)$.

Some preliminary observations are in order.

In the same fashion as for $P$, we define set polarization operators for $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$. Let $v = x_{ir_1} \cdots x_{ir_a}$ be a monomial containing only variables of the form $x_{ir}$, where $i$ is fixed and $r$ is arbitrary. Suppose $p$ is a positive integer, and $E$ a subset of $\{1, \ldots, a\}$. Then, $v^{E,p}$ [130] is the monomial obtained from $v$ by replacing the variable $x_{ir_b}$ by the variable $s_{pr_b}$ whenever $b \in E$. [131] Now, for a given nonnegative[132] integer $l$, the *set polarization operator* $\widehat{D}^l(s_p, x_i)$ acts on the monomial $v$ as follows:

$$\widehat{D}^l(s_p, x_i) v = \sum_E v^{E,p},$$

where the summation is over all the $l$-subsets[133] of $\{1, \ldots, a\}$ [134]. Consider now an arbitrary monomial $w$. We can write $w$ as the product of two monomials $w'$ and $w''$, where $w'$ is the product of all the variables in $w$ of the form $x_{ir}$, and $w''$ is the product of the remaining variables in $w$ [135]. Then, we set

$$\widehat{D}^l(s_p, x_i) w = \left(\widehat{D}^l(s_p, x_i) w'\right) w''. \tag{8}$$

The operator $\widehat{D}^l(s_p, x_i)$ is extended to all of $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$ by linearity.

The operators $\widehat{D}^l(t_q, u_j)$ are defined analogously. It is clear that, as in the case of the operators $D$, the operators $\widehat{D}$ commute[136].

We have the following identity:

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[130] *Correction:* Replaced "$V^{E,p}$" by "$v^{E,p}$".

[131] *Comment:* This monomial $v^{E,p}$ depends not just on the monomial $v$, but also on the order of the factors in the product $x_{ir_1} \cdots x_{ir_a}$; we can thus only use this notation when this order is fixed.

[132] *Correction:* Replaced "positive" by "nonnegative".

[133] *Comment:* "$l$-subset" means "$l$-element subset".

[134] *Comment:* We can rewrite this equality (without using the slippery notation $v^{E,p}$) as follows:

$$\widehat{D}^l(s_p, x_i) v = \sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E|=l}} \left(\prod_{b \in E} s_{pr_b}\right) \left(\prod_{\substack{b \in \{1,2,\ldots,a\}; \\ b \notin E}} x_{ir_b}\right) \tag{6}$$

$$= \sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E|=l}} \left(\prod_{b \in \{1,2,\ldots,a\}} \begin{cases} s_{pr_b}, & \text{if } b \in E \\ x_{ir_b}, & \text{if } b \notin E \end{cases}\right). \tag{7}$$

(It is easy to see that this does not depend on the order of the factors in $x_{ir_1} \cdots x_{ir_a}$, and thus the image $\widehat{D}^l(s_p, x_i) v$ is really well-defined.)

[135] *Correction:* Added "in $w$".

[136] *Comment:* I.e., the operators $\widehat{D}^l(s_p, x_i)$ for all $p$ and $i$ and the operators $\widehat{D}^l(t_q, u_j)$ for all $q$ and $j$ all commute.

**Lemma 5.2.**

$$\widehat{D}^l\left(s_p,x_i\right)\phi=\phi D^l\left(s_p,x_i\right).$$

*Proof of Lemma 5.2.* [137] It suffices to verify the identity for monomials of the form[138]

$$m=\left(x_i\mid u_{j_1}\right)\cdots\left(x_i\mid u_{j_a}\right),$$

where all the letters $x$ have index $i$. [139] Let $m^{E,p}$ denote the monomial obtained from $m$ by replacing each variable $\left(x_i\mid u_{j_b}\right)$ by $\left(s_p\mid u_{j_b}\right)$ whenever $b\in E$. Then, we observe (notation as earlier):

$$\left(m^{E,p}\right)_f=\left(m_f\right)^{E,p}.$$

We can now finish the proof through the following computation:

$$\begin{aligned}\phi D^l\left(s_p,x_i\right)m&=\phi\sum_E m^{E,p}=\sum_E\phi m^{E,p}\\&=\sum_E\sum_f\left(m^{E,p}\right)_f=\sum_E\sum_f\left(m_f\right)^{E,p}=\sum_f\widehat{D}^l\left(s_p,x_i\right)m_f\\&=\widehat{D}^l\left(s_p,x_i\right)\left(\sum_f m_f\right)=\widehat{D}^l\left(s_p,x_i\right)\phi m.\end{aligned}$$

□

*Alternative proof of Lemma 5.2.* [140] Introduce a new indeterminate $T$, and consider the polynomial rings $P[T]$, $(K[\mathbf{x},\mathbf{u},\mathbf{s},\mathbf{t}])[T]$ and $\widehat{P}[T]$. The $K$-algebra homomorphism $\phi$ gives rise to a $K$-algebra homomorphism $P[T]\to\widehat{P}[T]$ which sends $T$ to $T$ while acting on $P$ as $\phi$. This latter $K$-algebra homomorphism $P[T]\to\widehat{P}[T]$ shall be denoted by $\phi[T]$.

Let $\mathcal{V}_i$ be the set of all indeterminates in the polynomial ring $K[\mathbf{x},\mathbf{u},\mathbf{s},\mathbf{t}]$ that are **not** of the form $x_{ir}$ with $1\leq r\leq d$. In other words,

$$\begin{aligned}\mathcal{V}_i=&\left\{x_{i'r}\mid 1\leq i'\leq n\text{ and }1\leq r\leq d\text{ satisfying }i'\neq i\right\}\\&\cup\left\{u_{jr}\mid 1\leq j\leq k\text{ and }1\leq r\leq d\right\}\\&\cup\left\{s_{p'r}\mid 1\leq p'\text{ and }1\leq r\leq d\right\}\\&\cup\left\{t_{qr}\mid 1\leq q\text{ and }1\leq r\leq d\right\}.\end{aligned}$$

[137] *Comment:* I don't like this proof, as it somewhat lacks rigor (in fact, it uses the ambiguous notation $\left(m_f\right)^{E,p}$ without properly explaining how precisely it is defined). Thus, I have added an alternative proof of Lemma 5.2 further below.

[138] *Comment:* This is somewhat imprecise, since $m$ should also be allowed to contain factors of the form $\left(x_i\mid t_q\right)$.

[139] *Correction:* Replaced a comma by a period here.

[140] *Correction:* I have added this proof, since I do not fully trust the original proof given above.

Let $\widehat{\mathbf{D}}$ be the $K$-algebra homomorphism $K[\mathbf{x},\mathbf{u},\mathbf{s},\mathbf{t}] \to (K[\mathbf{x},\mathbf{u},\mathbf{s},\mathbf{t}])[T]$ that sends

$$x_{ir} \mapsto x_{ir} + Ts_{pr} \qquad \text{for all } 1 \leq r \leq d;$$
$$\eta \mapsto \eta \qquad \text{for each } \eta \in \mathcal{V}_i.$$

Then, each $w \in K[\mathbf{x},\mathbf{u},\mathbf{s},\mathbf{t}]$ satisfies

$$\widehat{\mathbf{D}}w = \sum_{l \geq 0} \widehat{D}^l \left(s_p, x_i\right) w \cdot T^l \tag{9}$$

[141].

Let $\mathbf{D}$ be the $K$-algebra homomorphism $P \to P[T]$ that sends

$$(x_i \mid \eta) \mapsto (x_i \mid \eta) + T\left(s_p \mid \eta\right) \qquad \text{for each } \eta \in \mathcal{U} \cup \mathcal{T};$$
$$(\xi \mid \eta) \mapsto (\xi \mid \eta) \qquad \text{for each } \xi \in (\mathcal{X} \cup \mathcal{S}) \setminus \{x_i\} \text{ and } \eta \in \mathcal{U} \cup \mathcal{T}.$$

Then, each $w \in P$ satisfies

$$\mathbf{D}w = \sum_{l \geq 0} D^l \left(s_p, x_i\right) w \cdot T^l. \tag{11}$$

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[141] *Proof of (9):* It is clearly enough to prove the equality (9) in the case when $w$ is a monomial. Thus, assume that $w$ is a monomial. Write $w$ in the form $w = w'w''$, where $w'$ is the product of all the variables in $w$ of the form $x_{ir}$, and $w''$ is the product of the remaining variables in $w$.

The monomial $w''$ contains no variables of the form $x_{ir}$ (because of how $w''$ was defined). Thus, $w''$ is a product of variables from the set $\mathcal{V}_i$. But the $K$-algebra homomorphism $\widehat{\mathbf{D}}$ leaves all variables from the set $\mathcal{V}_i$ fixed (by the definition of $\widehat{\mathbf{D}}$). Thus, $\widehat{\mathbf{D}}$ also leaves any product of such variables fixed. Hence, $\widehat{\mathbf{D}}$ leaves $w''$ fixed (since $w''$ is a product of variables from the set $\mathcal{V}_i$). In other words, $\widehat{\mathbf{D}}w'' = w''$.

Now, write the monomial $w'$ in the form $w' = \prod_{b=1}^{a} x_{ir_b}$. (This is clearly possible due to how $w'$ was defined.) Thus, $w' = \prod_{b=1}^{a} x_{ir_b} = x_{ir_1} x_{ir_2} \cdots x_{ir_a}$. Hence, (6) (applied to $v = w'$) yields

$$\widehat{D}^l \left(s_p, x_i\right) \left(w'\right) = \sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E|=l}} \left(\prod_{b \in E} s_{pr_b}\right) \left(\prod_{\substack{b \in \{1,2,\ldots,a\}; \\ b \notin E}} x_{ir_b}\right) \tag{10}$$

for each $l \geq 0$.

Applying the map $\widehat{\mathbf{D}}$ to the equality $w' = \prod_{b=1}^{a} x_{ir_b}$, we obtain

$$\begin{aligned}
\widehat{\mathbf{D}} w' &= \widehat{\mathbf{D}} \left( \prod_{b=1}^{a} x_{ir_b} \right) = \prod_{b=1}^{a} \underbrace{\widehat{\mathbf{D}} \left( x_{ir_b} \right)}_{= x_{ir_b} + T s_{pr_b}} \\
&\qquad \left( \text{since } \widehat{\mathbf{D}} \text{ is a } K\text{-algebra homomorphism} \right) \\
&= \prod_{b=1}^{a} \left( x_{ir_b} + T s_{pr_b} \right) = \underbrace{\sum_{E \subseteq \{1,2,\ldots,a\}}}_{= \sum_{l \geq 0} \sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E| = l}}} \underbrace{\left( \prod_{b \in E} \left( T s_{pr_b} \right) \right)}_{= T^{|E|} \prod_{b \in E} s_{pr_b}} \left( \prod_{\substack{b \in \{1,2,\ldots,a\}; \\ b \notin E}} x_{ir_b} \right) \\
&\qquad \left( \text{here, we have expanded the product} \right) \\
&= \sum_{l \geq 0} \sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E| = l}} \underbrace{T^{|E|}}_{\substack{= T^l \\ (\text{since } |E| = l)}} \left( \prod_{b \in E} s_{pr_b} \right) \left( \prod_{\substack{b \in \{1,2,\ldots,a\}; \\ b \notin E}} x_{ir_b} \right) \\
&= \sum_{l \geq 0} \sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E| = l}} T^l \left( \prod_{b \in E} s_{pr_b} \right) \left( \prod_{\substack{b \in \{1,2,\ldots,a\}; \\ b \notin E}} x_{ir_b} \right) \\
&= \sum_{l \geq 0} T^l \underbrace{\sum_{\substack{E \subseteq \{1,2,\ldots,a\}; \\ |E| = l}} \left( \prod_{b \in E} s_{pr_b} \right) \left( \prod_{\substack{b \in \{1,2,\ldots,a\}; \\ b \notin E}} x_{ir_b} \right)}_{\substack{= \widehat{D}^l (s_p, x_i)(w') \\ (\text{by (10)})}} \\
&= \sum_{l \geq 0} T^l \widehat{D}^l \left( s_p, x_i \right) \left( w' \right).
\end{aligned}$$

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Now, it is easy to see that

$$\widehat{\mathbf{D}} \circ \phi = \phi [T] \circ \mathbf{D}.$$

Indeed, both sides of this equality are $K$-algebra homomorphisms from $P$ to $(K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}])[T]$, and thus their equality can be comfortably proven by verifying it only on the generators $(\xi \mid \eta)$ of the $K$-algebra $P$ (which verification is straightforward and easy, requiring only two cases depending on whether $\xi = x_i$ or not).

Now, let $w \in P$. Then,

$$\left(\widehat{\mathbf{D}} \circ \phi\right)(w) = \widehat{\mathbf{D}}(\phi w) = \sum_{l \geq 0} \widehat{D}^l \left(s_p, x_i\right) \phi w \cdot T^l$$

(by (9), applied to $\phi w$ instead of $w$). Comparing this with

$$\begin{aligned}\underbrace{\left(\widehat{\mathbf{D}} \circ \phi\right)}_{=\phi[T]\circ\mathbf{D}}(w) &= (\phi[T] \circ \mathbf{D})(w) = (\phi[T]) \underbrace{(\mathbf{D}w)}_{\substack{=\sum\limits_{l\geq 0} D^l(s_p,x_i)w\cdot T^l \\ \text{(by (11))}}} \\ &= (\phi[T])\left(\sum_{l\geq 0} D^l\left(s_p, x_i\right) w \cdot T^l\right) \\ &= \sum_{l\geq 0} \phi D^l\left(s_p, x_i\right) w \cdot T^l,\end{aligned}$$

we obtain

$$\sum_{l\geq 0} \widehat{D}^l\left(s_p, x_i\right) \phi w \cdot T^l = \sum_{l\geq 0} \phi D^l\left(s_p, x_i\right) w \cdot T^l.$$

Comparing coefficients in front of $T^l$ on both sides of this equality, we obtain $\widehat{D}^l\left(s_p, x_i\right) \phi w = \phi D^l\left(s_p, x_i\right) w$. Since this holds for all $w \in P$, we thus find $\widehat{D}^l\left(s_p, x_i\right) \phi = \phi D^l\left(s_p, x_i\right)$. This proves Lemma 5.2. $\square$

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Applying the map $\widehat{\mathbf{D}}$ to the equality $w = w'w''$, we obtain

$$\begin{aligned}\widehat{\mathbf{D}}w = \widehat{\mathbf{D}}\left(w'w''\right) &= \underbrace{\left(\widehat{\mathbf{D}}w'\right)}_{=\sum\limits_{l\geq 0} T^l \widehat{D}^l(s_p,x_i)(w')} \underbrace{\left(\widehat{\mathbf{D}}w''\right)}_{=w''} \\ &\qquad \left(\text{since } \widehat{D}_T\left(s_p, x_i\right) \text{ is a } K\text{-algebra homomorphism}\right) \\ &= \left(\sum_{l\geq 0} T^l \widehat{D}^l\left(s_p, x_i\right)\left(w'\right)\right) w'' = \sum_{l\geq 0} T^l \underbrace{\widehat{D}^l\left(s_p, x_i\right)\left(w'\right) w''}_{\substack{=\widehat{D}^l(s_p,x_i)w \\ \text{(by (8))}}} \\ &= \sum_{l\geq 0} T^l \widehat{D}^l\left(s_p, x_i\right) w = \sum_{l\geq 0} \widehat{D}^l\left(s_p, x_i\right) w \cdot T^l.\end{aligned}$$

This proves (9).

[142] The proof of this is similar to the proof of (9).

An analogue of Lemma 5.2 for the operators $\widehat{D}^l\left(t_q, u_j\right)$ and $D^l\left(t_q, u_j\right)$ also holds.[143]

Consider now a bitableau $[T, T']$. The *Capelli operator* $\widehat{C}\left(T, T'\right)$ is[144] defined on $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$ by mimicking the definition of $C\left(T, T'\right)$ with $\widehat{D}$ instead of $D$.[145] Lemma 5.2 (and its analogue for the operators $\widehat{D}^l\left(t_q, u_j\right)$ and $D^l\left(t_q, u_j\right)$) yields as a corollary the following identity:[146]

**Corollary 5.3.**

$$\widehat{C}\left(T, T'\right)\phi = \phi C\left(T, T'\right).$$

Let us next prove some further observations.[147]

**Lemma 5.4.** Each inner product $\left(x_{i_1}\cdots x_{i_l} \mid u_{j_1}\cdots u_{j_l}\right)$ with $l > d$ satisfies $\phi\left(x_{i_1}\cdots x_{i_l} \mid u_{j_1}\cdots u_{j_l}\right) = 0$.

*Proof of Lemma 5.4.* Let $\left(x_{i_1}\cdots x_{i_l} \mid u_{j_1}\cdots u_{j_l}\right)$ be an inner product with $l > d$. It is straightforward to see that

$$\begin{pmatrix} \left\langle x_{i_1} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_1} \mid u_{j_l}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_{i_l} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_l} \mid u_{j_l}\right\rangle \end{pmatrix} = \begin{pmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_l,1} & \cdots & x_{i_l,d} \end{pmatrix} \cdot \begin{pmatrix} u_{j_1,1} & \cdots & u_{j_l,1} \\ \vdots & \ddots & \vdots \\ u_{j_1,d} & \cdots & u_{j_l,d} \end{pmatrix}.$$

Hence, the matrix $\begin{pmatrix} \left\langle x_{i_1} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_1} \mid u_{j_l}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_{i_l} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_l} \mid u_{j_l}\right\rangle \end{pmatrix}$ is the product of an $l \times d$-matrix with a $d \times l$-matrix. Thus, the matrix $\begin{pmatrix} \left\langle x_{i_1} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_1} \mid u_{j_l}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_{i_l} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_l} \mid u_{j_l}\right\rangle \end{pmatrix}$ has determinant 0 (since $d < l$). In other words, $\begin{vmatrix} \left\langle x_{i_1} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_1} \mid u_{j_l}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_{i_l} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_l} \mid u_{j_l}\right\rangle \end{vmatrix} = 0$.

[143] *Correction:* Added this sentence.

[144] *Correction:* Replaced “are” by “is”.

[145] *Comment:* Explicitly, this means that $\widehat{C}\left(T, T'\right)$ is defined as follows: Let $(\lambda)$ be the shape of $T$. Let $\alpha_i(q)$ (resp. $\beta_j(q)$) be the number of occurrences of $x_i$ (resp. $u_j$) in the $q$th column of $T$ (resp. $T'$). The Capelli operator $\widehat{C}\left(T, T'\right)$ is defined by the following formula:

$$\widehat{C}\left(T, T'\right) = \prod_{1\leq q\leq \lambda_1}\left(\prod_{1\leq i\leq n}\widehat{D}^{\alpha_i(q)}\left(s_q, x_i\right)\right)\left(\prod_{1\leq j\leq k}\widehat{D}^{\beta_j(q)}\left(t_q, u_j\right)\right).$$

The right hand side of this equality is a product of commuting operators (since the operators $\widehat{D}$ all commute). Notice that $\lambda_1$ is the number of columns of $T$.

[146] *Correction:* Slightly rewrote this sentence in order to make the identity an actual corollary.

[147] *Correction:* Everything between this sentence and the proof of Theorem 5.1 has been added by me.

But

$$\left(x_{i_1} \cdots x_{i_l} \mid u_{j_1} \cdots u_{j_l}\right) = \begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_l}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_l} \mid u_{j_1}\right) & \cdots & \left(x_{i_l} \mid u_{j_l}\right) \end{vmatrix}.$$

Applying the map $\phi$ to this identity, we obtain

$$\begin{aligned} \phi\left(x_{i_1} \cdots x_{i_l} \mid u_{j_1} \cdots u_{j_l}\right) &= \phi \begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_l}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_l} \mid u_{j_1}\right) & \cdots & \left(x_{i_l} \mid u_{j_l}\right) \end{vmatrix} \\ &= \begin{vmatrix} \phi\left(x_{i_1} \mid u_{j_1}\right) & \cdots & \phi\left(x_{i_1} \mid u_{j_l}\right) \\ \vdots & \ddots & \vdots \\ \phi\left(x_{i_l} \mid u_{j_1}\right) & \cdots & \phi\left(x_{i_l} \mid u_{j_l}\right) \end{vmatrix} \\ &\qquad \text{(since } \phi \text{ is a } K\text{-algebra homomorphism)} \\ &= \begin{vmatrix} \left\langle x_{i_1} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_1} \mid u_{j_l}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_{i_l} \mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_l} \mid u_{j_l}\right\rangle \end{vmatrix} \\ &\qquad \left(\text{since } \phi\left(x_i \mid u_j\right) = \left\langle x_i \mid u_j\right\rangle \text{ for all } i \text{ and } j\right) \\ &= 0. \end{aligned}$$

This proves Lemma 5.4. □

**Lemma 5.5.** Let $\omega : K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}] \to K$ be the $K$-algebra homomorphism given by

$$\begin{aligned} x_{ir} &\mapsto 0 && \text{for all } 1 \leq i \leq n \text{ and } 1 \leq r \leq d; \\ u_{jr} &\mapsto 0 && \text{for all } 1 \leq j \leq k \text{ and } 1 \leq r \leq d; \\ s_{pr} &\mapsto \delta_{p,r} && \text{for all } 1 \leq p \text{ and } 1 \leq r \leq d; \\ t_{qr} &\mapsto \delta_{q,r} && \text{for all } 1 \leq q \text{ and } 1 \leq r \leq d \end{aligned}$$

(where $\delta_{u,v}$ stands for $\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}$). Then, each $k \in \{0, 1, \ldots, d\}$ satisfies $\omega\left(\phi\left(s_1 s_2 \cdots s_k \mid t_1 t_2 \cdots t_k\right)\right) = 1$.

*Proof of Lemma 5.5.* Let $k \in \{0,1,\ldots,d\}$. Then, for each $p,q \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned}
\omega\left\langle s_p \mid t_q\right\rangle &= \omega\left(\sum_{1\leq r\leq d} s_{pr}t_{qr}\right) \qquad \left(\text{since } \left\langle s_p \mid t_q\right\rangle = \sum_{1\leq r\leq d} s_{pr}t_{qr}\right) \\
&= \sum_{1\leq r\leq d} \underbrace{\omega\left(s_{pr}\right)}_{\substack{=\delta_{p,r} \\ \text{(by the definition of } \omega)}} \underbrace{\omega\left(t_{qr}\right)}_{\substack{=\delta_{q,r} \\ \text{(by the definition of } \omega)}} \\
&\qquad (\text{since } \omega \text{ is a } K\text{-algebra homomorphism}) \\
&= \sum_{1\leq r\leq d} \delta_{p,r}\delta_{q,r} = \begin{cases} 1, & \text{if } p=q\leq d; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } p=q; \\ 0, & \text{otherwise} \end{cases} \\
&\qquad \left(\begin{array}{c} \text{since } p=q\leq d \text{ holds if and only if } p=q \\ \text{(because } q\leq d \text{ always holds (since } q\leq k\leq d)) \end{array}\right) \\
&= \delta_{p,q}. \qquad (12)
\end{aligned}$$

Now,

$$\left(s_1s_2\cdots s_k \mid t_1t_2\cdots t_k\right) = \begin{vmatrix} \left(s_1 \mid t_1\right) & \cdots & \left(s_1 \mid t_k\right) \\ \vdots & \ddots & \vdots \\ \left(s_k \mid t_1\right) & \cdots & \left(s_k \mid t_k\right) \end{vmatrix}.$$

Applying the map $\phi$ to this identity, we obtain

$$\begin{aligned}
\phi\left(s_1s_2\cdots s_k \mid t_1t_2\cdots t_k\right) &= \phi\begin{vmatrix} \left(s_1 \mid t_1\right) & \cdots & \left(s_1 \mid t_k\right) \\ \vdots & \ddots & \vdots \\ \left(s_k \mid t_1\right) & \cdots & \left(s_k \mid t_k\right) \end{vmatrix} = \begin{vmatrix} \phi\left(s_1 \mid t_1\right) & \cdots & \phi\left(s_1 \mid t_k\right) \\ \vdots & \ddots & \vdots \\ \phi\left(s_k \mid t_1\right) & \cdots & \phi\left(s_k \mid t_k\right) \end{vmatrix} \\
&\qquad (\text{since } \phi \text{ is a } K\text{-algebra homomorphism}) \\
&= \begin{vmatrix} \left\langle s_1 \mid t_1\right\rangle & \cdots & \left\langle s_1 \mid t_k\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle s_k \mid t_1\right\rangle & \cdots & \left\langle s_k \mid t_k\right\rangle \end{vmatrix} \\
&\qquad \left(\text{since } \phi\left(s_p \mid t_q\right) = \left\langle s_p \mid t_q\right\rangle \text{ for all } p \text{ and } q\right).
\end{aligned}$$

Applying the map $\omega$ to this identity, we find

$$\begin{aligned}
\omega\left(\phi\left(s_1 s_2 \cdots s_k \mid t_1 t_2 \cdots t_k\right)\right) &= \omega \begin{vmatrix} \langle s_1 \mid t_1 \rangle & \cdots & \langle s_1 \mid t_k \rangle \\ \vdots & \ddots & \vdots \\ \langle s_k \mid t_1 \rangle & \cdots & \langle s_k \mid t_k \rangle \end{vmatrix} \\
&= \begin{vmatrix} \omega \langle s_1 \mid t_1 \rangle & \cdots & \omega \langle s_1 \mid t_k \rangle \\ \vdots & \ddots & \vdots \\ \omega \langle s_k \mid t_1 \rangle & \cdots & \omega \langle s_k \mid t_k \rangle \end{vmatrix} \\
&\qquad \text{(since } \omega \text{ is a } K\text{-algebra homomorphism)} \\
&= \left| \underbrace{\begin{matrix} \delta_{1,1} & \cdots & \delta_{1,k} \\ \vdots & \ddots & \vdots \\ \delta_{k,1} & \cdots & \delta_{k,k} \end{matrix}}_{=I_k} \right| \qquad \text{(by (12))} \\
&= \left| I_k \right| = 1.
\end{aligned}$$

This proves Lemma 5.5. □

**Lemma 5.6.** Let $[T, T']$ be a standard $(\mathcal{X}, \mathcal{U})$-bitableau of shape $(\lambda)$, where $(\lambda)$ is not longer than $(d)$. Then,

$$\phi C\left(T, T'\right)\left(T \mid T'\right) \neq 0. \tag{13}$$

*Proof of Lemma 5.6.* Consider the $K$-algebra homomorphism $\omega : K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}] \to K$ defined in Lemma 5.5.

We have assumed that $(\lambda)$ is not longer than $(d)$. Thus, $\lambda_1 \leq d$. Hence, $\lambda_r \leq d$ for all $r \in \{1, 2, 3, \ldots\}$. Therefore, for each $r \in \{1, 2, 3, \ldots\}$, we have $\lambda_r \in \{0, 1, \ldots, d\}$ and thus

$$\omega\left(\phi\left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right)\right) = 1 \tag{14}$$

(by Lemma 5.5, applied to $k = \lambda_r$).

Define two tableaux $U$ and $U'$ as in the proof of Theorem 4.5 (i). Then, $C\left(T, T'\right)\left(T \mid T'\right) = \left(U \mid U'\right)$ (this was shown in the proof of Theorem 4.5 (i)). But from the definition of $U$ and $U'$, we obtain

$$\left(U \mid U'\right) = \prod_{r \geq 1} \left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right).$$

Applying the map $\phi$ to both sides of this equality, we obtain

$$\phi\left(U \mid U'\right) = \phi\left(\prod_{r \geq 1} \left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right)\right) = \prod_{r \geq 1} \phi\left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right)$$

(since $\phi$ is a $K$-algebra homomorphism). Applying the map $\phi$ to the equality $C(T,T')(T \mid T') = (U \mid U')$, we now find

$$\phi C\left(T,T'\right)\left(T \mid T'\right) = \phi\left(U \mid U'\right) = \prod_{r\geq 1} \phi\left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right).$$

Applying the map $\omega$ to both sides of this equality, we find

$$\begin{aligned}
\omega\left(\phi C\left(T,T'\right)\left(T \mid T'\right)\right) &= \omega\left(\prod_{r\geq 1} \phi\left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right)\right) \\
&= \prod_{r\geq 1} \underbrace{\omega\left(\phi\left(s_1 s_2 \cdots s_{\lambda_r} \mid t_1 t_2 \cdots t_{\lambda_r}\right)\right)}_{\substack{=1 \\ \text{(by (14))}}} \\
&\qquad \text{(since } \omega \text{ is a } K\text{-algebra homomorphism)} \\
&= \prod_{r\geq 1} 1 = 1 \neq 0.
\end{aligned}$$

Therefore, $\phi C(T,T')(T \mid T') \neq 0$. This proves Lemma 5.6. $\square$

**Lemma 5.7.** Let $[T,T']$ be a standard $(\mathcal{X},\mathcal{U})$-bitableau of shape $(\lambda)$, where $(\lambda)$ is longer than $(d)$. Then,

$$\phi\left(T \mid T'\right) = 0. \tag{15}$$

*Proof of Lemma 5.7.* We have assumed that $(\lambda)$ is longer than $(d)$. Thus, $\lambda_1 > d$. Now, for each $p \geq 1$ and $q \in \{1,2,\ldots,\lambda_p\}$, let $x_{i(p,q)}$ be the $(p,q)$-th entry of the tableau $T$, and let $u_{j(p,q)}$ be the $(p,q)$-th entry of the tableau $T'$. Then,

$$\left(T \mid T'\right) = \prod_{r\geq 1} \left(x_{i(r,1)} x_{i(r,2)} \cdots x_{i(r,\lambda_r)} \mid u_{j(r,1)} u_{j(r,2)} \cdots u_{j(r,\lambda_r)}\right). \tag{16}$$

But $\lambda_1 > d$ and thus $\phi\left(x_{i(1,1)} x_{i(1,2)} \cdots x_{i(1,\lambda_1)} \mid u_{j(1,1)} u_{j(1,2)} \cdots u_{j(1,\lambda_1)}\right) = 0$ (by Lemma 5.4, applied to $l = \lambda_1$, $i_k = i(1,k)$ and $j_k = j(1,k)$). Hence, the product $\prod_{r\geq 1} \phi\left(x_{i(r,1)} x_{i(r,2)} \cdots x_{i(r,\lambda_r)} \mid u_{j(r,1)} u_{j(r,2)} \cdots u_{j(r,\lambda_r)}\right)$ has at least one factor equal to 0 (namely, the factor for $r = 1$), and thus equals 0 itself. In other words,

$$\prod_{r\geq 1} \phi\left(x_{i(r,1)} x_{i(r,2)} \cdots x_{i(r,\lambda_r)} \mid u_{j(r,1)} u_{j(r,2)} \cdots u_{j(r,\lambda_r)}\right) = 0.$$

But applying the map $\phi$ to the equality (16), we obtain

$$\begin{aligned}
\phi\left(T \mid T'\right) &= \phi\left(\prod_{r\geq 1} \left(x_{i(r,1)} x_{i(r,2)} \cdots x_{i(r,\lambda_r)} \mid u_{j(r,1)} u_{j(r,2)} \cdots u_{j(r,\lambda_r)}\right)\right) \\
&= \prod_{r\geq 1} \phi\left(x_{i(r,1)} x_{i(r,2)} \cdots x_{i(r,\lambda_r)} \mid u_{j(r,1)} u_{j(r,2)} \cdots u_{j(r,\lambda_r)}\right) \\
&\qquad \text{(since } \phi \text{ is a } K\text{-algebra homomorphism)} \\
&= 0.
\end{aligned}$$

This proves Lemma 5.7. □

With these tools in hand, we can begin the proof of the theorem.

*Proof of Theorem 5.1.* [148] Indeed[149], consider an element $M \in P(\alpha, \beta)$ [150] in the kernel of $\phi$. Using Corollary 3.4[151], write $M$ as a linear combination of standard $(\mathcal{X}, \mathcal{U})$-bideterminants[152]:

$$M = a\left(T \mid T'\right) + N,$$

where $(\lambda)$ is the shortest shape occurring in the expansion, and, of all the bideterminants of shape $(\lambda)$ occurring in the expansion, $(T \mid T')$ is the one with the lexicographically smallest column sequence[153]. Applying the Capelli operator $C(T, T')$, and observing that, by Theorem 4.5, $C(T, T')\, N = 0$, we obtain

$$C\left(T, T'\right) M = aC\left(T, T'\right)\left(T \mid T'\right).$$

[154] Applying $\phi$ and using the identity in Corollary 5.3[155], we have

$$a\phi C\left(T, T'\right)\left(T \mid T'\right) = \phi C\left(T, T'\right) M = \widehat{C}\left(T, T'\right) \phi M.$$

But $\phi M = 0$; hence, $C(T, T')(T \mid T')$ must be a bideterminant of shape strictly longer than $(d)$ [156]. As $C(T, T')(T \mid T')$ and $(T \mid T')$ have the same shape, we

[148] *Correction:* I have removed the first two paragraphs of this proof, since they were a distraction from the proof. (To be more precise, the useful parts of these two paragraphs have been preserved in Lemmas 5.4, 5.5, 5.6, and 5.7. The two paragraphs I removed can be found in Remark 5.8.)

[149] *Correction:* Replaced "Finally" by "Indeed".

[150] *Correction:* Replaced "$M$" by "$M \in P(\alpha, \beta)$".

[151] *Correction:* Replaced "the straightening formula" by "Corollary 3.4".

[152] *Correction:* Replaced "bideterminants" by "$(\mathcal{X}, \mathcal{U})$-bideterminants".

[153] *Comment:* ... and, of course, $a$ is a nonzero element of $K$.

[154] *Comment:* Here is this argument in slightly more detail:

By the definition of $(\lambda)$ and $(T \mid T')$, we know that $N$ is a $K$-linear combination of bideterminants $(V \mid V')$ for standard $(\mathcal{X}, \mathcal{U})$-bitableaux $[V, V']$ which either have shape longer than $(\lambda)$ or have the same shape as $(\lambda)$ but have a greater column sequence than $[T, T']$. All such bideterminants $(V \mid V')$ satisfy $C(T, T')(V \mid V') = 0$ (indeed, in the former case, this follows from Theorem 4.5 (ii), whereas in the latter case it follows from Theorem 4.5 (iii)). Hence, $N$ (being a $K$-linear combination of such bideterminants) must also satisfy $C(T, T')\, N = 0$. Now, applying the operator $C(T, T')$ to the equality $M = a(T \mid T') + N$, we obtain

$$C\left(T, T'\right) M = C\left(T, T'\right)\left(a\left(T \mid T'\right) + N\right) = aC\left(T, T'\right)\left(T \mid T'\right) + \underbrace{C\left(T, T'\right) N}_{=0} = aC\left(T, T'\right)\left(T \mid T'\right),$$

qed.

[155] *Correction:* Replaced "the previous corollary" by "Corollary 5.3".

[156] *Comment:* Here is why this holds:

Define two tableaux $U$ and $U'$ as in the proof of Theorem 4.5 (i). Then, $C(T, T')(T \mid T') = (U \mid U')$ (this was shown in the proof of Theorem 4.5 (i)). Thus, $C(T, T')(T \mid T')$ is a bideterminant. It only remains to show that this bideterminant $C(T, T')(T \mid T')$ has shape longer than $(d)$. In other words, it remains to prove that $(\lambda)$ is longer than $(d)$ (since this bideterminant has conclude that $(\lambda)$ is strictly longer than $(d)$. But we have chosen $(\lambda)$ to be the shortest shape occurring in the expansion of $M$. Therefore, $M$ is a linear combination of standard $(\mathcal{X},\mathcal{U})$-bideterminants[157] of shape strictly longer than $(d)$. This concludes the proof of the theorem[158]. □

**Remark 5.8.** For the sake of completeness, here are the first two paragraphs from the original proof of Theorem 5.1 (which have been omitted from our above version of this proof)[159]:

The image under $\phi$ of an inner product $\left(x_{i_1}\cdots x_{i_l}\mid u_{j_1}\cdots u_{j_l}\right)$ is the determinant[160]

$$\left|\begin{array}{ccc} \left\langle x_{i_1}\mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_1}\mid u_{j_l}\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle x_{i_l}\mid u_{j_1}\right\rangle & \cdots & \left\langle x_{i_l}\mid u_{j_l}\right\rangle \end{array}\right|.$$

This is the determinant of the matrix

$$\left[\begin{array}{ccc} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_l,1} & \cdots & x_{i_l,d} \end{array}\right]\cdot\left[\begin{array}{ccc} u_{j_1,1} & \cdots & u_{j_l,1} \\ \vdots & \ddots & \vdots \\ u_{j_1,d} & \cdots & u_{j_l,d} \end{array}\right].$$

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shape $(\lambda)$).

Assume the contrary. Thus, $(\lambda)$ is not longer than $(d)$. Hence, (13) yields $\phi C\left(T,T'\right)\left(T\mid T'\right)\neq 0$.

But $a\phi C\left(T,T'\right)\left(T\mid T'\right)=\widehat{C}\left(T,T'\right)\underbrace{\phi M}_{=0}=0$, so that $a=0$ (since $\phi C\left(T,T'\right)\left(T\mid T'\right)\neq 0$). This contradicts $a\neq 0$. This contradiction shows that our assumption was false. Hence, we have shown that $(\lambda)$ is longer than $(d)$.

[157] *Correction:* Replaced "bideterminants" by "$(\mathcal{X},\mathcal{U})$-bideterminants".

[158] *Comment:* Let me explain why this concludes the proof of Theorem 5.1:

We have just shown that each $M\in P(\alpha,\beta)$ lying in the kernel of $\phi$ must be a linear combination of standard $(\mathcal{X},\mathcal{U})$-bideterminants of shape strictly longer than $(d)$. In other words, each $M$ in the kernel of the Pascal homomorphism must be a linear combination of standard $(\mathcal{X},\mathcal{U})$-bideterminants of shape strictly longer than $(d)$.

On the other hand, each standard $(\mathcal{X},\mathcal{U})$-bideterminant of shape strictly longer than $(d)$ lies in the kernel of $\phi$ (by (15)), thus in the kernel of the Pascal homomorphism. Hence, each linear combination of standard $(\mathcal{X},\mathcal{U})$-bideterminants of shape strictly longer than $(d)$ lies in the kernel of the Pascal homomorphism.

Thus, we have shown the following two facts:

- Each $M$ in the kernel of the Pascal homomorphism must be a linear combination of standard $(\mathcal{X},\mathcal{U})$-bideterminants of shape strictly longer than $(d)$;
- Each linear combination of standard $(\mathcal{X},\mathcal{U})$-bideterminants of shape strictly longer than $(d)$ lies in the kernel of the Pascal homomorphism.

Combining these two facts, we conclude that the elements of the kernel of the Pascal homomorphism are precisely the linear combinations of standard $(\mathcal{X},\mathcal{U})$-bideterminants of shape strictly longer than $(d)$. This proves Theorem 5.1.

[159] As already mentioned, they are not necessary to the proof.

[160] *Correction:* Replaced "$x_{j_1}$" by "$u_{j_1}$" in this determinant.

[161] As the variables $x_{ir}, u_{jr}$ are algebraically independent, the above matrix is of maximum possible rank[162]; that is to say, its determinant is zero iff $l > d$.

Now, consider a bideterminant $(T \mid T')$. It is a product of inner products of lengths $\lambda_1, \ldots, \lambda_l$. As $\widehat{P}$ [163] (being a subring of $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$) is an integral domain, $\phi(T \mid T')$ is zero iff one of its constituent inner products has zero image. This happens iff $\lambda_1 > d$, or $(\lambda)$ is strictly longer than $(d)$.

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[161] *Comment:* This is because

$$\begin{bmatrix} \langle x_{i_1} \mid u_{j_1} \rangle & \cdots & \langle x_{i_1} \mid u_{j_l} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{i_l} \mid u_{j_1} \rangle & \cdots & \langle x_{i_l} \mid u_{j_l} \rangle \end{bmatrix} = \begin{bmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_l,1} & \cdots & x_{i_l,d} \end{bmatrix} \cdot \begin{bmatrix} u_{j_1,1} & \cdots & u_{j_l,1} \\ \vdots & \ddots & \vdots \\ u_{j_1,d} & \cdots & u_{j_l,d} \end{bmatrix}.$$

[162] *Comment:* "Maximum possible rank" here means "rank $\min\{l, d\}$" (where "rank" means the rank over the quotient field of $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$). Of course, the claim that this matrix has rank $\min\{l, d\}$ is only made under the tacit assumption that the indices $i_1, \ldots, i_l$ are distinct and the indices $j_1, \ldots, j_l$ are distinct.

Let me explain why this claim holds. We must prove that the matrix

$$\begin{bmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_l,1} & \cdots & x_{i_l,d} \end{bmatrix} \cdot \begin{bmatrix} u_{j_1,1} & \cdots & u_{j_l,1} \\ \vdots & \ddots & \vdots \\ u_{j_1,d} & \cdots & u_{j_l,d} \end{bmatrix}$$

has rank $\min\{l, d\}$. Indeed, it clearly has rank $\leq \min\{l, d\}$ (since it is the product of an $l \times d$-matrix with a $d \times l$-matrix). Thus, it remains to show that it has rank $\geq \min\{l, d\}$. To achieve this, it will suffice to prove that it has a nonvanishing minor of size $\min\{l, d\}$. We shall distinguish between the cases $l \geq d$ and $l < d$:

- If $l \geq d$, then the northwesternmost $d \times d$-minor of our matrix is

$$\det\left( \begin{bmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_d,1} & \cdots & x_{i_d,d} \end{bmatrix} \cdot \begin{bmatrix} u_{j_1,1} & \cdots & u_{j_d,1} \\ \vdots & \ddots & \vdots \\ u_{j_1,d} & \cdots & u_{j_d,d} \end{bmatrix} \right),$$

  which is nonzero (because if we substitute $\delta_{s,t}$ for each variable $x_{i_s,t}$ and for each variable $u_{j_s,t}$, then this $d \times d$-minor becomes $\det(I_d) = 1$); this is thus a nonvanishing minor of size $d = \min\{l, d\}$ (since $l \geq d$).

- If $l < d$, then the $l \times l$-minor of our matrix is

$$\det\left( \begin{bmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_l,1} & \cdots & x_{i_l,d} \end{bmatrix} \cdot \begin{bmatrix} u_{j_1,1} & \cdots & u_{j_l,1} \\ \vdots & \ddots & \vdots \\ u_{j_1,d} & \cdots & u_{j_l,d} \end{bmatrix} \right),$$

  which is nonzero (because if we substitute $\delta_{s,t}$ for each variable $x_{i_s,t}$ and for each variable $u_{j_s,t}$, then this $l \times l$-minor becomes $\det(I_l) = 1$); this is thus a nonvanishing minor of size $l = \min\{l, d\}$ (since $l < d$).

[163] *Correction:* Replaced "$P$" by "$\widehat{P}$".

## 5.2. The First Fundamental Theorem: statement

Consider the algebra $P$, defined on the alphabets $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{U} = \{u_1, \ldots, u_d\}$. [164] In $P$, all the bideterminants of shape strictly longer than $(d)$ are identically 0 [165]; hence, by Theorem 5.1, the map[166] $\phi$ is an injection. This allows us to transfer questions about forms to questions about elements of $P$.

(Recall that $P$ is the $K$-algebra of polynomials in the indeterminates $\left(x_i \mid u_j\right)$ with $1 \leq i \leq n$ and $1 \leq j \leq d$ over $K$.) [167]

Let $L$ be a linear transformation from the vector space $V_d$ to itself.[168] If $F$ is a form on $V_d$, then[169] $L$ acts on $F$ by

$$LF\left(\mathbf{x}_1, \ldots, \mathbf{x}_m\right) = F\left(L\mathbf{x}_1, \ldots, L\mathbf{x}_m\right).$$

A form $F$ [170] is *invariant* if, for all invertible linear transformations $L$, there exists a scalar $a\left(L\right)$ such that $LF = a\left(L\right)F$.

Transferring to the algebra $P$, a *form* $F\left(\mathbf{x}_1, \ldots, \mathbf{x}_n\right)$ is an element of $P$, that is, a polynomial in the variables $\left(x_i \mid u_j\right)$ with $1 \leq i \leq n$ and $1 \leq j \leq d$. [171] A $d \times d$-matrix[172] $L = \left(l_{kj}\right)_{1 \leq j,k \leq d} \in K^{d \times d}$ acts upon $P$ as an algebra homomorphism as follows:

$$L\left(x_i \mid u_j\right) = \sum_{1 \leq k \leq d} l_{jk}\left(x_i \mid u_k\right), \qquad 1 \leq i \leq n, \qquad 1 \leq j \leq d.$$

A form $F$ is *invariant* if, for all invertible $d \times d$-matrices $L \in K^{d \times d}$, there exists a scalar $a\left(L\right)$ such that $LF = a\left(L\right)F$.

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[164] *Comment:* In other words, from here on, we return to the original version of the $K$-algebra $P$, which does not include the indeterminates from the alphabets $\mathcal{S}$ and $\mathcal{T}$; we furthermore set $d = k$.

From here on, the letter $k$ shall no longer stand for the size of the alphabet $\mathcal{U}$ (as it did in "$\mathcal{U} = \{u_1, \ldots, u_k\}$"). Indeed, we do not need it in this role any more, since the letter $d$ serves the same purpose (because $\mathcal{U} = \{u_1, \ldots, u_d\}$). Thus, the letter $k$ is now free for any other use.

[165] *Comment:* This is because if $(T \mid T')$ is a bideterminant of shape strictly longer than $(d)$, then the first row of $T'$ has length $> d$ and thus contains two equal letters.

[166] *Correction:* Added the words "the map".

[167] *Correction:* Added the preceding sentence.

[168] *Comment:* This paragraph, again, just serves the purpose of providing motivation.

[169] *Correction:* Added the word "then".

[170] *Correction:* Replaced "form" by "form $F$".

[171] *Correction:* I have simplified the preceding sentence, removing some confusing generality. I re-introduce said generality later on where it becomes necessary (in the second proof of Theorem 5.9).

[172] *Correction:* Here and in the following, I have replaced the word "transformation" (or "linear transformation") by "$d \times d$-matrix $L \in K^{d \times d}$". Indeed, the algebra $P$ is not defined in a basis-free fashion, so the use of transformations instead of matrices does not serve any useful purpose (but it does create an extra level of indirection).

Also, in this specific place, I have replaced "An invertible linear transformation $L$, given by an invertible $d \times d$ square matrix $\left(l_{jk}\right)$" by "A a $d \times d$-matrix $L = \left(l_{kj}\right)_{1 \leq j,k \leq d} \in K^{d \times d}$". (In particular, I have replaced the matrix $\left(l_{jk}\right)$ by $\left(l_{kj}\right)$ in order for the action to be left-associative.)

An example of an invariant form is the inner product $\left(x_{i_1}\cdots x_{i_d} \mid u_1\cdots u_d\right)$ [173]; it is the determinant $\left|\left(x_{i_j} \mid u_k\right)\right|_{1\leq j,k\leq d}$, and in this case $a(L) = \det\left(l_{jk}\right)$ [174]. Similarly, the bideterminants of shape $(d,\ldots,d)$ are invariant forms[175]; these bide-

[173] *Correction:* Replaced "$\left(x_{i_1}\cdots d_{i_d} \mid u_1\cdots u_d\right)$" by "$\left(x_{i_1}\cdots x_{i_d} \mid u_1\cdots u_d\right)$".

[174] *Comment:* Let me show why this is true:

Let $F = \left(x_{i_1}\cdots x_{i_d} \mid u_1\cdots u_d\right)$ for some choice of indices $i_1,\ldots,i_d \in \{1,2,\ldots,n\}$. Let $L = \left(l_{kj}\right)_{1\leq j,k\leq d}$ be any $d\times d$-matrix. (It needs not be invertible.) Set $a(L) = \det\left(l_{jk}\right)$. We must then prove that $LF = a(L)F$.

From $L = \left(l_{kj}\right)_{1\leq j,k\leq d}$, we obtain $L^T = \left(l_{jk}\right)_{1\leq j,k\leq d}$, so that $\det\left(L^T\right) = \det\left(l_{jk}\right)_{1\leq j,k\leq d} = \det\left(l_{jk}\right)$. Hence, $\det\left(l_{jk}\right) = \det\left(L^T\right) = \det L = \det\left(\left(l_{kj}\right)_{1\leq j,k\leq d}\right)$ (since $L = \left(l_{kj}\right)_{1\leq j,k\leq d}$).

We have $F = \left(x_{i_1}\cdots x_{i_d} \mid u_1\cdots u_d\right) = \left|\left(x_{i_j} \mid u_k\right)\right|_{1\leq j,k\leq d}$ and thus

$$\begin{aligned}
LF &= L\left|\left(x_{i_j} \mid u_k\right)\right|_{1\leq j,k\leq d} \\
&= \left|L\left(x_{i_j} \mid u_k\right)\right|_{1\leq j,k\leq d} \qquad \text{(since } L \text{ acts on } P \text{ by an algebra homomorphism)} \\
&= \left|\underbrace{L\left(x_{i_j} \mid u_r\right)}_{\substack{=\sum_{1\leq k\leq d} l_{rk}\left(x_{i_j}\mid u_k\right) \\ \text{(by the definition of the action of } L)}}\right|_{1\leq j,r\leq d} = \left|\sum_{1\leq k\leq d} l_{rk}\left(x_{i_j} \mid u_k\right)\right|_{1\leq j,r\leq d} \\
&= \det\left(\underbrace{\left(\sum_{1\leq k\leq d} l_{rk}\left(x_{i_j} \mid u_k\right)\right)_{1\leq j,r\leq d}}_{=\left(\left(x_{i_j}\mid u_k\right)\right)_{1\leq j,k\leq d}\cdot\left(l_{kj}\right)_{1\leq j,k\leq d}}\right) = \det\left(\left(\left(x_{i_j} \mid u_k\right)\right)_{1\leq j,k\leq d}\cdot\left(l_{kj}\right)_{1\leq j,k\leq d}\right) \\
&= \underbrace{\det\left(\left(x_{i_j} \mid u_k\right)_{1\leq j,k\leq d}\right)}_{=\left|\left(x_{i_j}\mid u_k\right)\right|_{1\leq j,k\leq d}=F}\cdot\underbrace{\det\left(\left(l_{kj}\right)_{1\leq j,k\leq d}\right)}_{=\det\left(l_{jk}\right)=a(L)} \\
&= F\cdot a(L) = a(L)F,
\end{aligned}$$

qed.

[175] *Comment:* This holds for the following reason: Let $[T,T']$ be a bitableau of shape $(d,\ldots,d)$. Then, we claim that the bideterminant $(T \mid T')$ is an invariant form.

*Proof of the claim:* We are in one of the following two cases:

*Case 1:* Each row of $T'$ consists of the $d$ entries $u_1,\ldots,u_d$ (possibly reordered).

*Case 2:* At least one row of $T'$ does not consist of the $d$ entries $u_1,\ldots,u_d$ (possibly reordered).

Let us first consider Case 1. In this case, each row of $T'$ consists of the $d$ entries $u_1,\ldots,u_d$

terminants are called *rectangular*. For a rectangular bideterminant with $g$ rows, $a(L) = \left(\det\left(l_{jk}\right)\right)^{g}$. Note that since any bideterminant of shape longer than $(d)$ is zero, a rectangular bideterminant is a linear combination of standard rectangular bideterminants (with the same number of rows)[176].

These examples are in fact paradigmatic.[177]

**Theorem 5.9** (the first fundamental theorem of invariant theory)**.** Over an infinite field $K$, a form in $P$ is invariant iff it is a linear combination of standard rectangular bideterminants.

Note that the "if" part of Theorem 5.9 is clearly true (since rectangular bideterminants are always invariant). Thus, only the "only if" part remains to be proven.[178]

## 5.3. The First Fundamental Theorem: first proof

Before we prove this theorem in two different ways, let us see how certain matrices act on inner products:[179]

**Lemma 5.10.** Let $j \in \{2,3,\ldots,d\}$. Let $L$ be the the $d \times d$-matrix $\left(l_{qp}\right)_{1\leq p,q\leq d}$, where

$$\begin{aligned} l_{j-1,j} &= 1; \\ l_{pp} &= 1 \qquad \text{for all } p; \\ l_{qp} &= 0 \qquad \text{for all } p,q \text{ satisfying } p \neq q \text{ and } (p,q) \neq (j,j-1). \end{aligned}$$

Thus, $L$ acts upon $P$ as an algebra homomorphism.

(possibly reordered). Hence, upon reordering the entries in the rows of $T'$, we can ensure that each row of $T'$ has the form $u_1 \cdots u_d$. Thus, the bideterminant $(T \mid T')$ is a product of inner products of the form of the form $\left(x_{i_1} \cdots x_{i_d} \mid u_1 \cdots u_d\right)$. Since all inner products of the latter form are invariant forms, we therefore conclude that the bideterminant $(T \mid T')$ is an invariant form as well (since any product of invariant forms must be an invariant form). Thus, the claim is proven in Case 1.

Let us now consider Case 2. In this case, at least one row of $T'$ does not consist of the $d$ entries $u_1, \ldots, u_d$ (possibly reordered). Hence, this row must have two equal entries (because it has $d$ entries chosen from the set $\mathcal{U} = \{u_1, \ldots, u_d\}$, and thus it either consists of the $d$ entries $u_1, \ldots, u_d$ or has two equal entries). Consequently, the inner product in $(T \mid T')$ corresponding to this row is 0. Therefore, the bideterminant $(T \mid T')$ is 0. Thus, of course, $(T \mid T')$ is an invariant form (since 0 is an invariant form). Hence, the claim is proven in Case 2.

We thus have proven the claim in both possible cases.

[176] *Comment:* This is a consequence of Theorem 3.2. For a more detailed proof, see Lemma 5.15 further below.

[177] *Correction:* Replaced "has" by "have" in the following theorem. Also, removed the "all of which have the same shape $(d, \ldots, d)$" part, since it only holds for homogeneous invariants.

[178] *Correction:* Added the preceding two sentences.

[179] *Correction:* I have added this sentence, the next lemma and its proof, in order to make the subsequent first proof of Theorem 5.9 clearer.

Let $p \in \mathbb{N}$. Let $x_{i_1}, \ldots, x_{i_p}$ be any $p$ letters in $\mathcal{X}$, and let $u_{j_1}, \ldots, u_{j_p}$ be $p$ distinct letters in $\mathcal{U}$.

**(a)** If no $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$, then

$$L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right).$$

**(b)** If some $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$, then this $g$ satisfies

$$\begin{aligned} & L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \\ & = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) + \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right). \end{aligned}$$

**(c)** If some $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$, and if some $h \in \{1, 2, \ldots, p\}$ satisfies $j_h = j$, then

$$L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right).$$

*Proof of Lemma 5.10.* The action of $L$ on $P$ satisfies the equality

$$L\left(x_i \mid u_{j-1}\right) = \left(x_i \mid u_{j-1}\right) + \left(x_i \mid u_j\right) \tag{17}$$

for all $i \in \{1, 2, \ldots, n\}$, as well as

$$L\left(x_i \mid u_k\right) = \left(x_i \mid u_k\right) \tag{18}$$

for all $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, d\}$ satisfying $k \neq j - 1$.

Applying the action of $L$ to the equality (1), we obtain

$$\begin{aligned} & L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \\ & = L \begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix} \\ & = \begin{vmatrix} L\left(x_{i_1} \mid u_{j_1}\right) & \cdots & L\left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ L\left(x_{i_p} \mid u_{j_1}\right) & \cdots & L\left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix} \end{aligned} \tag{19}$$

(since $L$ acts upon $P$ as an algebra homomorphism).

**(b)** Assume that some $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$. Consider this $g$.

The equality (17) rewrites as

$$L\left(x_i \mid u_{j_g}\right) = \left(x_i \mid u_{j_g}\right) + \left(x_i \mid u_j\right) \tag{20}$$

(since $j-1=j_g$). Furthermore, every $h\in\{1,2,\ldots,p\}$ satisfying $h\neq g$ satisfies

$$L\left(x_i \mid u_{j_h}\right)=\left(x_i \mid u_{j_h}\right) \tag{21}$$

[180].

Applying the equality (1) to $\left(j_1,\ldots,j_{g-1},j,j_{g+1},\ldots,j_p\right)$ instead of $\left(j_1,\ldots,j_p\right)$, we obtain

$$\begin{aligned}&\left(x_{i_1}\cdots x_{i_p} \mid u_{j_1}\cdots u_{j_{g-1}}u_j u_{j_{g+1}}\cdots u_{j_p}\right)\\&=\left|\begin{array}{ccccc}\left(x_{i_1}\mid u_{j_1}\right)&\cdots&\left(x_{i_1}\mid u_j\right)&\cdots&\left(x_{i_1}\mid u_{j_p}\right)\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \left(x_{i_p}\mid u_{j_1}\right)&\cdots&\left(x_{i_p}\mid u_j\right)&\cdots&\left(x_{i_p}\mid u_{j_p}\right)\end{array}\right|,\end{aligned} \tag{22}$$

where the matrix on the right hand side is to be understood as the $p\times p$-matrix whose $(s,t)$-th entry is $\begin{cases}\left(x_{i_s}\mid u_{j_t}\right), & \text{if } t\neq g;\\ \left(x_{i_s}\mid u_j\right), & \text{if } t=g\end{cases}$.

But the equalities (20) and (21) show that the matrix $\begin{bmatrix}L\left(x_{i_1}\mid u_{j_1}\right)&\cdots&L\left(x_{i_1}\mid u_{j_p}\right)\\ \vdots&\ddots&\vdots\\ L\left(x_{i_p}\mid u_{j_1}\right)&\cdots&L\left(x_{i_p}\mid u_{j_p}\right)\end{bmatrix}$ differs from the matrix $\begin{bmatrix}\left(x_{i_1}\mid u_{j_1}\right)&\cdots&\left(x_{i_1}\mid u_{j_p}\right)\\ \vdots&\ddots&\vdots\\ \left(x_{i_p}\mid u_{j_1}\right)&\cdots&\left(x_{i_p}\mid u_{j_p}\right)\end{bmatrix}$ only in its $g$-th column, which is $\begin{bmatrix}\left(x_{i_1}\mid u_{j_g}\right)+\left(x_{i_1}\mid u_j\right)\\ \vdots\\ \left(x_{i_p}\mid u_{j_g}\right)+\left(x_{i_p}\mid u_j\right)\end{bmatrix}$. Since the determinant of a matrix is multilinear in its columns (in particular, linear in its $g$-th column), we thus conclude that the determinant of the matrix $\begin{bmatrix}L\left(x_{i_1}\mid u_{j_1}\right)&\cdots&L\left(x_{i_1}\mid u_{j_p}\right)\\ \vdots&\ddots&\vdots\\ L\left(x_{i_p}\mid u_{j_1}\right)&\cdots&L\left(x_{i_p}\mid u_{j_p}\right)\end{bmatrix}$ can be expressed

[180]*Proof:* Let $h\in\{1,2,\ldots,p\}$ be such that $h\neq g$. The numbers $j_1,\ldots,j_p$ are distinct (since $u_{j_1},\ldots,u_{j_p}$ are distinct letters). Thus, from $h\neq g$, we obtain $j_h\neq j_g=j-1$. Hence, (18) (applied to $k=j_h$) shows that $L\left(x_i\mid u_{j_h}\right)=\left(x_i\mid u_{j_h}\right)$. This proves (21).

as follows:

$$\begin{vmatrix} L\left(x_{i_1} \mid u_{j_1}\right) & \cdots & L\left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ L\left(x_{i_p} \mid u_{j_1}\right) & \cdots & L\left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix}}_{\substack{=\left(x_{i_1}\cdots x_{i_p} \mid u_{j_1}\cdots u_{j_p}\right) \\ \text{(by (1))}}} + \underbrace{\begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix}}_{\substack{=\left(x_{i_1}\cdots x_{i_p} \mid u_{j_1}\cdots u_{j_{g-1}}u_j u_{j_{g+1}}\cdots u_{j_p}\right) \\ \text{(by (22))}}}$$

$$\left(\begin{array}{c}\text{where the second matrix is the same as the} \\ \text{matrix on the right hand side of (22)}\end{array}\right)$$

$$= \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) + \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right).$$

Hence, (19) becomes

$$\begin{aligned} & L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \\ & = \begin{vmatrix} L\left(x_{i_1} \mid u_{j_1}\right) & \cdots & L\left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ L\left(x_{i_p} \mid u_{j_1}\right) & \cdots & L\left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix} \\ & = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) + \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right). \end{aligned}$$

This proves Lemma 5.10 **(b)**.

**(c)** Assume that some $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$. Consider this $g$.

Assume that some $h \in \{1, 2, \ldots, p\}$ satisfies $j_h = j$. Consider this $h$.

We have $j_g = j - 1 \neq j = j_h$ and thus $g \neq h$. Hence, $h \neq g$; this shows that $j_h$ is one of the numbers $j_1, \ldots, j_{g-1}, j_{g+1}, \ldots, j_p$. Hence, two of the numbers $j_1, \ldots, j_{g-1}, j, j_{g+1}, \ldots, j_p$ are equal (namely, $j_h$ and $j$). Thus, two of the letters $u_{j_1}, \ldots, u_{j_{g-1}}, u_j, u_{j_{g+1}}, \ldots, u_{j_p}$ are equal. Therefore, the inner product

$$\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right) = 0$$

(since the inner product is antisymmetric in the $u$'s). Now, Lemma 5.10 **(b)** yields

$$\begin{aligned} & L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \\ & = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) + \underbrace{\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right)}_{=0} \\ & = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right). \end{aligned}$$

This proves Lemma 5.10 **(c)**.

**(a)** Assume that no $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$. Then, every $h \in \{1, 2, \ldots, p\}$ satisfies

$$L\left(x_i \mid u_{j_h}\right) = \left(x_i \mid u_{j_h}\right) \tag{23}$$

[181]. Thus,

$$\begin{bmatrix} L\left(x_{i_1} \mid u_{j_1}\right) & \cdots & L\left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ L\left(x_{i_p} \mid u_{j_1}\right) & \cdots & L\left(x_{i_p} \mid u_{j_p}\right) \end{bmatrix} = \begin{bmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{bmatrix}. \tag{24}$$

Now, (19) becomes

$$\begin{aligned} & L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \\ &= \begin{vmatrix} L\left(x_{i_1} \mid u_{j_1}\right) & \cdots & L\left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ L\left(x_{i_p} \mid u_{j_1}\right) & \cdots & L\left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix} \\ &= \begin{vmatrix} \left(x_{i_1} \mid u_{j_1}\right) & \cdots & \left(x_{i_1} \mid u_{j_p}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{i_p} \mid u_{j_1}\right) & \cdots & \left(x_{i_p} \mid u_{j_p}\right) \end{vmatrix} \qquad \text{(by (24))} \\ &= \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \qquad \text{(by (1))}. \end{aligned}$$

This proves Lemma 5.10 **(a)**. □

A further lemma will help us analyze bitableaux:[182]

**Lemma 5.11.** Let $T$ be a standard tableau with values in the alphabet $\mathcal{U}$. Let $j > 1$ be an integer. Let $g$ be a positive integer.

An entry $u_{j-1}$ in $T$ shall be called *malleable* if the letter $u_j$ does not appear in the same row of $T$ as this entry.

Let $\widetilde{T}$ be the tableau obtained from $T$ by replacing each malleable entry $u_{j-1}$ in the $g$-th column of $T$ by $u_j$.

Then, the tableau $\widetilde{T}$ is again standard.

[181] *Proof:* Let $h \in \{1, 2, \ldots, p\}$. Recall that no $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j - 1$. In other words, each $g \in \{1, 2, \ldots, p\}$ satisfies $j_g \neq j - 1$. Applying this to $g = h$, we obtain $j_h \neq j - 1$. Hence, (18) (applied to $k = j_h$) shows that $L\left(x_i \mid u_{j_h}\right) = \left(x_i \mid u_{j_h}\right)$. This proves (23).

[182] *Correction:* I have added this sentence, the next lemma and its proof, in order to make the subsequent first proof of Theorem 5.9 clearer.

*Proof of Lemma 5.11.* [183] Recall that the entries of $T$ and $\widetilde{T}$ belong to the totally ordered set $\mathcal{U}$.

The tableau $T$ is standard; in other words, the entries in each row of $T$ are increasing from left to right, and the entries in each column of $T$ are nondecreasing downward. We must prove the same for the tableau $\widetilde{T}$.

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the lengths of the rows of $T$ (from top to bottom), and let $\lambda_1', \lambda_2', \lambda_3', \ldots$ be the lengths of the columns of $T$ (from left to right). Thus, of course, $\lambda_1, \lambda_2, \lambda_3, \ldots$ are also the lengths of the rows of $\widetilde{T}$, whereas $\lambda_1', \lambda_2', \lambda_3', \ldots$ are also the lengths of the columns of $\widetilde{T}$.

Recall that the tableau $\widetilde{T}$ is obtained from $T$ by replacing each malleable entry $u_{j-1}$ in the $g$-th column of $T$ by $u_j$. Thus, each entry of $\widetilde{T}$ either equals the corresponding entry of $T$, or is larger than it (because the only entries that were replaced were entries $u_{j-1}$, and these were replaced by the larger letter $u_j$). In other words, each cell $(p, q)$ of $T$ satisfies

$$\widetilde{T}(p,q) \geq T(p,q). \tag{25}$$

Recall again that the tableau $\widetilde{T}$ is obtained from $T$ by replacing each malleable entry $u_{j-1}$ in the $g$-th column of $T$ by $u_j$. Thus, each malleable entry $u_{j-1}$ in the $g$-th column of $T$ was replaced by $u_j$. In other words, if some cell $(p, q)$ in the $g$-th column of $T$ contains a malleable entry $u_{j-1}$, then the corresponding entry of $\widetilde{T}$ is $u_j$. In other words, if some cell $(p, q)$ in the $g$-th column of $T$ contains a malleable entry $u_{j-1}$, then

$$\widetilde{T}(p,q) = u_j. \tag{26}$$

(Of course, under this condition, we must have $q = g$; but we nevertheless chose to call the cell $(p, q)$ and not $(p, g)$ just for the sake of uniform notations.)

Recall again that the tableau $\widetilde{T}$ is obtained from $T$ by replacing each malleable entry $u_{j-1}$ in the $g$-th column of $T$ by $u_j$. Thus, each entry of $T$ has been preserved unchanged in $\widetilde{T}$ unless it was a malleable entry $u_{j-1}$ in the $g$-th column of $T$. In other words, if $(p, q)$ is a cell of $T$, then

$$\widetilde{T}(p,q) = T(p,q), \tag{27}$$

unless the entry of $T$ in cell $(p, q)$ is a malleable entry $u_{j-1}$ in the $g$-th column of $T$.

Let us make the following simple observation:

*Observation 0:* Let $(p, q)$ be a cell of $T$ satisfying $T(p,q) \neq \widetilde{T}(p,q)$. Then:

- We have $q = g$ and $T(p,q) = u_{j-1}$ and $\widetilde{T}(p,q) = u_j$.
- Furthermore, the letter $u_j$ does not appear in the $p$-th row of $T$.

[183] This proof is here purely for the sake of completeness. It is a long-winded writeup of a fairly straightforward combinatorial argument, which you will probably have found faster than you can read my proof.

[*Proof of Observation 0:* We have $T(p,q) \neq \widetilde{T}(p,q)$. In other words, $\widetilde{T}(p,q) \neq T(p,q)$.

If the entry of $T$ in cell $(p,q)$ was not a malleable entry $u_{j-1}$ in the $g$-th column of $T$, then we would have $\widetilde{T}(p,q) = T(p,q)$ (by (27)); but this would contradict $\widetilde{T}(p,q) \neq T(p,q)$. Hence, the entry of $T$ in cell $(p,q)$ is a malleable entry $u_{j-1}$ in the $g$-th column of $T$.

Thus, in particular, the entry of $T$ in cell $(p,q)$ lies in the $g$-th column of $T$. In other words, cell $(p,q)$ lies in the $g$-th column. In other words, $q = g$.

Moreover, the entry of $T$ in cell $(p,q)$ is a malleable entry $u_{j-1}$ in the $g$-th column of $T$. In particular, this entry is $u_{j-1}$. In other words, $T(p,q) = u_{j-1}$.

The cell $(p,q)$ in the $g$-th column of $T$ contains a malleable entry $u_{j-1}$ (since the entry of $T$ in cell $(p,q)$ is a malleable entry $u_{j-1}$ in the $g$-th column of $T$). Thus, (26) shows that $\widetilde{T}(p,q) = u_j$.

Recall that the entry of $T$ in cell $(p,q)$ is a malleable entry $u_{j-1}$ in the $g$-th column of $T$. In particular, this entry is malleable. In other words, the letter $u_j$ does not appear in the same row of $T$ as this entry (by the definition of "malleable"). In other words, the letter $u_j$ does not appear in the $p$-th row of $T$ (since clearly, the row of $T$ that contains the entry in cell $(p,q)$ must be the $p$-th row of $T$). Thus, the proof of Observation 0 is complete.]

We know that the entries of each row of $T$ are increasing from left to right. In other words, each positive integer $p$ satisfies

$$T(p,1) < T(p,2) < \cdots < T(p,\lambda_p). \tag{28}$$

We know that the entries of each column of $T$ are nondecreasing downward. In other words, each positive integer $q$ satisfies

$$T(1,q) \leq T(2,q) \leq \cdots \leq T\left(\lambda'_q, q\right). \tag{29}$$

We now make the following two claims:

*Claim 1:* Each positive integer $p$ satisfies

$$\widetilde{T}(p,1) < \widetilde{T}(p,2) < \cdots < \widetilde{T}(p,\lambda_p).$$

*Claim 2:* Each positive integer $q$ satisfies

$$\widetilde{T}(1,q) \leq \widetilde{T}(2,q) \leq \cdots \leq \widetilde{T}\left(\lambda'_q, q\right).$$

[*Proof of Claim 1:* Let $p$ be a positive integer. Let $q \in \{1,2,\ldots,\lambda_p - 1\}$. We are going to show that $\widetilde{T}(p,q) < \widetilde{T}(p,q+1)$.

Indeed, assume the contrary. Thus, $\widetilde{T}(p,q) \geq \widetilde{T}(p,q+1)$.

However, $(p,q+1)$ is also a cell of $T$ (since $q \in \{1,2,\ldots,\lambda_p - 1\}$). Thus, (25) (applied to $(p,q+1)$ instead of $(p,q)$) yields $\widetilde{T}(p,q+1) \geq T(p,q+1)$. Hence, $\widetilde{T}(p,q) \geq \widetilde{T}(p,q+1) \geq T(p,q+1)$.

But (28) shows that $T(p,q) < T(p,q+1) \leq \widetilde{T}(p,q)$ (since $\widetilde{T}(p,q) \geq T(p,q+1)$). Hence, $T(p,q) \neq \widetilde{T}(p,q)$. Thus, Observation 0 shows that:

- We have $q = g$ and $T(p,q) = u_{j-1}$ and $\widetilde{T}(p,q) = u_j$.
- Furthermore, the letter $u_j$ does not appear in the $p$-th row of $T$.

From $T(p,q) = u_{j-1}$, we obtain $u_{j-1} = T(p,q) < T(p,q+1)$. Therefore, $T(p,q+1) > u_{j-1}$, so that $T(p,q+1) = u_h$ for some $h \in \{j, j+1, \ldots, d\}$. Consider this $h$. We have $h \geq j$ (since $h \in \{j, j+1, \ldots, d\}$). But we have $\widetilde{T}(p,q) = u_j$. Hence, $u_j = \widetilde{T}(p,q) \geq T(p,q+1) = u_h$. In other words, $j \geq h$. Combined with $h \geq j$, this yields $h = j$. Hence, $u_h = u_j$. Thus, $T(p,q+1) = u_h = u_j$. Thus, the letter $u_j$ appears in the $p$-th row of $T$ (namely, in cell $(p,q+1)$). This contradicts the fact that the letter $u_j$ does not appear in the $p$-th row of $T$.

This contradiction shows that our assumption was false. Hence, $\widetilde{T}(p,q) < \widetilde{T}(p,q+1)$ is proven.

Now, forget that we fixed $q$. We thus have shown that $\widetilde{T}(p,q) < \widetilde{T}(p,q+1)$ for each $q \in \{1, 2, \ldots, \lambda_p - 1\}$. In other words, $\widetilde{T}(p,1) < \widetilde{T}(p,2) < \cdots < \widetilde{T}(p, \lambda_p)$. This proves Claim 1.]

[*Proof of Claim 2:* Let $q$ be a positive integer. Let $p \in \left\{1, 2, \ldots, \lambda'_q - 1\right\}$. We are going to show that $\widetilde{T}(p,q) \leq \widetilde{T}(p+1,q)$.

Indeed, assume the contrary. Thus, $\widetilde{T}(p,q) > \widetilde{T}(p+1,q)$.

However, $(p+1,q)$ is also a cell of $T$ (since $p \in \left\{1, 2, \ldots, \lambda'_q - 1\right\}$). Thus, (25) (applied to $(p+1,q)$ instead of $(p,q)$) yields $\widetilde{T}(p+1,q) \geq T(p+1,q)$. Hence, $\widetilde{T}(p,q) > \widetilde{T}(p+1,q) \geq T(p+1,q)$.

But (29) shows that $T(p,q) \leq T(p+1,q) < \widetilde{T}(p,q)$ (since $\widetilde{T}(p,q) > T(p+1,q)$). Hence, $T(p,q) \neq \widetilde{T}(p,q)$. Thus, Observation 0 shows that:

- We have $q = g$ and $T(p,q) = u_{j-1}$ and $\widetilde{T}(p,q) = u_j$.
- Furthermore, the letter $u_j$ does not appear in the $p$-th row of $T$.

From $T(p,q) = u_{j-1}$, we obtain $u_{j-1} = T(p,q) \leq T(p+1,q)$. Therefore, $T(p+1,q) \geq u_{j-1}$, so that $T(p+1,q) = u_h$ for some $h \in \{j-1, j, \ldots, d\}$. Consider this $h$. We have $h \geq j-1$ (since $h \in \{j-1, j, \ldots, d\}$). But $u_h = T(p+1,q) < \widetilde{T}(p,q) = u_j$. In other words, $h < j$. Hence, $h \leq j-1$. Combining this with $h \geq j-1$, we obtain $h = j-1$. Now, $T(p+1,q) = u_h = u_{j-1}$ (since $h = j-1$). Thus, there is an entry $u_{j-1}$ in cell $(p+1,q)$ of $T$.

The letter $u_j$ does not appear in the $(p+1)$-st row of $T$ [184]. In other words, the letter $u_j$ does not appear in the same row of $T$ as the entry of $T$ in cell $(p+1,q)$.

[184] *Proof.* Assume the contrary. Thus, the letter $u_j$ appears in the $(p+1)$-st row of $T$. In other words, there exists an $r \in \{1, 2, \ldots, \lambda_{p+1}\}$ such that $T(p+1,r) = u_j$. Consider this $r$.

Applying (28) to $p+1$ instead of $p$, we obtain $T(p+1,1) < T(p+1,2) < \cdots <$

Hence, the entry $u_{j-1}$ in cell $(p+1,q)$ of $T$ has the property that the letter $u_j$ does not appear in the same row of $T$ as this entry. In other words, this entry $u_{j-1}$ is malleable (by the definition of "malleable").

Also, $(p+1,q)$ is a cell in the $g$-th column of $T$ (since $q=g$). Hence, $(p+1,q)$ is a cell in the $g$-th column of $T$ containing a malleable entry $u_{j-1}$ (since the entry $u_{j-1}$ in cell $(p+1,q)$ of $T$ is malleable). Thus, (26) (applied to $(p+1,q)$ instead of $(p,q)$) yields $\widetilde{T}(p+1,q)=u_j$. Now, recall that $\widetilde{T}(p,q)=u_j$; hence, $u_j=\widetilde{T}(p,q)>\widetilde{T}(p+1,q)=u_j$. This is clearly absurd.

This contradiction shows that our assumption was false. Hence, $\widetilde{T}(p,q)\leq\widetilde{T}(p+1,q)$ is proven.

Now, forget that we fixed $p$. We thus have shown that $\widetilde{T}(p,q)\leq\widetilde{T}(p+1,q)$ for each $p\in\left\{1,2,\ldots,\lambda'_q-1\right\}$. In other words, $\widetilde{T}(1,q)\leq\widetilde{T}(2,q)\leq\cdots\leq\widetilde{T}\left(\lambda'_q,q\right)$. This proves Claim 2.]

Now, consider the tableau $\widetilde{T}$. The entries in each row of $\widetilde{T}$ are increasing from left to right (by Claim 1), and the entries in each column of $\widetilde{T}$ are nondecreasing downward (by Claim 2). In other words, the tableau $\widetilde{T}$ is standard. This proves Lemma 5.11. $\square$

We shall say that a standard tableau with entries from the alphabet $\mathcal{U}$ is *rectangular* if and only if it has shape $(d,\ldots,d)$ (for some number of entries $d$, possibly zero). Notice that such a rectangular tableau must necessarily look like this:

$$\begin{matrix} u_1 & u_2 & \cdots & u_d \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_d \end{matrix}$$

(because each of its rows has length $d$ and is strictly increasing, whence it has the form $u_1u_2\cdots u_d$). Notice also that a bideterminant $(T\mid T')$ is rectangular if and only if the tableau $T'$ is rectangular.[185]

$T(p+1,\lambda_{p+1})$. Hence, if we had $r\leq q$, then we would have $T(p+1,r)\leq T(p+1,q)=u_{j-1}$, which would contradict $T(p+1,r)=u_j>u_{j-1}$. Thus, we cannot have $r\leq q$. Hence, we have $r>q$. In other words, $q<r$.

Thus, the cell $(p,r)$ lies "between" the cells $(p,q)$ and $(p+1,r)$ (in the sense that there is a shortest northwest-to-southeast lattice path from $(p,q)$ to $(p+1,r)$ passing through $(p,r)$). Thus, the tableau $T$ must have a cell $(p,r)$ (since $T$ has a cell $(p,q)$ and a cell $(p+1,r)$). Now, from (28), we obtain $T(p,q)<T(p,r)$ (since $q<r$). Hence, $T(p,r)>T(p,q)=u_{j-1}$.

But (29) (applied to $r$ instead of $q$) yields $T(1,r)\leq T(2,r)\leq\cdots\leq T(\lambda'_r,r)$. Hence, $T(p,r)\leq T(p+1,r)=u_j$. In other words, $T(p,r)=u_i$ for some $i\in\{1,2,\ldots,j\}$. Consider this $i$.

But $u_i=T(p,r)>u_{j-1}$, so that $i>j-1$. In other words, $i\geq j$. Combining this with $i\leq j$ (since $i\in\{1,2,\ldots,j\}$), we obtain $i=j$. Hence, $T(p,r)=u_i=u_j$ (since $i=j$). Hence, the letter $u_j$ appears in the $p$-th row of $T$ (namely, in cell $(p,r)$). This contradicts the fact that the letter $u_j$ does not appear in the $p$-th row of $T$. This contradiction shows that our assumption was false, qed.

[185] *Correction:* I have added this paragraph in order to clarify what the word "rectangular" means when it is applied to a single tableau (as it is in the following proof).

*First proof of Theorem 5.9.* Suppose $F$ is invariant. WLOG assume that $F \neq 0$ (otherwise, everything is clear).[186] Using the straightening formula, we can express $F$ as a linear combination of standard bideterminants:

$$F = \sum_s \alpha_s \left(T_s \mid T'_s\right), \tag{30}$$

where the $[T_s, T'_s]$ are distinct standard bitableaux and where the $\alpha_s$ are nonzero scalars for all $s$ [187]. Our goal is to prove that all tableaux $T'_s$ appearing in (30) are rectangular.[188]

Assume the contrary.[189] We shall probe $F$ with appropriate $d \times d$-matrices.

Consider first the $d \times d$-matrix $L \in K^{d \times d}$ defined by

$$L\left(x_i \mid u_j\right) = c\left(x_i \mid u_j\right), \tag{31}$$
$$L\left(x_i \mid u_k\right) = \left(x_i \mid u_k\right), \qquad \text{for } k \neq j, \tag{32}$$

where $j$ is a fixed element of $\{1, 2, \ldots, d\}$ and[190] where $c$ is a nonzero scalar[191]. If $b_j^{(s)}$ is the number of occurrences of $u_j$ in $T'_s$, then

$$L\left(T_s \mid T'_s\right) = c^{b_j^{(s)}}\left(T_s \mid T'_s\right)$$

and therefore[192]

$$LF = \sum_s \alpha_s c^{b_j^{(s)}}\left(T_s \mid T'_s\right). \tag{33}$$

As $F$ is invariant, we also have

$$LF = a\left(L\right) F = \sum_s \alpha_s a\left(L\right)\left(T_s \mid T'_s\right). \tag{34}$$

But the expansion into standard bideterminants is unique. Hence, comparing (33) with (34), we obtain[193]

$$\alpha_s c^{b_j^{(s)}} = \alpha_s a\left(L\right) \qquad \text{for all } s, \tag{35}$$

[186] *Correction:* Added this sentence.

[187] *Correction:* Added "where the $[T_s, T'_s]$ are distinct standard bitableaux and where the $\alpha_s$ are nonzero scalars for all $s$" (this later becomes important).

[188] *Correction:* Added this sequence.

[189] *Correction:* Added this sequence.

[190] *Correction:* Added "where $j$ is a fixed element of $\{1, 2, \ldots, d\}$ and".

[191] *Comment:* Here is a more precise way to say this: Fix a $j \in \{1, 2, \ldots, d\}$ and a nonzero scalar $c$. Let $L$ be the $d \times d$-matrix $\left(l_{qp}\right)_{1 \leq p,q \leq d} \in K^{d \times d}$, where

$$\begin{aligned} l_{jj} &= c; \\ l_{kk} &= 1 \qquad \text{for } k \neq j; \\ l_{qp} &= 0 \qquad \text{for } p \neq q. \end{aligned}$$

Clearly, this $d \times d$-matrix $L$ is invertible (since it is a diagonal matrix with nonzero entries on the diagonal). It is now straightforward to see that this $L$ satisfies (31) and (32).

[192] *Correction:* Added the word "therefore".

[193] *Correction:* Added this sequence for clarity.

and therefore

$$c^{b_j^{(s)}} = a(L) \qquad \text{for all } s \tag{36}$$

(since the $\alpha_s$ are nonzero). Hence, we must have, for all $s$ and $t$,

$$c^{b_j^{(s)}} = c^{b_j^{(t)}} = a(L).$$

This equality holds for all the nonzero[194] scalars $c$ in the infinite field $K$ [195]. Therefore,

$$b_j^{(s)} = b_j^{(t)} \qquad \text{for all } s \text{ and } t \tag{37}$$

[196]. We shall write $b_j$ for the common value of the integers $b_j^{(s)}$. Hence, each tableau $T_s'$ contains the letter $u_j$ exactly $b_j$ times.[197]

Now, fix two distinct elements $j$ and $k$ of $\{1, 2, \ldots, d\}$, and[198] let $L$ be the $d \times d$-matrix in $K^{d \times d}$ defined by

$$L(x_i \mid u_j) = (x_i \mid u_k), \tag{38}$$

$$L(x_i \mid u_k) = (x_i \mid u_j), \tag{39}$$

$$L(x_i \mid u_p) = (x_i \mid u_p), \qquad \text{for } p \neq j \text{ and } p \neq k \tag{40}$$

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[194] *Correction:* Added the word "nonzero".

[195] *Correction:* Replaced "$k$" by "$K$".

[196] *Comment:* Here is this argument in more detail:

Comparing (33) with (34), we obtain

$$\sum_s \alpha_s c^{b_j^{(s)}} (T_s \mid T_s') = \sum_s \alpha_s a(L) (T_s \mid T_s').$$

Since the standard bideterminants $(T_s \mid T_s')$ are $K$-linearly independent (by Theorem 4.6, since the $[T_s, T_s']$ are distinct standard bitableaux), this shows that

$$\alpha_s c^{b_j^{(s)}} = \alpha_s a(L) \qquad \text{for each } s.$$

We can divide this equality by $\alpha_s$ (since $\alpha_s$ is nonzero), and thus obtain

$$c^{b_j^{(s)}} = a(L) \qquad \text{for each } s.$$

Hence, $\underbrace{c^{b_j^{(s)}}}_{=a(L)} - \underbrace{c^{b_j^{(t)}}}_{=a(L)} = a(L) - a(L) = 0$ for any $s$ and $t$.

Now, let $s$ and $t$ be arbitrary indices. Then, we have just shown that $c^{b_j^{(s)}} - c^{b_j^{(t)}} = 0$. We have proven this for every nonzero scalar $c \in K$. Thus, $c^{b_j^{(s)}} - c^{b_j^{(t)}} = 0$ holds for infinitely many values of $c \in K$ (since there are infinitely many nonzero scalars $c \in K$).

Now, consider the polynomial $T^{b_j^{(s)}} - T^{b_j^{(t)}} \in K[T]$. This polynomial has infinitely many roots in $K$ (since $c^{b_j^{(s)}} - c^{b_j^{(t)}} = 0$ holds for infinitely many values of $c \in K$), and thus must be identically zero. Therefore, $b_j^{(s)} = b_j^{(t)}$, qed.

[197] *Correction:* Added this sentence.

[198] *Correction:* Added "fix two distinct elements $j$ and $k$ of $\{1, 2, \ldots, d\}$, and".

[199]. Each of the bideterminants $L(T_s \mid T'_s)$ contains[200] the letter $u_k$ exactly[201] $b_j$ times. As the content of a bideterminant is preserved under straightening[202], $LF$ is therefore[203] a linear combination of standard bideterminants each containing the letter $u_k$ exactly[204] $b_j$ times. But $F$ is invariant, and therefore[205] $LF = \sum_s \alpha_s a(L)(T_s \mid T'_s)$, where each of the bideterminants in this expansion contains[206] the letter $u_k$ exactly[207] $b_k$ times. Applying the basis theorem[208], we conclude that $b_j = b_k$ [209]. We shall denote by $b$ the common value of the integers $b_j$. Thus,

$$\text{each tableau } T'_s \text{ contains the letter } u_j \text{ exactly } b \text{ times} \tag{41}$$

[210].

Since each letter $u_j$ is repeated $b$ times, the minimum number of rows in $T'_s$ is $b$. The number of rows is exactly $b$ if and only if $T'_s$ is rectangular[211].

Now, suppose that $T'_s$ is not rectangular; that is, the number of rows in $T'_s$ is strictly greater than $b$. In $T'_s$, all the letters $u_1$ are in the first column. Let $u_l$ be the first letter in the first column following the run of letters $u_1$ [212]. Then, all

[199] *Comment:* Here is a more precise way to say this: Fix two distinct elements $j$ and $k$ of $\{1, 2, \ldots, d\}$. Let $L$ be the $d \times d$-matrix $(l_{qp})_{1 \leq p,q \leq d} \in K^{d \times d}$, where

$$\begin{aligned}
l_{jk} &= 1; \\
l_{jp} &= 0 && \text{for } p \neq k; \\
l_{kj} &= 1; \\
l_{kp} &= 0 && \text{for } p \neq j; \\
l_{pp} &= 1 && \text{for } p \neq j \text{ and } p \neq k; \\
l_{qp} &= 0 && \text{for } q \notin \{j, k\} \text{ and } p \neq q.
\end{aligned}$$

Clearly, this matrix $L$ is invertible (since it is a permutation matrix). It is now straightforward to see that this $L$ satisfies (38), (39) and (40).

[200] *Correction:* Replaced "contain" by "contains".

[201] *Correction:* Added the word "exactly".

[202] *Comment:* This is just a reference to Theorem 3.2.

[203] *Correction:* Added the word "therefore".

[204] *Correction:* Added the word "exactly".

[205] *Correction:* Added the word "therefore".

[206] *Correction:* Replaced "contain" by "contains".

[207] *Correction:* Added the word "exactly".

[208] *Comment:* i.e., Corollary 4.8

[209] *Comment:* Here, we are using the fact that $a(L) \neq 0$. Why is this the case?

Indeed, assume the contrary. Thus, $a(L) = 0$. But since $F$ is invariant, we have $LF = a(L)F$ and $L^{-1}F = a(L^{-1})F$. Hence, $L^{-1} \underbrace{LF}_{=a(L)F} = \underbrace{a(L)}_{=0} L^{-1}F = 0$, which contradicts $L^{-1}LF = F \neq 0$.

This contradiction shows that our assumption was false; hence, $a(L) \neq 0$.

[210] *Correction:* Added this sentence.

[211] *Comment:* Recall that a standard tableau $T'_s$ is rectangular if and only if each of its rows has the form $u_1 u_2 \cdots u_d$. (Thus, if $T'_s$ is rectangular, then the bideterminant $(T_s \mid T'_s)$ is rectangular.)

[212] *Comment:* Such a letter exists, because the run of letters $u_1$ cannot cover the whole first column

the letters from $u_1$ to $u_{l-1}$ occur in the first $b$ rows. Hence, each of the letters from $u_1$ to $u_{l-1}$ occurs in each of the first $b$ rows (since it must occur $b$ times in total, but can only occur in the first $b$ rows, and therefore has to occur at least once in each of these $b$ rows). Thus, each of the first $b$ rows begins with $u_1 u_2 \cdots u_{l-1}$ [213]. Moreover, if $q$ is the number of occurrences of $u_l$ in the first column, then[214] the remaining $b - q$ letters $u_l$ must all occur in the first $b - q$ rows[215]. The situation is

(since the number of rows is greater than $b$, but the letter $u_1$ appears only $b$ times).

[213] *Correction:* Added this and the preceding sentence.

[214] *Correction:* Added "then".

[215] *Comment:* Let me justify this:

We must prove that each occurrence of $u_l$ in $T'_s$ other than in the first column must be in one of the first $b - q$ rows. In other words, we must prove that if the $(r, c)$-th entry of $T'_s$ equals $u_l$ for some integers $r \geq 1$ and $s \geq 2$, then $r \leq b - q$.

So let us assume that the $(r, c)$-th entry of $T'_s$ equals $u_l$ for some integers $r \geq 1$ and $c > 1$. We must prove that $r \leq b - q$.

We are in one of the following two cases:

*Case 1:* We have $r \leq b$.

*Case 2:* We have $r > b$.

Let us first consider Case 1. In this case, we have $r \leq b$. Thus, the $r$-th row of $T'_s$ begins with $u_1 u_2 \cdots u_{l-1}$ (since each of the first $b$ rows begins with $u_1 u_2 \cdots u_{l-1}$). The next entry of this $r$-th row after $u_1 u_2 \cdots u_{l-1}$ must then be $u_l$ (since $u_l$ occurs in the $r$-th row of $T'_s$ (since the $(r, c)$-th entry of $T'_s$ equals $u_l$)). Hence, the $(r, l)$-th entry of $T'_s$ is $u_l$. Therefore, the first $r$ entries of the $l$-th column of $T'_s$ must be of the form $u_g$ with $g \leq l$ (since $T'_s$ is standard). However, these entries must also be of the form $u_g$ with $g \geq l$ (since any entry in the $l$-th column of $T'_s$ has this form (again since $T'_s$ is standard)). Hence, these entries must be of the form $u_g$ with $g$ satisfying both $g \leq l$ and $g \geq l$ simultaneously. In other words, these entries must be of the form $u_l$. Hence, there are at least $r$ entries $u_l$ in the $l$-th column of $T'_s$. Since there are also $q$ entries $u_l$ in the 1-st column of $T'_s$, we conclude that there are at least $r + q$ entries $u_l$ in $T'_s$ (since the $l$-th column of $T'_s$ and the 1-st column of $T'_s$ are two different columns). In other words, $b \geq r + q$ (since the total number of entries $u_l$ in $T'_s$ is $b$). Hence, $r \leq b - q$. Thus, we have proven $r \leq b - q$ in Case 1.

Now, let us consider Case 2. In this case, we have $r > b$. Hence, the first entry of the $r$-th row of $T'_s$ has the form $u_g$ with $g \geq l$. Consider this $g$.

Recall that $c > 1$. Hence, the $c$-th entry of the $r$-th row of $T'_s$ is larger than the first entry of the $r$-th row of $T'_s$. In other words, the $c$-th entry of the $r$-th row of $T'_s$ has the form $u_h$ for some $h > g$ (since the first entry of the $r$-th row of $T'_s$ is $u_g$). Consider this $h$. Thus,

$$u_h = \left(\text{the } c\text{-th entry of the } r\text{-th row of } T'_s\right) = \left(\text{the } (r, c)\text{-th entry of } T'_s\right) = u_l$$

(since the $(r, c)$-th entry of $T'_s$ equals $u_l$). Hence, $h = l \leq g$ (since $g \geq l$). This contradicts $h > g$. Thus, we have obtained a contradiction in Case 2. Hence, Case 2 cannot occur. Thus, $r \leq b - q$ is proven (since we have proven $r \leq b - q$ in Case 1). This completes the proof.

summarized by:[216]

$$
T'_s = \begin{array}{r}
b\text{ rows}\left\{\begin{array}{r}
\begin{array}{ccccc} u_1 & \cdots & u_{l-1} & u_l & \cdots \\ \vdots & \ddots & \vdots & \vdots & \cdots \\ u_1 & \cdots & u_{l-1} & u_l & \cdots \end{array} \\
q\text{ rows}\left\{\begin{array}{ccccc} u_1 & \cdots & u_{l-1} & u_m & \cdots \\ \vdots & \ddots & \vdots & {\cdot}^{\cdot^{\cdot}} & \\ u_1 & \cdots & u_{l-1} & & \end{array}\right.
\end{array}\right. \\
q\text{ rows}\left\{\begin{array}{lll} u_l & \cdots & \cdots \\ \vdots & \ddots & {\cdot}^{\cdot^{\cdot}} \\ u_l & \cdots & \end{array}\right. \\
\vdots
\end{array} \tag{42}
$$

where $m \geq l+1$. Such a tableau is called an *l-critical tableau*, and its *parameter* is $q$.

Let $j$ be the smallest index such that there exists a $j$-critical tableau in the expansion (30)[217] of $F$. We break up the expansion (30)[218] of $F$ into

$$F = \sum_s \alpha_s \left(T_s \mid T'_s\right) + \sum_t \alpha_t \left(T_t \mid T'_t\right) + G, \tag{43}$$

where the first summation is over all the indices $s$ such that $T'_s$ is $j$-critical, the second summation is over all $t$ such that $T'_t$ is $l$-critical[219] for some $l > j$ [220], and $G$ is the linear combination of all the $b \times d$ rectangular standard bideterminants.

Now, fix any $j \in \{2, 3, \ldots, d\}$, and[221] let $L$ be the $d \times d$-matrix in $K^{d \times d}$ defined by

$$L\left(x_i \mid u_{j-1}\right) = \left(x_i \mid u_{j-1}\right) + \left(x_i \mid u_j\right), \tag{44}$$

$$L\left(x_i \mid u_k\right) = \left(x_i \mid u_k\right), \qquad \text{for } k \neq j-1 \tag{45}$$

[222]. Under $L$, those bideterminants in which all the letters $u_{j-1}$ and $u_j$ occur in the first $b$ rows are unchanged[223]; in particular, the rectangular bideterminants

[216] *Correction:* Replaced "$T'$" by "$T'_s$" in the display. As usual, added some "$\cdots$"s and the words "rows".

[217] *Correction:* Added "(30)".

[218] *Correction:* Added "(30)".

[219] *Correction:* Removed a redundant comma here.

[220] *Correction:* Replaced "$l > q$" by "$l > j$".

[221] *Correction:* Added "Now, fix any $j \in \{2, 3, \ldots, d\}$, and".

[222] *Comment:* Here is a more precise way to say this: Fix any $j \in \{2, 3, \ldots, d\}$. Let $L$ be the $d \times d$-matrix $\left(l_{qp}\right)_{1 \leq p,q \leq d} \in K^{d \times d}$, where

$$\begin{aligned} l_{j-1,j} &= 1; & \\ l_{pp} &= 1 & \text{for all } p; \\ l_{qp} &= 0 & \text{for all } p, q \text{ satisfying } p \neq q \text{ and } (p,q) \neq (j, j-1). \end{aligned}$$

Clearly, this matrix $L$ is invertible (since it is lower-unitriangular). It is now straightforward to see that this $L$ satisfies (44) and (45).

[223] *Comment:* This is because

in $G$ and the $l$-critical bideterminants in the second summation in (43)[224] remain unaltered. In other words,[225]

$$LG = G, \tag{46}$$

and

$$L\left(T_t \mid T_t'\right) = \left(T_t \mid T_t'\right) \qquad \text{for each } t \text{ in the second summation in (43).} \tag{47}$$

For the $j$-critical bideterminants,[226]

$$L\left(T_s \mid T_s'\right) = \sum_r \left(T_s \mid T_s^r\right), \tag{48}$$

where $T_s^r$ is a tableau (not necessarily standard)[227] of the following form:[228]

$$\begin{array}{l}
b\text{ rows}\left\{\begin{array}{l}
\begin{array}{ccccc} u_1 & \cdots & u_{j-1} & u_j & \cdots \\ \vdots & \ddots & \vdots & \vdots & \cdots \\ u_1 & \cdots & u_{j-1} & u_j & \cdots \end{array} \\
q\text{ rows}\left\{\begin{array}{ccccc} u_1 & \cdots & * & u_m & \cdots \\ \vdots & \ddots & \vdots & \iddots & \\ u_1 & \cdots & * & & \end{array}\right.
\end{array}\right. \\
q\text{ rows}\left\{\begin{array}{ccc} u_j & \cdots & \cdots \\ \vdots & \ddots & \iddots \\ u_j & \cdots & \end{array}\right. \\
\vdots
\end{array} \tag{49}$$

where any of the $*$ symbols in column $j-1$ [229] may be $u_{j-1}$ or $u_j$, and where[230] $m \geq j+1$. [231]

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- any inner product containing both letters $u_{j-1}$ and $u_j$ is unchanged under $L$ (by Lemma 5.10 **(c)**);
- any inner product that does not contain the letter $u_j$ is unchanged under $L$ (by Lemma 5.10 **(a)**);
- $L$ acts upon $P$ as an algebra homomorphism.

(Here, we are only talking about inner products in which the indices of the $u$ are distinct. But all the inner products forming a bideterminant have this property.)

[224] *Correction:* Added "in (43)".

[225] *Correction:* Inserted this sentence.

[226] *Correction:* In the following display, I have removed the "$(T_s \mid T_s')$" addend, because I prefer to regard this addend as part of the sum $\sum_r (T_s \mid T_s^r)$.

[227] *Correction:* Added "(not necessarily standard)".

[228] *Correction:* As usual, added some "$\cdots$"s and the words "rows" in the display below.

[229] *Correction:* Replaced "$*$" by "any of the $*$ symbols in column $j-1$".

[230] *Correction:* Added "where".

[231] *Comment:* This follows from the following argument:

Fix any $s$.

Let $w$ be the number of rows of the tableau $T_s$; thus, $w$ is also the number of rows of the tableau $T_s'$.

The tableau $T_s'$ is $j$-critical; thus, it looks as follows:

$$T_s' = \begin{array}{l} b\text{ rows}\left\{\begin{array}{l} \begin{array}{ccccc} u_1 & \cdots & u_{j-1} & u_j & \cdots \\ \vdots & \ddots & \vdots & \vdots & \cdots \\ u_1 & \cdots & u_{j-1} & u_j & \cdots \end{array} \\ q\text{ rows}\left\{\begin{array}{ccccc} u_1 & \cdots & u_{j-1} & u_m & \cdots \\ \vdots & \ddots & \vdots & \iddots & \\ u_1 & \cdots & u_{j-1} & & \end{array}\right. \end{array}\right. \\ \qquad q\text{ rows}\left\{\begin{array}{ccc} u_j & \cdots & \cdots \\ \vdots & \ddots & \iddots \\ u_j & \cdots & \end{array}\right. \\ \qquad\qquad \vdots \end{array} . \tag{50}$$

In particular, all entries of $T_s'$ in row $b+1$ and the rows below have the form $u_h$ for $h \geq j$ (because the first entry of row $b+1$ is $u_j$, and the entries are nondecreasing along rows and column).

For each $k \in \{1,2,\ldots,w\}$, we set $r_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$, where $x_{i_1} \cdots x_{i_p}$ is the $k$-th row of the tableau $T_s$, and where $u_{j_1} \cdots u_{j_p}$ is the $k$-th row of the tableau $T_s'$. Thus, $(T_s \mid T_s') = r_1 r_2 \cdots r_w$. Hence,

$$L\left(T_s \mid T_s'\right) = L\left(r_1 r_2 \cdots r_w\right) = \left(Lr_1\right)\left(Lr_2\right)\cdots\left(Lr_w\right) \tag{51}$$

(since $L$ acts upon $P$ as an algebra homomorphism).

Fix $k \in \{1,2,\ldots,w\}$. We shall now compute the value $Lr_k$. First, let $x_{i_1} \cdots x_{i_p}$ be the $k$-th row of the tableau $T_s$, and let $u_{j_1} \cdots u_{j_p}$ be the $k$-th row of the tableau $T_s'$. Thus, $r_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$ (by the definition of $r_k$). Notice that $j_1 < j_2 < \cdots < j_p$ (since $T_s'$ is a standard tableau); thus, the letters $u_{j_1}, \ldots, u_{j_p}$ are distinct. Now, let us compute $Lr_k$ depending on the value of $k$:

- Let us first assume that $k \leq b-q$. Thus, the $k$-th row of $T_s'$ contains both $u_{j-1}$ and $u_j$ (because of (50)). In other words, $u_{j_1} \cdots u_{j_p}$ contains both $u_{j-1}$ and $u_j$ (since $u_{j_1} \cdots u_{j_p}$ is the $k$-th row of $T_s'$). In other words, some $g \in \{1,2,\ldots,p\}$ satisfies $j_g = j-1$, and some $h \in \{1,2,\ldots,p\}$ satisfies $j_h = j$. Hence, Lemma 5.10 **(c)** shows that $L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$. Since $r_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$, this rewrites as $Lr_k = r_k$. Thus, we have computed $Lr_k$ in the case when $k \leq b-q$.
- Let us now assume that $b-q < k \leq b$. Thus, the $k$-th row of $T_s'$ contains $u_{j-1}$ but not $u_j$ (because of (50)). In other words, $u_{j_1} \cdots u_{j_p}$ contains $u_{j-1}$ but not $u_j$ (since $u_{j_1} \cdots u_{j_p}$ is the $k$-th row of $T_s'$). In particular, $u_{j_1} \cdots u_{j_p}$ contains $u_{j-1}$. In other words, some $g \in \{1,2,\ldots,p\}$ satisfies $j_g = j-1$. Consider this $g$. Hence, Lemma 5.10 **(b)** shows that

$$\begin{aligned} & L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) \\ & = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) + \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right). \end{aligned} \tag{52}$$

Let $T''_s$ be the tableau such that all the $*$'s are $u_j$ [232]; thus, $T''_s$ is a standard

Set $r'_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right)$. Thus, $r'_k$ is the same inner product $r_k$, except that the letter $u_{j_g}$ (that is, the $g$-th letter to the right of the vertical bar) has been replaced by $u_j$. In other words, $r'_k$ is the result of replacing the letter $u_{j-1}$ by $u_j$ in the inner product $r_k$ (since $j_g = j-1$). Now, (52) rewrites as $Lr_k = r_k + r'_k$ (since $r_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$ and $r'_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_{g-1}} u_j u_{j_{g+1}} \cdots u_{j_p}\right)$). Thus, we have computed $Lr_k$ in the case when $b-q < k \leq b$.

- Let us finally assume that $k > b$. Thus, all entries in the $k$-th row of $T'_s$ have the form $u_h$ for $h \geq j$ (because all entries of $T'_s$ in row $b+1$ and the rows below have the form $u_h$ for $h \geq j$). In particular, the $k$-th row of $T'_s$ does not contain $u_{j-1}$. In other words, $u_{j_1} \cdots u_{j_p}$ does not contain $u_{j-1}$ (since $u_{j_1} \cdots u_{j_p}$ is the $k$-th row of $T'_s$). In other words, no $g \in \{1, 2, \ldots, p\}$ satisfies $j_g = j-1$. Hence, Lemma 5.10 **(a)** shows that $L\left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right) = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$. Since $r_k = \left(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}\right)$, this rewrites as $Lr_k = r_k$. Thus, we have computed $Lr_k$ in the case when $k > q$.

Now, forget that we fixed $k$. Thus, for each $k \in \{1, 2, \ldots, w\}$, we have obtained a formula for $Lr_k$, namely:

- If $k$ satisfies either $k \leq b-q$ or $k > b$, then $Lr_k = r_k$.
- If $k$ satisfies $b-q < k \leq b$, then $Lr_k = r_k + r'_k$, where $r'_k$ is the result of replacing the letter $u_{j-1}$ by $u_j$ in the inner product $r_k$.

Multiplying all of these formulas, we obtain

$$\begin{aligned}
&(Lr_1)(Lr_2)\cdots(Lr_w) \\
&= r_1 r_2 \cdots r_{b-q}\left(r_{b-q+1} + r'_{b-q+1}\right)\left(r_{b-q+2} + r'_{b-q+2}\right)\cdots\left(r_b + r'_b\right) r_{b+1} r_{b+2} \cdots r_w.
\end{aligned}$$

Hence, (51) becomes

$$\begin{aligned}
L\left(T_s \mid T'_s\right) &= (Lr_1)(Lr_2)\cdots(Lr_w) \\
&= r_1 r_2 \cdots r_{b-q}\left(r_{b-q+1} + r'_{b-q+1}\right)\left(r_{b-q+2} + r'_{b-q+2}\right)\cdots\left(r_b + r'_b\right) r_{b+1} r_{b+2} \cdots r_w.
\end{aligned}$$

Expanding the right hand side, we thus conclude that $L(T_s \mid T'_s)$ is the sum of all products obtained from $r_1 r_2 \cdots r_w$ by replacing some of the factors $r_{b-q+1}, r_{b-q+2}, \ldots, r_b$ by $r'_{b-q+1}, r'_{b-q+2}, \ldots, r'_b$ (respectively). But of course, these products are precisely the bideterminants of all bitableaux that are obtained from $(T_s \mid T'_s)$ by replacing some of the letters $u_{j-1}$ in rows $b-q+1, b-q+2, \ldots, b$ by $u_j$. Hence, $L(T_s \mid T'_s)$ is the sum of the bideterminants of all these bitableaux. Hence, $L(T_s \mid T'_s) = \sum_r (T_s \mid T^r_s)$, where each $T^r_s$ is a tableau of the form (49), where any of the $*$ symbols in column $j-1$ may be $u_{j-1}$ or $u_j$, and where $m \geq j+1$. This proves (48).

[232] *Comment:* More precisely: Let $T''_s$ be the tableau obtained from $T'_s$ by replacing **all** the letters $u_{j-1}$ in rows $b-q+1, b-q+2, \ldots, b$ by $u_j$. Thus, $T''_s$ is one of the tableaux $T^r_s$ on the right hand side of (48). Actually, among all those tableaux $T^r_s$, the tableau $T''_s$ is the one with the most

tableau containing $u_j$ exactly[233] $b+q$ times[234]. Applying the operator $L$ to the equality (43), we obtain[235]

$$\begin{aligned} LF &= \sum_s \alpha_s \underbrace{L\left(T_s \mid T_s'\right)}_{\substack{=\sum_r (T_s \mid T_s^r) \\ \text{(by (48))}}} + \sum_t \alpha_t \underbrace{L\left(T_t \mid T_t'\right)}_{\substack{=(T_t \mid T_t') \\ \text{(by (47))}}} + \underbrace{LG}_{\substack{=G \\ \text{(by (46))}}} \\ &= \sum_s \alpha_s \sum_r \left(T_s \mid T_s^r\right) + \sum_t \alpha_t \left(T_t \mid T_t'\right) + G \\ &= \sum_{s,r} \alpha_s \left(T_s \mid T_s^r\right) + \sum_t \alpha_t \left(T_t \mid T_t'\right) + G, \end{aligned}$$

so that

$$\begin{aligned} &\sum_{s,r} \alpha_s \left(T_s \mid T_s^r\right) + \sum_t \alpha_t \left(T_t \mid T_t'\right) + G \\ &= LF = a\left(L\right) F \qquad \text{(since } F \text{ is invariant)} \\ &= \sum_s a\left(L\right) \alpha_s \left(T_s \mid T_s'\right) + \sum_t a\left(L\right) \alpha_t \left(T_t \mid T_t'\right) + a\left(L\right) G. \end{aligned} \tag{53}$$

(by (43)).

[236] Let $q'$ be the largest parameter for the $j$-critical tableaux $T_s'$ appearing in (53)[237]. The only bideterminants in the equality (53)[238] containing $u_j$ exactly $b+q'$ times are the bideterminants $\left(T_v \mid T_v''\right)$, where $T_v'$ is a $j$-critical tableau with param-

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occurrences of the letter $u_j$.

[233] *Correction:* Added the word "exactly".

[234] *Comment:* Here is why this is true:

First, let us see why $T_s''$ is a standard tableau.

An entry $u_{j-1}$ in $T_s'$ shall be called *malleable* if the letter $u_j$ does not appear in the same row of $T_s'$ as this entry.

Indeed, recall that the tableau $T_s'$ is standard. The tableau $T_s''$ is obtained from it by replacing some entries (namely, the entries $u_{j-1}$ in rows $b-q+1, b-q+2, \ldots, b$ of the $(j-1)$-th column) of $T_s'$ by $u_j$. It is easy to see (from a look at (50)) that these entries are precisely the malleable entries $u_{j-1}$ in the $(j-1)$-th column of $T_s'$. Thus, the tableau $T_s''$ is obtained from $T_s'$ by replacing each malleable entry $u_{j-1}$ in the $(j-1)$-th column of $T$ by $u_j$. Hence, Lemma 5.11 (applied to $T_s'$, $j-1$ and $T_s''$ instead of $T$, $g$ and $\widetilde{T}$) yields the tableau $T_s''$ is standard.

It now remains to prove that the tableau $T_s''$ contains $u_j$ exactly $b+q$ times.

Indeed, we know that the tableau $T_s'$ contains $u_j$ exactly $b$ times. The tableau $T_s''$ differs from $T_s'$ in having $q$ further entries equal to $u_j$ (in fact, $T_s''$ was obtained from $T_s'$ by replacing a total of $q$ entries by $u_j$, and none of these entries had been $u_j$ in $T_s'$). Thus, the tableau $T_s''$ contains $u_j$ exactly $b+q$ times. This completes our proof.

[235] *Correction:* This sentence has been refactored and expanded. Also, I have again removed the "$\left(T_s \mid T_s'\right)$" addend, because I prefer to regard this addend as part of the sum $\sum\limits_r \left(T_s \mid T_s^r\right)$.

[236] *Correction:* Removed the sentence "Each bideterminant $\left(T_s \mid T_s^r\right)$ contains the letter $u_j$ at least $b+1$ times.". This sentence was useless and does not hold with my definition of $\left(T_s \mid T_s^r\right)$.

[237] *Correction:* Added the words "$T_s'$ appearing in (53)".

[238] *Correction:* Replaced "the above equality" by "the equality (53)".

eter $q'$ [239]. Therefore, the projection of the equality (53)[240] onto the subspace spanned by the bideterminants containing $u_j$ exactly $b + q'$ times yields

$$\sum_v \alpha_v \left( T_v \mid T''_v \right) = 0 \tag{54}$$

[239]*Comment:* Let me explain why this is true.

First, we recall that a $j$-critical tableau must have at least one entry $u_j$ in its first column; thus, its parameter is positive (since its parameter is the number of entries $u_j$ in its first column). Hence, $q'$ is positive (since $q'$ is the parameter of a $j$-critical tableau). Thus, $q' \neq 0$, so that $b + q' \neq b$.

Each of the bideterminants appearing in (30) contains $u_j$ exactly $b$ times (because of (41)). In other words, each of the bideterminants appearing in (43) contains $u_j$ exactly $b$ times (since these bideterminants are the same as the bideterminants appearing in (30)). Hence, none of the bideterminants appearing in (43) contains $u_j$ exactly $b + q'$ times (since $b + q' \neq b$). Thus, none of the bideterminants $(T_s \mid T'_s)$ and $(T_t \mid T'_t)$ in (53) contains $u_j$ exactly $b + q'$ times. The same holds for the bideterminants contained in $G$ (for the same reason).

Therefore, any bideterminants in (53) that contain $u_j$ exactly $b + q'$ times must be of the form $(T_s \mid T^r_s)$. However, not every bideterminant of this form actually contains $u_j$ exactly $b + q'$ times. Let us see which ones do:

Consider a bideterminant $(T_s \mid T^r_s)$ that appears in (53) and contains $u_j$ exactly $b + q'$ times. The corresponding tableau $T'_s$ is $j$-critical; let $q$ be its parameter. Thus, $q' \geq q$ (since $q'$ is the largest parameter for the $j$-critical tableaux appearing in (53)). Furthermore, the tableau $T^r_s$ has the form (49), where any of the $*$ symbols in column $j - 1$ may be $u_{j-1}$ or $u_j$. Let $h$ be the number of $*$ symbols that are $u_j$; thus, $h \leq q$ (since the total number of $*$ symbols in (49) is $q$). Notice that the equality $h = q$ holds if and only if each of the $*$ symbols is a $u_j$.

From (41), we know that the tableau $T'_s$ contains the letter $u_j$ exactly $b$ times. The tableau $T^r_s$ is obtained from $T'_s$ by replacing some of the letters $u_{j-1}$ by $u_j$ (indeed, the positions of the $*$ symbols in (49) were occupied by $u_{j-1}$ in $T'_s$). Hence, the tableau $T^r_s$ contains the letter $u_j$ exactly $b + h$ times (since there are precisely $h$ letters $u_{j-1}$ that are replaced by $u_j$ in $T^r_s$). Comparing this with the fact that the tableau $T^r_s$ contains the letter $u_j$ exactly $b + q'$ times, we conclude that $b + h = b + q'$. Thus, $h = q'$. Combining $h = q' \geq q$ with $h \leq q$, we obtain $h = q$. In other words, exactly $q$ of the $*$ symbols in (49) are $u_j$. This means that **all** the $*$ symbols in (49) are $u_j$ (because $q$ is the total number of $*$ symbols in (49)). Hence, $T^r_s$ is the tableau $T''_s$ (because $T''_s$ was defined as the one tableau $T^r_s$ for which all the $*$ symbols in (49) are $u_j$).

Furthermore, $q' = h = q$. Hence, the $j$-critical tableau $T'_s$ has parameter $q'$ (since it has parameter $q$). Thus, we have shown that $T^r_s$ is the tableau $T''_s$, and that the $j$-critical tableau $T'_s$ has parameter $q'$.

Now, forget that we fixed $(T_s \mid T^r_s)$. We thus have shown that if $(T_s \mid T^r_s)$ is a bideterminant that appears in (53) and contains $u_j$ exactly $b + q'$ times, then $T^r_s$ is the tableau $T''_s$, and the $j$-critical tableau $T'_s$ has parameter $q'$. Since any bideterminants in (53) that contain $u_j$ exactly $b + q'$ times must be of the form $(T_s \mid T^r_s)$, we can therefore summarize: Any bideterminants in (53) that contain $u_j$ exactly $b + q'$ times must be of the form $(T_s \mid T^r_s)$, where $T^r_s$ is the tableau $T''_s$, and the $j$-critical tableau $T'_s$ has parameter $q'$. In other words, any bideterminants in (53) that contain $u_j$ exactly $b + q'$ times must be of the form $(T_s \mid T''_s)$, where the $j$-critical tableau $T'_s$ has parameter $q'$. In other words, any bideterminants in (53) that contain $u_j$ exactly $b + q'$ times must be of the form $(T_v \mid T''_v)$, where $T'_v$ is a $j$-critical tableau with parameter $q'$.

Conversely, any bideterminant of this form must contain $u_j$ exactly $b + q'$ times (as one can easily show by following the above argument backwards). Thus, the bideterminants in the equality (53) containing $u_j$ exactly $b + q'$ times are **exactly** the bideterminants $(T_v \mid T''_v)$, where $T'_v$ is a $j$-critical tableau with parameter $q'$. Qed.

[240]*Correction:* Replaced "this equality" by "the equality (53)".

[241] (where the sum is over all $v$ for which $T'_v$ is a $j$-critical tableau with parameter $q'$) [242].

All the bitableaux[243] $[T_v, T'_v]$, and hence $[T_v, T''_v]$, are distinct and standard[244]. By the basis theorem (specifically, the part of Corollary 4.8 that says that the standard $(\mathcal{X}, \mathcal{U})$-bideterminants are linearly independent)[245], we thus conclude that all the coefficients $\alpha_v$ in (54) must be 0. We conclude that there cannot be a $j$-critical bideterminant in the expansion of $F$; in particular, there cannot be a nonrectangular bideterminant in the expansion. This contradicts our assumption that not all tableaux $T'_s$ appearing in (30) are rectangular[246]. This completes the first proof of Theorem 5.9. $\square$

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[241] *Correction:* I have switched the left and the right hand side of this equality, to match my modified version of (53).

[242] *Correction:* Added the preceding parenthetical.

[243] *Correction:* Just as many times before, I have replaced bideterminants by bitableaux.

[244] *Comment:* Let me explain why this is true.

We already know that the tableaux $T''_v$ are standard; thus, the bitableaux $[T_v, T''_v]$ are standard. It remains to show that these bitableaux $[T_v, T''_v]$ are distinct.

Assume the contrary. Thus, there exist two distinct indices $v$ and $w$ (having the property that $T'_v$ and $T'_w$ are $j$-critical tableaux with parameters $q'$) such that $[T_v, T''_v] = [T_w, T''_w]$. Hence, $T_v = T_w$ and $T''_v = T''_w$.

Recall that $T'_v$ is a $j$-critical tableau with parameter $q'$. Hence, the tableau $T''_v$ is obtained from $T'_v$ by replacing the entries $u_{j-1}$ in rows $b - q' + 1, b - q' + 2, \ldots, b$ of the $(j-1)$-th column by $u_j$ (by the definition of $T''_v$). Therefore, the tableau $T'_v$ can be obtained from the tableau $T''_v$ by replacing the entries $u_j$ in rows $b - q' + 1, b - q' + 2, \ldots, b$ of the $(j-1)$-th column by $u_{j-1}$ (since the entries in rows $b - q' + 1, b - q' + 2, \ldots, b$ of the $(j-1)$-th column of $T'_v$ are $u_{j-1}$ (again, because $T'_v$ is $j$-critical)). Similarly, the tableau $T'_w$ can be obtained from the tableau $T''_w$ by the exact same replacements. Therefore, $T'_v = T'_w$ (since $T''_v = T''_w$). Combined with $T_v = T_w$, this yields $[T_v, T'_v] = [T_w, T'_w]$. But this contradicts the fact that the bitableaux $[T_s, T'_s]$ were chosen to be distinct. This contradiction completes our proof.

[245] *Correction:* I have expanded this sentence to be clearer.

[246] *Correction:* Inserted this sentence.

## 5.4. The First Fundamental Theorem: second proof

In preparation for the second proof of Theorem 5.9, let us state a few lemmas.[247]

**Lemma 5.12.** Let $K$ be a field. Let $d \geq 1$. Considered as a polynomial in the $d^2$ indeterminates $l_{ij}$ (for $1 \leq i \leq d$ and $1 \leq j \leq d$), the determinant $\Delta = \left|l_{ij}\right|$ is irreducible.

*Proof of Lemma 5.12.* Suppose that $\Delta = AB$. Since $\Delta$ is linear in each variable $l_{ij}$, the variable $l_{11}$ cannot occur in both $A$ and $B$. Suppose that $l_{11}$ occurs in $A$. In the expansion of $\Delta$ into monomials, each monomial contains exactly one variable from each row and column. Hence, none of the variables $l_{1r}, l_{s1}$ with $1 \leq r \leq d$ and $1 \leq s \leq d$ can occur in $B$. [248]

If $B$ is not a constant polynomial, then $B$ contains a variable $l_{pq}$, where, a fortiori, $p, q > 1$. By a similar argument, $A$ cannot contain the variables $l_{pr}, l_{sq}$ with $r \neq q$ and $s \neq p$. But this implies that neither $A$ nor $B$ contain the variables $l_{p1}$ and

[247] *Correction:* I have rewritten this second proof in many ways. Here are the most important changes:

- Lemma 5.12 and Lemma 5.17 have been moved out of the proof, as they are easier to understand as separate results.
- I have made "Weyl's principle of the irrelevance of algebraic inequalities" into a separate proposition (Proposition 5.13), and fixed it by requiring $g_1, \ldots, g_r$ to be nonzero. I have also added a proof outline.
- I have explicitly stated two corollaries of Weyl's principle (Corollaries 5.14 and 5.19) as well as two further auxiliary facts (Propositions 5.15 and 5.16) which were mostly implicit in the original proof.
- Multiple wrong indices have been fixed in the treatment of the adjugate $L^*$.
- Arguments such as the proofs of (59) and (60) have been expanded significantly, and I have clarified why $a(L)$ is a polynomial in the entries of the matrix $L$.
- I have introduced an extra polynomial ring $K[\lambda]$, which allows me to clarify the various applications of Weyl's principle to matrices as well as make Lemma 5.22 more precise (the $\lambda_{ij}$ in Lemma 5.22 are now polynomial indeterminates rather than elements of $K$). This has the additional advantage that an important puzzle piece (the invariance of the polynomials $F_{i(1,1),\ldots,i(m,d)}$) could be proved in higher generality and in a simpler way. As a consequence of the latter, a part of the original proof (the argument that the $B_{jk}$ are algebraically independent, and its consequences) has been removed for uselessness.
- Lemma 5.20 and Lemma 5.21 have been switched, thus allowing me to deduce the latter from the former, sidestepping the somewhat complicated proof in the original paper.
- The definition of the evaluation $\epsilon_L$ and various other parts of the proof has been rewritten in a clearer fashion.
- Generally, details have been added.

[248] *Comment:* Here is why:

Assume (for the sake of contradiction) that one of these variables does occur in $B$. Let us assume that it is a variable of the form $l_{1r}$ with $1 \leq r \leq d$ (since the case when it is a variable of the form $l_{s1}$ with $1 \leq s \leq d$ is analogous).

$l_{1q}$, both of which appear in the expansion of $\Delta$. Hence, $B$ must be a constant polynomial, and our lemma is proved. $\square$

A technical result we shall use time and again is *Weyl's principle of the irrelevance of algebraic inequalities*:

**Proposition 5.13** (Weyl's principle of the irrelevance of algebraic inequalities)**.** Let $K$ be an infinite field. Let $\{z_1, \ldots, z_p\}$ be a finite set of indeterminates. Let $f, g_1, \ldots, g_r$ be polynomials in $K[z_1, \ldots, z_p]$ such that $g_1, \ldots, g_r$ are nonzero. Suppose that

$$f(s_1, \ldots, s_p) = 0 \tag{55}$$

for all $s_1, \ldots, s_p \in K$ such that

$$g_i(s_1, \ldots, s_p) \neq 0 \qquad \text{for all } 1 \leq i \leq r. \tag{56}$$

Then, $f$ is identically zero.

*Proof of Proposition 5.13.* The proof is routine commutative algebra (see Weyl [4, Chapter I, Lemma (1.1.A)]). For the sake of completeness, let us nevertheless sketch it: We know that $K$ is a field; thus, the ring $K[z_1, \ldots, z_p]$ is an integral domain. Let

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Let $S$ be the ring of all polynomials in the indeterminates

$$\begin{aligned}&l_{1,1}, l_{1,2}, \ldots, l_{1,d},\\ &l_{2,1}, l_{2,2}, \ldots, l_{2,d},\\ &\ldots,\\ &l_{d,1}, l_{d,2}, \ldots, l_{d,d}\end{aligned}$$

over $K$. For any polynomial $g \in S$, we let $\overline{\deg} g$ denote the total degree of $g$ with respect to the indeterminates $l_{1,1}, l_{1,2}, \ldots, l_{1,d}$ (that is, the total degree of $g$ when $g$ is considered as a polynomial in the indeterminates $l_{1,1}, l_{1,2}, \ldots, l_{1,d}$, while all the remaining indeterminates

$$\begin{aligned}&l_{2,1}, l_{2,2}, \ldots, l_{2,d},\\ &l_{3,1}, l_{3,2}, \ldots, l_{3,d},\\ &\ldots,\\ &l_{d,1}, l_{d,2}, \ldots, l_{d,d}\end{aligned}$$

are regarded as constants). Then, any two nonzero polynomials $g$ and $h$ in $S$ satisfy $\overline{\deg}(gh) = \overline{\deg} g + \overline{\deg} h$. Applying this to $g = A$ and $h = B$, we obtain $\overline{\deg}(AB) = \overline{\deg} A + \overline{\deg} B$.

The determinant $\Delta = |l_{ij}|$ is homogeneous of degree 1 in the variables $l_{1,1}, l_{1,2}, \ldots, l_{1,d}$ (since the determinant of a $d \times d$-matrix is linear with respect to its first row). Thus, $\overline{\deg} \Delta \leq 1$.

But the variable $l_{11}$ occurs in $A$; thus, $\overline{\deg} A \geq 1$. Also, a variable of the form $l_{1r}$ with $1 \leq r \leq d$ occurs in $B$ (according to our assumption); thus, $\overline{\deg} B \geq 1$. Hence, $\overline{\deg}(AB) = \underbrace{\overline{\deg} A}_{\geq 1} + \underbrace{\overline{\deg} B}_{\geq 1} \geq 1 + 1 > 1$. This contradicts $\overline{\deg}\left(\underbrace{AB}_{=\Delta}\right) = \overline{\deg} \Delta \leq 1$. This contradiction shows that our assumption was false. This completes the proof.

$g$ be the polynomial $g_1 g_2 \cdots g_r$. Since the polynomials $g_1, \ldots, g_r$ are nonzero, their product $g_1 g_2 \cdots g_r$ must also be nonzero (since $K\left[z_1, \ldots, z_p\right]$ is an integral domain). In other words, $g$ is nonzero (since $g = g_1 g_2 \cdots g_r$).

Every $s_1, \ldots, s_p \in K$ satisfy

$$(fg)\left(s_1, \ldots, s_p\right) = 0 \tag{57}$$

[249].

But the field $K$ is infinite. Thus, the subset $K^p$ of $K^p$ is Zariski-dense. In elementary terms, this says the following: If $h \in K\left[z_1, \ldots, z_p\right]$ is a polynomial such that every $s_1, \ldots, s_p \in K$ satisfy $h\left(s_1, \ldots, s_p\right) = 0$, then $h = 0$. We can apply this to $h = fg$ (since every $s_1, \ldots, s_p \in K$ satisfy (57)). Thus, we conclude that $fg = 0$. Since the ring $K\left[z_1, \ldots, z_p\right]$ is an integral domain, we thus have either $f = 0$ or $g = 0$ (or both). Since $g$ is nonzero, we thus conclude that $f = 0$. This proves Proposition 5.13. □

For the sake of convenience, let us specifically state the particular case of Proposition 5.13 for $r = 1$:

**Corollary 5.14.** Let $K$ be an infinite field. Let $\left\{z_1, \ldots, z_p\right\}$ be a finite set of indeterminates. Let $f$ be a polynomial in $K\left[z_1, \ldots, z_p\right]$. Let $g$ be a nonzero polynomial in $K\left[z_1, \ldots, z_p\right]$. Assume that $f\left(s_1, \ldots, s_p\right) = 0$ for all $s_1, \ldots, s_p \in K$ such that $g\left(s_1, \ldots, s_p\right) \neq 0$. Then, $f$ is identically zero.

For each $d$-tuple $\left(i_1, i_2, \ldots, i_d\right) \in \{1, 2, \ldots, n\}^d$, we adopt the following *bracket* notation:

$$\left[x_{i_1}, \ldots, x_{i_d}\right] = \left(x_{i_1} \cdots x_{i_d} \mid u_1 \cdots u_d\right) \in P.$$

[249] *Proof of (57):* Let $s_1, \ldots, s_p \in K$. If all $1 \leq i \leq r$ satisfy $g_i\left(s_1, \ldots, s_p\right) \neq 0$, then

$$(fg)\left(s_1, \ldots, s_p\right) = \underbrace{f\left(s_1, \ldots, s_p\right)}_{\substack{=0 \\ \text{(by (55))}}} g\left(s_1, \ldots, s_p\right) = 0.$$

Thus, if all $1 \leq i \leq r$ satisfy $g_i\left(s_1, \ldots, s_p\right) \neq 0$, then (57) is proven. Hence, for the rest of the proof of (57), we WLOG assume that not all $1 \leq i \leq r$ satisfy $g_i\left(s_1, \ldots, s_p\right) \neq 0$. Thus, at least one $1 \leq i \leq r$ satisfies $g_i\left(s_1, \ldots, s_p\right) = 0$. Therefore, the product $\prod_{i=1}^{r} g_i\left(s_1, \ldots, s_p\right)$ has at least one factor equal to 0; consequently, the whole product is 0. In other words, $\prod_{i=1}^{r} g_i\left(s_1, \ldots, s_p\right) = 0$. But $g = g_1 g_2 \cdots g_r = \prod_{i=1}^{r} g_i$ and therefore

$$g\left(s_1, \ldots, s_p\right) = \prod_{i=1}^{r} g_i\left(s_1, \ldots, s_p\right) = 0.$$

Hence,

$$(fg)\left(s_1, \ldots, s_p\right) = f\left(s_1, \ldots, s_p\right) \underbrace{g\left(s_1, \ldots, s_p\right)}_{=0} = 0.$$

This proves (57).

Any such polynomial $\left[x_{i_1}, \ldots, x_{i_d}\right]$ will be called a *bracket*. A *bracket monomial* shall mean a product of brackets. The following is easy to see:

**Lemma 5.15.** Any bracket monomial is a linear combination of standard rectangular bideterminants, all of which have the same shape $(d, \ldots, d)$.

*Proof of Lemma 5.15.* Clearly, any bracket monomial is a rectangular bideterminant. It thus remains to show that each rectangular bideterminant is a linear combination of standard rectangular bideterminants, all of which have the same shape $(d, \ldots, d)$.

In other words, it remains to show that if $[T, T']$ is a bitableau such that the tableau $T'$ is rectangular, then $(T \mid T')$ is a linear combination of standard rectangular bideterminants, all of which have the same shape $(d, \ldots, d)$.

So let $[T, T']$ be a bitableau such that the tableau $T'$ is rectangular. Let $(\lambda)$ be the shape of $T'$. Thus, $(\lambda) = \underbrace{(d, \ldots, d)}_{g \text{ entries}}$ for some $g \in \mathbb{N}$ (since $T'$ is rectangular).

Now, we notice the following:

> *Claim 1:* Let $[W, W']$ be a standard bitableau of the same content as $[T, T']$ and of the same or longer shape. Then, the tableau $W'$ has shape $(\lambda)$.

[*Proof of Claim 1:* Assume the contrary. Thus, $W'$ does not have shape $(\lambda)$. In other words, the tableau $W'$ does not have the same shape as $T'$ (because the shape of $T'$ is $(\lambda)$).

The tableau $T'$ has shape $(\lambda) = \underbrace{(d, \ldots, d)}_{g \text{ entries}}$. Hence, it has $gd$ entries.

The bitableau $[W, W']$ has the same content as $[T, T']$. Thus, the tableau $W'$ has equally many entries as $T'$. Therefore, the tableau $W'$ has $gd$ entries (since the tableau $T'$ has $gd$ entries).

Let $(\mu) = (\mu_1, \mu_2, \ldots, \mu_h)$ be the shape of the tableau $W'$. Therefore, the tableau $W'$ has $\mu_1 + \mu_2 + \cdots + \mu_h$ entries. Since the tableau $W'$ has $gd$ entries, we thus conclude that $\mu_1 + \mu_2 + \cdots + \mu_h = gd$.

The bitableau $[W, W']$ has the same or longer shape than $[T, T']$. In other words, the tableau $W'$ has the same or longer shape than $T'$. Hence, the tableau $W'$ has longer shape than $T'$ (because the tableau $W'$ cannot have the same shape as $T'$). In other words, $(\mu_1, \mu_2, \ldots, \mu_h) > \underbrace{(d, \ldots, d)}_{g \text{ entries}}$ in lexicographic order (since the tableau $W'$ has shape $(\mu_1, \mu_2, \ldots, \mu_h)$, while the tableau $T'$ has shape $\underbrace{(d, \ldots, d)}_{g \text{ entries}}$). By the definition of lexicographic order, this means that we must be in one of the following two cases:

*Case 1:* There exists some $i \in \{1, 2, \ldots, \min\{h, g\}\}$ such that $\mu_i > d$ and $\left(\mu_j = d \text{ for each } j < i\right)$.

*Case 2:* We have $h < g$ and $\left(\mu_j = d \text{ for each } j \in \{1, 2, \ldots, h\}\right)$.

Let us first consider Case 1. In this case, there exists some $i \in \{1,2,\ldots,\min\{h,g\}\}$ such that $\mu_i > d$ and $\left(\mu_j = d \text{ for each } j < i\right)$. Consider this $i$. Then, $\mu_i > d$. But the $i$-th row of $W'$ has more than $d$ entries (since $(\mu_1,\mu_2,\ldots,\mu_h)$ is the shape of the tableau $W'$). Thus, the $i$-th row of $W'$ has more than $d$ entries (since $\mu_i > d$). Consequently, two of these entries must be equal (since the only available entries for $W'$ are the $d$ entries $u_1,u_2,\ldots,u_d$). This contradicts the fact that the entries in the $i$-th row of $W'$ are strictly increasing (since $W'$ is a standard tableau). Thus, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, we have $h < g$ and $\left(\mu_j = d \text{ for each } j \in \{1,2,\ldots,h\}\right)$. Now,

$$\begin{aligned}\mu_1+\mu_2+\cdots+\mu_h &= \underbrace{h}_{<g} d \qquad \left(\text{since } \mu_j = d \text{ for each } j \in \{1,2,\ldots,h\}\right) \\ &< gd \qquad (\text{since } d > 0).\end{aligned}$$

This contradicts $\mu_1+\mu_2+\cdots+\mu_h = gd$. Thus, we have obtained a contradiction in Case 2.

Hence, we have obtained a contradiction in both Cases 1 and 2. This contradiction shows that our assumption was wrong; hence, Claim 1 is proven.]

Now, Theorem 3.2 shows that $(T \mid T')$ is a linear combination, with integer coefficients, of bideterminants $(W \mid W')$ of standard bitableaux $[W,W']$ of the same content and of the same or longer shape. All these bideterminants $(W \mid W')$ have shape $(\lambda)$ (since Claim 1 yields that the respective tableaux $W'$ have shape $(\lambda)$) and thus are rectangular (since $(\lambda) = \underbrace{(d,\ldots,d)}_{g\text{ entries}}$) and have the same shape $(d,\ldots,d)$ (again since $(\lambda) = \underbrace{(d,\ldots,d)}_{g\text{ entries}}$). Thus, $(T \mid T')$ is a linear combination of standard rectangular bideterminants $(W \mid W')$, all of which have the same shape $(d,\ldots,d)$. This proves Lemma 5.15. ☐

**Lemma 5.16.** Assume that $n \geq d$. Let $F \in P$ be an invariant form, and let $g \in \mathbb{N}$ be such that $[x_1,\ldots,x_d]^g \cdot F$ is a polynomial in the brackets (i.e., a linear combination of bracket monomials). Then, $F$ is a linear combination of standard rectangular bideterminants.

*Proof of Lemma 5.16.* By Corollary 3.4, we can write $F$ as a linear combination of bideterminants of standard bitableaux:

$$F = \sum_{i\in I} b_i \left(U_i \mid U_i'\right), \tag{58}$$

where the $\left[U_i, U_i'\right]$ are distinct standard bitableaux, and where the $b_i$ are nonzero elements of $K$. For each $i \in I$, each row of the tableau $U_i'$ has length $\leq d$ [250], and

[250] *Proof.* Let $i \in I$. The tableau $U_i'$ is standard; thus, each row of $U_i'$ is strictly increasing. Hence, each

therefore each row of the tableau $U_i$ has length $\leq d$ as well (since the length of a row of $U_i$ equals the length of the corresponding row of $U_i'$).

If each of the bitableaux $\left[U_i, U_i'\right]$ is rectangular, then Lemma 5.16 is proven (because in this case, (58) shows that $F$ is a linear combination of standard rectangular bideterminants). Let us thus assume that this is not the case. Hence, not all of the bitableaux $\left[U_i, U_i'\right]$ are rectangular.

For each $i \in I$, we let $\widehat{U}_i$ be the tableau

$$g \text{ rows} \begin{cases} x_1 & x_2 & \cdots & x_d \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_d \end{cases} \\ \qquad\qquad U_i$$

(that is, the result of piling $g$ rows of the form $x_1 x_2 \cdots x_d$ on top of the tableau $U_i$). This tableau $\widehat{U}_i$ is standard (since $U_i$ is standard, since each row of the tableau $U_i$ has length $\leq d$, and since $x_k$ is the smallest possible entry that an entry in the $k$-th column of a standard tableau can have).

Similarly, for each $i \in I$, we let $\widehat{U}_i'$ be the tableau

$$g \text{ rows} \begin{cases} u_1 & u_2 & \cdots & u_d \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_d \end{cases} \\ \qquad\qquad U_i'$$

(that is, the result of piling $g$ rows of the form $u_1 u_2 \cdots u_d$ on top of the tableau $U_i'$). A similar argument shows that this tableau $\widehat{U}_i'$ is standard. Thus, $\left[\widehat{U}_i, \widehat{U}_i'\right]$ is a standard bitableau for each $i \in I$. Moreover, the bitableaux $\left[\widehat{U}_i, \widehat{U}_i'\right]$ for $i \in I$ are distinct (since the bitableaux $\left[U_i, U_i'\right]$ are distinct, and since the bitableaux $\left[\widehat{U}_i, \widehat{U}_i'\right]$ are obtained from $\left[U_i, U_i'\right]$ by piling $g$ new rows on top). Also notice that not all of the bitableaux $\left[\widehat{U}_i, \widehat{U}_i'\right]$ are rectangular (since not all of the bitableaux $\left[U_i, U_i'\right]$ are rectangular).

Clearly, each $i \in I$ satisfies

$$\begin{aligned} \left(\widehat{U}_i \mid \widehat{U}_i'\right) &= \left(\underbrace{(x_1 x_2 \cdots x_d \mid u_1 u_2 \cdots u_d)}_{=[x_1, x_2, \ldots, x_d]}\right)^g \left(U_i \mid U_i'\right) \\ &\qquad \left(\text{by the construction of } \widehat{U}_i \text{ and } \widehat{U}_i'\right) \\ &= [x_1, x_2, \ldots, x_d]^g \left(U_i \mid U_i'\right). \end{aligned}$$

row of $U_i'$ must have at most $d$ entries (because if it had more than $d$ entries, then two of these entries would be equal (since the only available entries for $U_i'$ are the $d$ letters $u_1, u_2, \ldots, u_d$), and this would contradict the fact that this row is strictly increasing). In other words, each row of $U_i'$ has length $\leq d$. Qed.

Thus,

$$\sum_{i\in I} b_i \underbrace{\left(\widehat{U}_i \mid \widehat{U}'_i\right)}_{=[x_1,x_2,\ldots,x_d]^g\left(U_i\mid U'_i\right)} = [x_1,x_2,\ldots,x_d]^g \cdot \underbrace{\sum_{i\in I} b_i \left(U_i \mid U'_i\right)}_{\substack{=F \\ \text{(by (58))}}}$$
$$= [x_1,x_2,\ldots,x_d]^g \cdot F.$$

Thus, the decomposition of $[x_1,x_2,\ldots,x_d]^g \cdot F$ into a linear combination of standard bideterminants is $\sum_{i\in I} b_i \left(\widehat{U}_i \mid \widehat{U}'_i\right)$. (We are allowed to speak of "the" decomposition, because Corollary 4.8 shows that there is a unique such decomposition.)

But $[x_1,\ldots,x_d]^g \cdot F$ is a polynomial in the brackets, i.e., a linear combination of bracket monomials. Hence, $[x_1,\ldots,x_d]^g \cdot F$ is a linear combination of standard rectangular bideterminants (since Lemma 5.15 shows that any bracket monomial is a linear combination of standard rectangular bideterminants). Thus, the decomposition of $[x_1,\ldots,x_d]^g \cdot F$ into a linear combination of standard bideterminants has the property that all bideterminants appearing in it are rectangular. Since this decomposition is $\sum_{i\in I} b_i \left(\widehat{U}_i \mid \widehat{U}'_i\right)$, this shows that all of the bideterminants $\left(\widehat{U}_i \mid \widehat{U}'_i\right)$ are rectangular. In other words, all of the bitableaux $\left[\widehat{U}_i, \widehat{U}'_i\right]$ are rectangular. This contradicts the fact that not all of the bitableaux $\left[\widehat{U}_i, \widehat{U}'_i\right]$ are rectangular. This contradiction shows that our assumption was wrong. As we have seen, this completes the proof of Lemma 5.16. □

**Lemma 5.17.** Assume that $n \geq d$. In the letter place algebra $P$, we have

$$[x_1,\ldots,x_d]\left(x_j \mid u_m\right) = \sum_{k=1}^{d} \left[x_1,\ldots,x_{k-1},x_j,x_{k+1},\ldots,x_d\right]\left(x_k \mid u_m\right)$$

for any $1 \leq j \leq n$ and $1 \leq m \leq d$.

*Proof of Lemma 5.17.* The identity follows from expanding, by Laplace's rule, the inner product $\left(x_1 \cdots x_d x_j \mid u_1 \cdots u_d u_m\right)$, which is identically zero in $P$ (since it is a determinant with two equal columns).[251] □

*Second proof of Theorem 5.9.* Recall that a form $F$ is invariant if, for all invertible $d \times d$-matrices $L \in K^{d\times d}$, there exists a scalar $a(L)$ such that $LF = a(L)F$. Let $F = F_0 + \cdots + F_t$ be the decomposition of the polynomial $F$ into homogeneous components with respect to the *total* degree (such that each component $F_i$ has total degree $i$). Then, if $F$ is invariant,

$$LF = LF_0 + \cdots + LF_t,$$

[251]In more detail:

Applying (2) to $p = d+1$, $\left(i_1,i_2,\ldots,i_p\right) = (j,1,2,\ldots,d)$ and $\left(j_1,j_2,\ldots,j_p\right) = (m,1,2,\ldots,d)$, whence

$$LF_0 + \cdots + LF_t = LF = a(L) F = a(L) F_0 + \cdots + a(L) F_t.$$

Since the action of $L$ on $P$ preserves the total degree, we must thus have

$$LF_i = a(L) F_i \qquad \text{for all } 0 \leq i \leq t.$$

In other words, all the homogeneous components $F_0, F_1, \ldots, F_t$ of $F$ are themselves invariant. It suffices, therefore, to consider only invariant forms that are homogeneous.

Now, fix a nonzero homogeneous invariant form $F$ of degree $t$. Then, for all invertible $d \times d$-matrices $L \in K^{d \times d}$, there exists a scalar $a(L)$ such that $LF = a(L) F$ (since $F$ is invariant). This scalar $a(L)$ is uniquely determined by $L$ and $F$ (since $F$ is nonzero), and is a polynomial in the entries $l_{jk}$ of the matrix $L$ [252]. If $c \in K$ is a

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we obtain

$$\begin{aligned}
&\left(x_j x_1 \cdots x_d \mid u_m u_1 \cdots u_d\right) \\
&= \left(x_j \mid u_m\right) \underbrace{\left(x_1 \cdots x_d \mid u_1 \cdots u_d\right)}_{=[x_1,\ldots,x_d]} \\
&\qquad + \sum_{k=1}^{d} (-1)^k \underbrace{\left(x_k \mid u_m\right) \left(x_j x_1 \cdots x_{k-1} x_{k+1} \cdots x_d \mid u_1 \cdots u_d\right)}_{=\left(x_j x_1 \cdots x_{k-1} x_{k+1} \cdots x_d \mid u_1 \cdots u_d\right)\left(x_k \mid u_m\right)} \\
&= \left(x_j \mid u_m\right) [x_1, \ldots, x_d] + \sum_{k=1}^{d} (-1)^k \underbrace{\left(x_j x_1 \cdots x_{k-1} x_{k+1} \cdots x_d \mid u_1 \cdots u_d\right)}_{=(-1)^{k-1}\left(x_1 \cdots x_{k-1} x_j x_{k+1} \cdots x_d \mid u_1 \cdots u_d\right)} \left(x_k \mid u_m\right) \\
&= \underbrace{\left(x_j \mid u_m\right) [x_1, \ldots, x_d]}_{=[x_1,\ldots,x_d]\left(x_j \mid u_m\right)} + \sum_{k=1}^{d} \underbrace{(-1)^k (-1)^{k-1}}_{=-1} \underbrace{\left(x_1 \cdots x_{k-1} x_j x_{k+1} \cdots x_d \mid u_1 \cdots u_d\right)}_{=\left[x_1,\ldots,x_{k-1},x_j,x_{k+1},\ldots,x_d\right]} \left(x_k \mid u_m\right) \\
&= [x_1, \ldots, x_d] \left(x_j \mid u_m\right) - \sum_{k=1}^{d} \left[x_1, \ldots, x_{k-1}, x_j, x_{k+1}, \ldots, x_d\right] \left(x_k \mid u_m\right).
\end{aligned}$$

But $\left(x_j x_1 \cdots x_d \mid u_m u_1 \cdots u_d\right) = 0$ (since $\left(x_j x_1 \cdots x_d \mid u_m u_1 \cdots u_d\right)$ is a determinant having two equal columns (since $u_m$ also appears among $u_1, u_2, \ldots, u_d$)). Thus,

$$\begin{aligned}
0 &= \left(x_j x_1 \cdots x_d \mid u_m u_1 \cdots u_d\right) \\
&= [x_1, \ldots, x_d] \left(x_j \mid u_m\right) - \sum_{k=1}^{d} \left[x_1, \ldots, x_{k-1}, x_j, x_{k+1}, \ldots, x_d\right] \left(x_k \mid u_m\right).
\end{aligned}$$

In other words,

$$[x_1, \ldots, x_d] \left(x_j \mid u_m\right) = \sum_{k=1}^{d} \left[x_1, \ldots, x_{k-1}, x_j, x_{k+1}, \ldots, x_d\right] \left(x_k \mid u_m\right).$$

This proves Lemma 5.17.

[252] *Comment:* To be more precise: There exists a polynomial map $\mathbf{a} : K^{d \times d} \to K$ such that every invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies $a(L) = \mathbf{a}(L)$.

nonzero scalar, then we have

$$a\left(cI\right) = c^t. \tag{59}$$

[*Proof:* Let $c \in K$ be a nonzero scalar. Consider the invertible $d \times d$-matrix $L \in K^{d\times d}$ defined by $L = cI$, where $I$ is the identity matrix. Under the action of this $L$ on $P$, we have $\left(x_i \mid u_j\right) \mapsto c\left(x_i \mid u_j\right)$. Thus, $LF = c^t F$ (since $F$ is homogeneous of degree $t$). Thus, the formula $LF = a\left(L\right)F$ becomes $c^t F = a\left(L\right)F = a\left(cI\right)F$ (since $L = cI$). Hence, $c^t = a\left(cI\right)$ (since $F$ is nonzero), qed.]

Moreover, the function $a\left(L\right)$ is multiplicative, in the sense that for any two invertible $d \times d$-matrices $L_1$ and $L_2$ in $K^{d\times d}$, we have

$$a\left(L_1 L_2\right) = a\left(L_1\right) a\left(L_2\right). \tag{60}$$

[*Proof:* Let $L_1$ and $L_2$ be two invertible $d \times d$-matrices in $K^{d\times d}$. The formula $LF = a\left(L\right)F$ then yields

$$L_1 F = a\left(L_1\right)F, \qquad L_2 F = a\left(L_2\right)F, \qquad \text{and} \qquad L_1 L_2 F = a\left(L_1 L_2\right)F.$$

Hence, $a\left(L_1 L_2\right)F = L_1 \underbrace{L_2 F}_{=a(L_2)F} = a\left(L_2\right)\underbrace{L_1 F}_{=a(L_1)F} = a\left(L_1\right)a\left(L_2\right)F$. Since $F$ is nonzero, this results in $a\left(L_1 L_2\right) = a\left(L_1\right)a\left(L_2\right)$, qed.]

Given a $d \times d$-matrix $L = \left(l_{jk}\right) \in K^{d\times d}$, its adjugate $L^*$ is the $d \times d$-matrix $\left(l^*_{jk}\right)$, where

$$l^*_{jk} = \left(\text{the } kj\text{th cofactor of the matrix } L\right) = \left(-1\right)^{j+k}\left|l_{pq}\right|_{p\neq k,\ q\neq j}.$$

The adjugate has the property

$$LL^* = \left|l_{ij}\right| I. \tag{61}$$

[*Proof of (61):* Fix $1 \leq s \leq d$ and $1 \leq t \leq d$. The $st$th entry of the matrix $LL^*$ is

$$\sum_m l_{sm} l^*_{mt} = \sum_m \left(-1\right)^{m+t} l_{sm}\left|l_{pq}\right|_{p\neq t,\ q\neq m}.$$

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This can be proven as follows:

Since $F$ is nonzero, there exists some monomial $\mathfrak{m}$ such that the coefficient of $\mathfrak{m}$ in $F$ is $\neq 0$. Fix such an $\mathfrak{m}$. Let $\lambda$ be the coefficient of $\mathfrak{m}$ in $F$; thus, $\lambda \neq 0$. Now, each invertible $d \times d$-matrix $L \in K^{d\times d}$ satisfies

$$\begin{aligned} &\left(\text{the coefficient of } \mathfrak{m} \text{ in } \underbrace{LF}_{=a(L)F}\right) \\ &= \left(\text{the coefficient of } \mathfrak{m} \text{ in } a\left(L\right)F\right) = a\left(L\right)\cdot\underbrace{\left(\text{the coefficient of } \mathfrak{m} \text{ in } F\right)}_{=\lambda} = a\left(L\right)\cdot\lambda \end{aligned}$$

and therefore

$$a\left(L\right) = \frac{1}{\lambda}\cdot\left(\text{the coefficient of } \mathfrak{m} \text{ in } LF\right).$$

Therefore, $a\left(L\right)$ is a polynomial in the entries of the matrix $L$ (since the coefficient of $\mathfrak{m}$ in $LF$ is a polynomial in the entries of the matrix $L$). This is what we wanted to prove.

By the Laplace expansion, the right-hand side is the determinant of the matrix $(l_{jk})$ with the $t$th row replaced by the row vector $(l_{sm})_{1\leq m\leq d}$. This determinant is zero if $s \neq t$ (since it has two equal rows in this case) and equals $|l_{ij}|$ if $s = t$. This is, of course, precisely the $st$th entry of the matrix $|l_{ij}| I$. Hence, (61) follows.]

As an immediate consequence of (61), we obtain

$$a(L)\, a(L^*) = |l_{ij}|^t$$

(because of (60) and (59)). As the determinant is irreducible (by Lemma 5.12)[253], each of the factors on the left must therefore also be a power of the determinant (since the polynomial ring $P$ is a unique factorization domain). In particular, $a(L)$ is a power of the determinant. In other words, there is a nonnegative integer $g$ such that every invertible $d \times d$-matrix $L \in K^{d\times d}$ satisfies $a(L) = (\det L)^g$. For this $g$, every invertible $d \times d$-matrix $L \in K^{d\times d}$ satisfies $LF = \underbrace{a(L)}_{=(\det L)^g} F = (\det L)^g F$. We have thus proved:

**Lemma 5.18.** Let $F \in P$ be a nonzero homogeneous invariant form. Then, there is a nonnegative integer $g$ such that every invertible $d \times d$-matrix $L \in K^{d\times d}$ satisfies $LF = (\det L)^g F$.

Let us now introduce a polynomial ring, which we will call $K[\lambda]$. Namely, we let $K[\lambda]$ be the polynomial ring over $K$ in the $d^2$ indeterminates

$$\begin{aligned}
&\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,d},\\
&\lambda_{2,1}, \lambda_{2,2}, \ldots, \lambda_{2,d},\\
&\vdots,\\
&\lambda_{d,1}, \lambda_{d,2}, \ldots, \lambda_{d,d}.
\end{aligned}$$

More generally, if $R$ is any commutative ring, then $R[\lambda]$ shall mean the polynomial ring over $R$ in these $d^2$ indeterminates.

If $f \in K[\lambda]$ is a polynomial, if $A$ is a commutative $K$-algebra, and if $M = (m_{j,k})_{1\leq j\leq d,\ 1\leq k\leq d} \in A^{d\times d}$ is a $d \times d$-matrix over $A$, then $f[M]$ shall mean the result of substituting $a_{j,k}$ for each indeterminate $\lambda_{j,k}$ in $f$. For example, if $f = \lambda_{1,1} + \lambda_{2,2} + \cdots + \lambda_{d,d}$, then $f[M] = m_{1,1} + m_{2,2} + \cdots + m_{d,d} = \operatorname{Tr} M$ for any $M = (m_{j,k})_{1\leq j\leq d,\ 1\leq k\leq d} \in A^{d\times d}$.

We let $\det\lambda$ be the polynomial $\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma)\, \lambda_{1,\sigma(1)}\lambda_{2,\sigma(2)} \cdots \lambda_{d,\sigma(d)} \in K[\lambda]$. Thus, $(\det\lambda)[M] = \det M$ for any $M = (m_{j,k})_{1\leq j\leq d,\ 1\leq k\leq d} \in A^{d\times d}$.

The following particular case of Corollary 5.14 will be particularly useful:

[253]*Comment:* This requires $d \geq 1$. In the (degenerate) case $d = 0$, everything is obvious anyway.

**Corollary 5.19.** Let $K$ be an infinite field. Let $f$ be a polynomial in $K[\lambda]$. Assume that $f[L] = 0$ for every invertible $d \times d$-matrix $L \in K^{d\times d}$. Then, $f = 0$.

*Proof of Corollary 5.19.* We have assumed that $f[L] = 0$ for every invertible $d \times d$-matrix $L \in K^{d\times d}$. In other words, $f[M] = 0$ for every invertible $d \times d$-matrix $M \in K^{d\times d}$ (here, we have renamed the index $L$ as $M$). In other words, $f[M] = 0$ for every matrix $M \in K^{d\times d}$ such that $\det M \neq 0$ (since a $d \times d$-matrix $M \in K^{d\times d}$ is invertible if and only if $\det M \neq 0$). In other words, $f[M] = 0$ for every matrix $M \in K^{d\times d}$ such that $(\det\lambda)[M] \neq 0$ (since $(\det\lambda)[M] = \det M$). In other words, $f\left(\left(m_{j,k}\right)_{1\leq j\leq d,\ 1\leq k\leq d}\right) = 0$ for every family $\left(m_{j,k}\right)_{1\leq j\leq d,\ 1\leq k\leq d} \in K^{d\times d}$ such that $(\det\lambda)\left(\left(m_{j,k}\right)_{1\leq j\leq d,\ 1\leq k\leq d}\right) \neq 0$. Therefore, Corollary 5.14 (applied to $\{z_1, \ldots, z_p\} = \{\lambda_{j,k} \mid 1 \leq j \leq d \text{ and } 1 \leq k \leq d\}$ and $g = \det\lambda$) shows that $f$ is identically zero (since $\det\lambda$ is nonzero). This proves Corollary 5.19. □

We also introduce the following notation: Let $L = (l_{jk})$ be a $d \times d$-matrix in $K^{d\times d}$. The evaluation $\epsilon_L$ is the $K$-algebra homomorphism from $P$ to $K$ given by

$$\left(x_j \mid u_k\right) \mapsto \begin{cases} l_{kj}, & \text{if } 1 \leq j \leq d; \\ 0, & \text{otherwise} \end{cases}.$$

An easy computation shows that the evaluation satisfies

$$\epsilon_L F = \epsilon_I L F \qquad \text{for any form } F \in P, \tag{62}$$

where $I$ is the identity matrix.[254]

[254]*Proof.* We need to show that the maps

$$\begin{aligned} &P \to K, \qquad F \mapsto \epsilon_L F \qquad \text{and} \\ &P \to K, \qquad F \mapsto \epsilon_I L F \end{aligned}$$

are identical. Since both of these maps are $K$-algebra homomorphism, it suffices to show that these maps send the form $\left(x_j \mid u_k\right)$ (for any $j$ and $k$) to the same image. In other words, it suffices to show that $\epsilon_L\left(x_j \mid u_k\right) = \epsilon_I L\left(x_j \mid u_k\right)$ for all $j$ and $k$.

So let us fix $j$ and $k$. The definition of the action of $L$ on $P$ yields $L\left(x_j \mid u_k\right) = \sum_{1\leq r\leq d} l_{kr}\left(x_j \mid u_r\right)$. Hence,

$$\epsilon_I L\left(x_j \mid u_k\right) = \epsilon_I\left(\sum_{1\leq r\leq d} l_{kr}\left(x_j \mid u_r\right)\right) = \sum_{1\leq r\leq d} l_{kr} \underbrace{\epsilon_I\left(x_j \mid u_r\right)}_{\substack{= \begin{cases} 1, & \text{if } j = r; \\ 0, & \text{if } j \neq r \end{cases} \\ \text{(by the definition of } \epsilon_I)}} = \sum_{1\leq r\leq d} l_{kr} \begin{cases} 1, & \text{if } j = r; \\ 0, & \text{if } j \neq r \end{cases}$$

$$= \begin{cases} l_{kj}, & \text{if } 1 \leq j \leq d; \\ 0, & \text{otherwise} \end{cases}.$$

For our next considerations, it will be helpful to consider the rings $P$ for various values of $n$ simultaneously. This necessitates the following notation: We denote the $K$-algebra $P$ by $P_n$ when we want to stress its dependence on $n$. Then,

$$\text{the } K\text{-algebra } P_m \text{ canonically becomes a } K\text{-subalgebra of } P_n \tag{63}$$

whenever $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfy $m \leq n$ (because each of the indeterminates $(x_i \mid u_j)$ appearing in $P_m$ is one of the indeterminates of $P_n$ as well). Any $d \times d$-matrix $L \in K^{d \times d}$ acts on $P_m$ for all $m \in \mathbb{N}$, and these actions are compatible (i.e., the canonical inclusions $P_m \to P_n$ commute with the actions of $L$).

Thus, for every $m \in \{0, 1, \ldots, n\}$, the $K$-algebra $P_m$ is a subalgebra of $P_n = P$, and the action of any $d \times d$-matrix $L \in K^{d \times d}$ on $P_m$ is the restriction of the corresponding action on $P_n$. When $F$ is an element of the $K$-algebra $P_m$, we shall write $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ for $F$ in order to stress that it belongs to $P_m$.

(For example, if $n = 4$ and $d = 3$, then the form $(x_2 \mid u_3)(x_3 \mid u_3)$ belongs not only to $P = P_4$ but also to $P_3$.)

**Lemma 5.20.** Assume that $n \geq d$. If $F(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in P_d$ is a nonzero homogeneous invariant, then

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_d) = c\,(x_1 \cdots x_d \mid u_1 \cdots u_d)^g$$

for some nonnegative integer $g$, and scalar $c$.

*Proof of Lemma 5.20.* The $K$-algebra homomorphism

$$\psi : K[\lambda] \to P_d, \qquad f \mapsto f\left(\left((x_j \mid u_k)\right)_{1 \leq j \leq d,\ 1 \leq k \leq d}\right)$$

is a $K$-algebra isomorphism[255]. Define $f \in K[\lambda]$ by $f = \psi^{-1}(F)$. Thus,

$$F = \psi(f) = f\left(\left((x_j \mid u_k)\right)_{1 \leq j \leq d,\ 1 \leq k \leq d}\right). \tag{64}$$

Hence, every $d \times d$-matrix $L \in K^{d \times d}$ satisfies

$$\epsilon_L F = f[L] \tag{65}$$

[256].

Comparing this with

$$\epsilon_L (x_j \mid u_k) = \begin{cases} l_{kj}, & \text{if } 1 \leq j \leq d; \\ 0, & \text{otherwise} \end{cases} \qquad \text{(by the definition of } \epsilon_L\text{)},$$

this yields $\epsilon_L (x_j \mid u_k) = \epsilon_I L (x_j \mid u_k)$. This completes our proof.

[255] In fact, $K[\lambda]$ is the polynomial ring in the $d^2$ indeterminates $\lambda_{j,k}$ for $1 \leq j \leq d$ and $1 \leq k \leq d$, whereas $P_d$ is the polynomial ring in the $d^2$ indeterminates $(x_j \mid u_k)$ for $1 \leq j \leq d$ and $1 \leq k \leq d$. The homomorphism $\psi$ sends each of the former indeterminates $\lambda_{j,k}$ to the corresponding $(x_j \mid u_k)$; thus, it can be viewed as just a renaming of the indeterminates.

[256] *Proof.* Let $L$ be a $d \times d$-matrix in $K^{d \times d}$. Write this matrix $L$ in the form $L = \left(l_{kj}\right)$.

Denote the scalar $\epsilon_I F \in K$ by $c$.

Lemma 5.18 shows that there is a nonnegative integer $g$ such that every invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies $LF = (\det L)^g F$. Consider this $g$.

Now, any invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies

$$\begin{aligned} f[L] &= \epsilon_L F \qquad \text{(by (65))} \\ &= \epsilon_I \underbrace{LF}_{=(\det L)^g F} \qquad \text{(by (62))} \\ &= (\det L)^g \underbrace{\epsilon_I F}_{=c} = (\det L)^g c = c \left( \underbrace{\det L}_{=(\det \lambda)[L]} \right)^g = c\left((\det \lambda)[L]\right)^g \\ &= \left(c (\det \lambda)^g\right)[L]. \end{aligned}$$

In other words, any invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies $\left(f - c(\det \lambda)^g\right)[L] = 0$. Therefore, Corollary 5.19 (applied to $f - c(\det \lambda)^g$ instead of $f$) shows that

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Let $\omega_L : K[\lambda] \to K$ be the $K$-algebra homomorphism sending each $g \in K[\lambda]$ to $g[L]$. Every $j$ and $k$ satisfy

$$\begin{aligned} (\epsilon_L \circ \psi)\left(\lambda_{j,k}\right) &= \epsilon_L \left( \underbrace{\psi\left(\lambda_{j,k}\right)}_{\substack{=(x_j | u_k) \\ \text{(by the definition of } \psi)}} \right) = \epsilon_L \left(x_j \mid u_k\right) \\ &= \begin{cases} l_{kj}, & \text{if } 1 \leq j \leq d; \\ 0, & \text{otherwise} \end{cases} \qquad \text{(by the definition of } \epsilon_L) \\ &= l_{kj} \qquad \text{(since } 1 \leq j \leq d) \\ &= \omega_L\left(\lambda_{j,k}\right) \end{aligned}$$

(since the definition of $\omega_L$ yields $\omega_L\left(\lambda_{j,k}\right) = \lambda_{j,k}[L] = l_{kj}$). In other words, the two $K$-algebra homomorphisms $\epsilon_L \circ \psi$ and $\omega_L$ from $K[\lambda]$ to $K$ are equal to each other on the generators $\lambda_{j,k}$ of the $K$-algebra $K[\lambda]$. Hence, these two homomorphisms must be identical. In other words, $\epsilon_L \circ \psi = \omega_L$. Applying both sides of this equality to $f$, we find $(\epsilon_L \circ \psi)(f) = \omega_L(f) = f[L]$ (by the definition of $\omega_L$). Hence, $f[L] = (\epsilon_L \circ \psi)(f) = \epsilon_L \left( \underbrace{\psi(f)}_{=F} \right) = \epsilon_L(F)$, qed.

$f - c(\det\lambda)^g = 0$. Thus, $f = c(\det\lambda)^g$. Hence, (64) becomes

$$\begin{aligned} F &= \underbrace{f}_{=c(\det\lambda)^g} \left( \left( (x_j \mid u_k) \right)_{1\leq j\leq d,\ 1\leq k\leq d} \right) \\ &= \left( c(\det\lambda)^g \right) \left( \left( (x_j \mid u_k) \right)_{1\leq j\leq d,\ 1\leq k\leq d} \right) \\ &= c \left( \underbrace{\det\left( (x_j \mid u_k) \right)_{1\leq j\leq d,\ 1\leq k\leq d}}_{=(x_1\cdots x_d \mid u_1\cdots u_d)} \right)^g = c\,(x_1\cdots x_d \mid u_1\cdots u_d)^g . \end{aligned}$$

This proves Lemma 5.20. □

**Lemma 5.21.** If $F(\mathbf{x}_1,\ldots,\mathbf{x}_m) \in P_m$ is a nonzero homogeneous invariant form with $m < d$, then $F$ is constant.

*Proof of Lemma 5.21.* Let $F(\mathbf{x}_1,\ldots,\mathbf{x}_m) \in P_m$ be a nonzero homogeneous invariant form with $m < d$.

If $n < d$, then $P_n$ canonically becomes a $K$-subalgebra of $P_d$ (by (63)). Hence, we WLOG assume that $n \geq d$ (because if $n < d$, then we can replace $n$ by $d$, and perform the argument below in $P_d$ instead of $P_n$).

Now, $m < d$. Hence, (63) (applied to $d$ instead of $n$) shows that $P_m$ canonically becomes a $K$-subalgebra of $P_d$. Thus, $P_m \subseteq P_d$.

Now, $F(\mathbf{x}_1,\ldots,\mathbf{x}_m) \in P_m \subseteq P_d$. Therefore, we can write $F(\mathbf{x}_1,\ldots,\mathbf{x}_d)$ for $F(\mathbf{x}_1,\ldots,\mathbf{x}_m)$. Since this form $F(\mathbf{x}_1,\ldots,\mathbf{x}_d) = F(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ is invariant, we can thus apply Lemma 5.20, and conclude that

$$F(\mathbf{x}_1,\ldots,\mathbf{x}_d) = c\,(x_1\cdots x_d \mid u_1\cdots u_d)^g \tag{66}$$

for some nonnegative integer $g$, and scalar $c$. Consider these $g$ and $c$.

Consider the $K$-algebra homomorphism

$$\eta : P_d \to P_m, \qquad (x_j \mid u_k) \mapsto \begin{cases} (x_j \mid u_k), & \text{if } 1 \leq j \leq m; \\ 0, & \text{otherwise} \end{cases}.$$

This homomorphism $\eta$ sends

$$(x_d \mid u_k) \mapsto \begin{cases} (x_d \mid u_k), & \text{if } 1 \leq d \leq m; \\ 0, & \text{otherwise} \end{cases} = 0 \qquad (\text{since } d > m)$$

for each $k$. Thus, it sends the whole $d$-th row of the determinant $(x_1\cdots x_d \mid u_1\cdots u_d)$ to 0. Consequently, it also sends this determinant to 0. In other words,

$$\eta\,(x_1\cdots x_d \mid u_1\cdots u_d) = 0.$$

But on the other hand, $\eta$ acts as the identity on the subalgebra $P_m$ of $P_d$ (since it fixes $(x_j \mid u_k)$ whenever $1 \leq j \leq m$); therefore, it fixes $F(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ (since $F(\mathbf{x}_1, \ldots, \mathbf{x}_d) = F(\mathbf{x}_1, \ldots, \mathbf{x}_m) \in P_m$). In other words, $\eta(F(\mathbf{x}_1, \ldots, \mathbf{x}_d)) = F(\mathbf{x}_1, \ldots, \mathbf{x}_d)$. Now, applying the map $\eta$ to the equality (66), we find

$$\begin{aligned}\eta(F(\mathbf{x}_1, \ldots, \mathbf{x}_d)) &= \eta\left(c(x_1 \cdots x_d \mid u_1 \cdots u_d)^g\right) = c\left(\underbrace{\eta(x_1 \cdots x_d \mid u_1 \cdots u_d)}_{=0}\right)^g \\ &\qquad \text{(since } \eta \text{ is a } K\text{-algebra homomorphism)} \\ &= c0^g.\end{aligned}$$

Hence, $c0^g = \eta(F(\mathbf{x}_1, \ldots, \mathbf{x}_d)) = F(\mathbf{x}_1, \ldots, \mathbf{x}_d) \neq 0$ (since $F$ is nonzero). Consequently, $0^g \neq 0$. Therefore, $g = 0$. Hence, (66) rewrites as $F(\mathbf{x}_1, \ldots, \mathbf{x}_d) = c(x_1 \cdots x_d \mid u_1 \cdots u_d)^0 = c$. Therefore, $F$ is constant. This proves Lemma 5.21. $\square$

The next lemma is a simple variation on the multinomial theorem.

**Lemma 5.22.** Let $F \in P_n$ be a homogeneous form of degree $t$. Let $F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \in P_d[\lambda]$ be the result of substituting $\sum_{i=1}^{d} \lambda_{j,i}(x_i \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ (with $1 \leq j \leq n$ and $1 \leq r \leq d$) in $F$.

**(a)** Then,

$$\begin{aligned}&F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \\ &= \sum \left(\prod_{j=1}^{n} \prod_{r=1}^{d} \lambda_{j,r}^{i(j,r)}\right) \cdot F_{i(1,1),i(1,2),\ldots,i(n,d)}(\mathbf{x}_1, \ldots, \mathbf{x}_d),\end{aligned} \tag{67}$$

where the summation is over all families $(i(1,1), i(1,2), \ldots, i(n,d)) = (i(j,r))_{1 \leq j \leq n,\ 1 \leq r \leq d} \in \mathbb{N}^{n \times d}$ of nonnegative integers satisfying

$$\sum_{q=1}^{d} \sum_{p=1}^{n} i(p,q) = t,$$

and where the polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)} \in P_d$ are homogeneous of degree $t$.

**(b)** Assume that $F$ is invariant. Then, each of the polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)} \in P_d$ is invariant.

*Proof of Lemma 5.22.* **(a)** The polynomial ring $P_d[\lambda]$ can be regarded as a polynomial ring over $K$ in the $nd + d^2$ indeterminates

$$\begin{aligned}(x_j \mid u_k) &\qquad \text{for } 1 \leq j \leq n \text{ and } 1 \leq k \leq d, \textbf{ as well as} \\ \lambda_{j,k} &\qquad \text{for } 1 \leq j \leq d \text{ and } 1 \leq k \leq d.\end{aligned}$$

Let us consider this polynomial ring $P_d[\lambda]$ as a $\mathbb{Z} \times \mathbb{Z}$-graded $K$-algebra, where each of the variables $(x_j \mid u_k)$ has degree $(1,0)$ whereas the variables $\lambda_{j,k}$ have degree $(0,1)$. Thus, the sums $\sum\limits_{i=1}^{d} \lambda_{j,i} (x_i \mid u_r) \in P_d[\lambda]$ (for all $1 \leq j \leq n$ and $1 \leq r \leq d$) are homogeneous of degree $(1,1)$. Therefore, substituting these sums for the indeterminates $(x_j \mid u_r)$ transforms any homogeneous polynomial $G \in P_n$ of degree $t$ into a homogeneous polynomial in $P_d[\lambda]$ of degree $(t,t)$ [257]. Therefore, the polynomial $F\left(\sum\limits_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum\limits_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \in P_d[\lambda]$ is homogeneous of degree $(t,t)$ (since it is obtained by this substitution from the polynomial $F \in P_n$, which is homogeneous of degree $t$). Hence, it can be written in the form

$$\begin{aligned} & F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \\ &= \sum \left(\prod_{j=1}^{n}\prod_{r=1}^{d} \lambda_{j,r}^{i(j,r)}\right) \cdot F_{i(1,1),i(1,2),\ldots,i(n,d)}(\mathbf{x}_1, \ldots, \mathbf{x}_d), \end{aligned}$$

where the summation is over all families $(i(1,1), i(1,2), \ldots, i(n,d)) = (i(j,r))_{1 \leq j \leq n,\ 1 \leq r \leq d} \in \mathbb{N}^{n \times d}$ of nonnegative integers satisfying

$$\sum_{q=1}^{d} \sum_{p=1}^{n} i(p,q) = t,$$

and where the polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)} \in P_d$ are homogeneous of degree $t$. This proves Lemma 5.22 **(a)**.

**(b)** Every $d \times d$-matrix $L \in K^{d \times d}$ acts on $P_d$ as a $K$-algebra endomorphism, and thus also acts on $P_d[\lambda]$ as a $K[\lambda]$-algebra endomorphism[258]. This leads to a notion of invariant elements of $P_d[\lambda]$: Namely, we say that a $G \in P_d[\lambda]$ is *invariant* if and only if, for all invertible $d \times d$-matrices $L \in K^{d \times d}$, there exists a scalar $a(L) \in K$ such that $LG = a(L)G$.

Now, let us show the following claim:

*Claim 1:* Each $G \in P_n$ satisfies

$$(LG)\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) = L \cdot G\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right).$$

[257] If you don't believe this, just check it in the case when $G$ is a monomial (from which the general case follows by linearity).

[258] The action of a $d \times d$-matrix $L \in K^{d \times d}$ on $P_d[\lambda]$ is obtained by extending the action of $L$ on $P_d$ as a $K[\lambda]$-algebra endomorphism. Thus, if we regard an element of $P_d[\lambda]$ as a polynomial in the $\lambda_{j,k}$ with coefficients in $P_d$, then $L$ acts on each of these coefficients separately.

[*Proof of Claim 1:* We consider the two maps

$$P_n \to P_d[\lambda], \qquad G \mapsto (LG)\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right)$$

and

$$P_n \to P_d[\lambda], \qquad G \mapsto L\cdot G\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right).$$

Our goal is to show that these two maps are equal. Since both of these maps are $K$-algebra homomorphisms[259], we only need to check that these two maps take equal values on the generators $(x_j \mid u_r)$ of $P_n$. In other words, we need to prove Claim 1 in the case when $G = (x_j \mid u_r)$ for some $1 \leq j \leq n$ and $1 \leq r \leq d$.

So let us fix $1 \leq j \leq n$ and $1 \leq r \leq d$. Let us set $G = (x_j \mid u_r)$. We must prove Claim 1 for this $G$.

Write the $d \times d$-matrix $L$ in the form $L = (l_{kj})_{1\leq j,k\leq d}$. We have $G = (x_j \mid u_r)$ and thus

$$LG = L(x_j \mid u_r) = \sum_{1\leq k\leq d} l_{rk}(x_j \mid u_k)$$

(by the definition of the action of $L$ on $P_n$). Hence, the definition of $(LG)\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right)$ yields

$$\begin{aligned}(LG)\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right) &= \sum_{1\leq k\leq d} l_{rk}\sum_{i=1}^d \lambda_{j,i}(x_i \mid u_k) \\ &= \sum_{1\leq k\leq d}\sum_{i=1}^d l_{rk}\lambda_{j,i}(x_i \mid u_k). \qquad (68)\end{aligned}$$

On the other hand, $G = (x_j \mid u_r)$. Hence, the definition of $G\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right)$ yields

$$G\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right) = \sum_{i=1}^d \lambda_{j,i}(x_i \mid u_r).$$

[259]This is because $L$ acts as $K$-algebra endomorphisms on both $P_d$ and $P_d[\lambda]$, and because the map

$$P_n \to P_d[\lambda], \qquad G \mapsto G\left(\sum_{i=1}^d \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^d \lambda_{n,i}\mathbf{x}_i\right)$$

is a $K$-algebra homomorphism as well.

Hence,

$$\begin{aligned}
L \cdot G\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) &= L \cdot \sum_{i=1}^{d} \lambda_{j,i}\left(x_i \mid u_r\right) \\
&= \sum_{i=1}^{d} \lambda_{j,i} \qquad \underbrace{L\left(x_i \mid u_r\right)}_{\substack{= \sum\limits_{1 \leq k \leq d} l_{rk}(x_i \mid u_k) \\ \text{(by the definition of the action of } L\text{)}}} \\
&= \sum_{i=1}^{d} \lambda_{j,i} \sum_{1 \leq k \leq d} l_{rk}\left(x_i \mid u_k\right) \\
&= \sum_{1 \leq k \leq d} \sum_{i=1}^{d} l_{rk}\lambda_{j,i}\left(x_i \mid u_k\right).
\end{aligned}$$

Comparing this with (68), we obtain

$$(LG)\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) = L \cdot G\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right).$$

Thus, Claim 1 is proven for $G = \left(x_j \mid u_r\right)$. As we have already explained, this completes the proof of Claim 1.]

Now, consider our invariant form $F \in P_n$. Thus, any invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies $LF = a(L) F$ for some scalar $a(L)$ (since $F$ is invariant). Now, any invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies

$$\begin{aligned}
L \cdot F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) &= \underbrace{(LF)}_{=a(L)F}\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \\
&\qquad \text{(by Claim 1, applied to } G = F\text{)} \\
&= a(L) F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right).
\end{aligned}$$

Hence, the element $F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \in P_d[\lambda]$ is invariant.

But it is easy to see that if a polynomial $G \in P_d[\lambda]$ is invariant, then each of its coefficients is invariant[260] (because $d \times d$-matrices $L \in K^{d \times d}$ act on $P_d[\lambda]$ by acting on each coefficient separately). Applying this to $G = F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)$, we conclude that each coefficient of $F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)$ is invariant. But

[260] Here, we are regarding $G$ as a polynomial in the indeterminates $\lambda_{k,j}$ over the ring $P_d$. Thus, the indeterminates in $P_d$ are regarded as constants, not as indeterminates.

the coefficients of $F\left(\sum_{i=1}^{d}\lambda_{1,i}\mathbf{x}_i,\ldots,\sum_{i=1}^{d}\lambda_{n,i}\mathbf{x}_i\right)$ are none other than the polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)}\in P_d$ (by the equality (67)). Hence, we have shown that each of the polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)}\in P_d$ is invariant. This proves Lemma 5.22 **(b)**. $\square$

The preliminary lemmas are now disposed of. Let $F\in P=P_n$ be a homogeneous invariant form of degree $t$. We must show that $F$ is a linear combination of standard rectangular bideterminants. Due to Lemma 5.15, it will suffice to show that $F$ is a linear combination of bracket monomials. We can WLOG assume that $F$ is nonconstant (since constant forms are clearly linear combinations of bracket monomials); thus, $t>0$. If we had $n<d$, then Lemma 5.21 (applied to $m=n$) would yield that $F$ is constant. Thus, we cannot have $n<d$. Hence, we have $n\geq d$.

Let $P^B$ be the $K$-subalgebra of $P=P_n$ generated by all brackets.

Let $P_d^I$ be the $K$-subalgebra of $P_d$ consisting of all forms polynomials in $P_d$. Then, $P_d^I$ is spanned (as a $K$-vector space) by all homogeneous invariants in $P_d$; but all the latter are scalar multiples of powers of $(x_1\cdots x_d\mid u_1\cdots u_d)$ (by Lemma 5.20). Hence, $P_d^I$ is spanned (as a $K$-vector space) by scalar multiples of powers of $(x_1\cdots x_d\mid u_1\cdots u_d)$. In other words, $P_d^I$ is generated as a $K$-algebra by the form $(x_1\cdots x_d\mid u_1\cdots u_d)$. In other words, $P_d^I$ is generated as a $K$-algebra by the bracket $[x_1,\ldots,x_d]$ (since $[x_1,\ldots,x_d]=(x_1\cdots x_d\mid u_1\cdots u_d)$). Hence, $P_d^I\subseteq P^B$ (since $P^B$ is the $K$-subalgebra of $P$ generated by **all** brackets).

For each $1\leq j\leq n$ and $1\leq k\leq d$, we define a bracket $B_{j,k}\in P$ by

$$B_{j,k}=\left[x_1,\ldots,x_{k-1},x_j,x_{k+1},\ldots,x_d\right]. \tag{69}$$

Then, every $1\leq j\leq n$ and $1\leq r\leq d$ satisfy

$$\begin{aligned}
&[x_1,\ldots,x_d]\left(x_j\mid u_r\right)\\
&=\sum_{k=1}^{d}\underbrace{\left[x_1,\ldots,x_{k-1},x_j,x_{k+1},\ldots,x_d\right]}_{\substack{=B_{j,k}\\ \text{(by (69))}}}\left(x_k\mid u_r\right) \qquad \text{(by Lemma 5.17, applied to } m=r\text{)}\\
&=\sum_{k=1}^{d}B_{j,k}\left(x_k\mid u_r\right)=\sum_{i=1}^{d}B_{j,i}\left(x_i\mid u_r\right).
\end{aligned} \tag{70}$$

Lemma 5.22 **(a)** yields that the polynomial $F\left(\sum_{i=1}^{d}\lambda_{1,i}\mathbf{x}_i,\ldots,\sum_{i=1}^{d}\lambda_{n,i}\mathbf{x}_i\right)\in P_d[\lambda]$ (defined as in Lemma 5.22) can be written in the form (67), where the summation is over all families $(i(1,1),i(1,2),\ldots,i(n,d))=(i(j,r))_{1\leq j\leq n,\ 1\leq r\leq d}\in\mathbb{N}^{n\times d}$ of nonnegative integers satisfying $\sum_{q=1}^{d}\sum_{p=1}^{n}i(p,q)=t$, and where the polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)}\in P_d$ are homogeneous of degree $t$. Consider these polynomials $F_{i(1,1),i(1,2),\ldots,i(n,d)}\in P_d$. Lemma 5.22 **(b)** shows that each of these polynomials

$F_{i(1,1),i(1,2),\ldots,i(n,d)} \in P_d$ is invariant, i.e., belongs to $P_d^I$. Thus, the equality (67) shows that

$$F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right) \in P_d^I[\lambda].$$

Let $\omega : P_d^I[\lambda] \to P^B$ be the unique $P_d^I$-algebra homomorphism sending each indeterminate $\lambda_{j,k}$ to $B_{j,k}$. This is well-defined, since both $B_{j,k} \in P^B$ for all $j$ and $k$ (since $B_{j,k}$ is a bracket) and $P_d^I \subseteq P^B$.

We claim that the polynomial

$$[x_1, \ldots, x_d]^t F$$

equals a polynomial in the brackets $\left[x_{i_1}, \ldots, x_{i_d}\right]$. Indeed, as $F$ is homogeneous of degree $t$, we have

$$[x_1, \ldots, x_d]^t \cdot F = F\left([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n\right) \tag{71}$$

[261]. But

$$F\left([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n\right) = \omega\left(F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)\right) \tag{72}$$

[262]. Hence, (71) becomes

$$\begin{aligned}[x_1, \ldots, x_d]^t \cdot F &= F\left([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n\right) \\ &= \omega\left(F\left(\sum_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)\right) \in P^B.\end{aligned}$$

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[261] We are here using the following notation: If $a_1, a_2, \ldots, a_n$ are $n$ elements of $P_n$, then $F(a_1\mathbf{x}_1, \ldots, a_n\mathbf{x}_n)$ denotes the result of substituting $a_j \cdot (x_j \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ (with $1 \leq j \leq n$ and $1 \leq r \leq d$) in $F$.

[262] *Proof of (72):* Recall that $F\left(\sum\limits_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum\limits_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)$ is the result of substituting $\sum\limits_{i=1}^{d} \lambda_{j,i}(x_i \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ in $F$. Applying the map $\omega$ further replaces each $\lambda_{j,i}$ by $B_{j,i}$ (by the definition of $\omega$). Combining these two facts, we conclude that $\omega\left(F\left(\sum\limits_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum\limits_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)\right)$ is the result of substituting $\sum\limits_{i=1}^{d} B_{j,i}(x_i \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ in $F$.

But the definition of $F([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n)$ shows that $F([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n)$ is the result of substituting $[x_1, \ldots, x_d](x_j \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ in $F$. In light of (70), this rewrites as follows: $F([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n)$ is the result of substituting $\sum\limits_{i=1}^{d} B_{j,i}(x_i \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ in $F$.

Hence, both forms $F([x_1, \ldots, x_d]\,\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\,\mathbf{x}_n)$ and $\omega\left(F\left(\sum\limits_{i=1}^{d} \lambda_{1,i}\mathbf{x}_i, \ldots, \sum\limits_{i=1}^{d} \lambda_{n,i}\mathbf{x}_i\right)\right)$ have been characterized in the same way (namely, as the result of substituting $\sum\limits_{i=1}^{d} B_{j,i}(x_i \mid u_r)$ for each indeterminate $(x_j \mid u_r)$ in $F$). Thus, these two forms are equal. In other words, (72) is proven.

In other words, $[x_1, \ldots, x_d]^t \cdot F$ belongs to the $K$-subalgebra of $P$ generated by all brackets (since $P^B$ is the $K$-subalgebra of $P$ generated by all brackets). In other words, $[x_1, \ldots, x_d]^t \cdot F$ is a polynomial in the brackets. Hence, Lemma 5.16 (applied to $g = t$) shows that $F$ is a linear combination of standard rectangular bideterminants. This completes the second proof of Theorem 5.9. □

## 5.5. A note on homogeneous forms

Theorem 5.9 quickly yields the following corollary:

**Corollary 5.23.** Let $K$ be an infinite field. A homogeneous form in $P$ is invariant iff it is a linear combination of standard rectangular bideterminants, all of which have the same shape $(d, \ldots, d)$.

*Proof of Corollary 5.23.* The "if" part of Corollary 5.23 is clearly true (since rectangular bideterminants are always invariant). Thus, it remains to prove the "only if" part.

Let $F$ be an invariant homogeneous form in $P$. We must show that $F$ is a linear combination of standard rectangular bideterminants, all of which have the same shape $(d, \ldots, d)$.

The form $F$ is homogeneous; let $t$ be its degree.

Theorem 5.9 shows that $F$ is a linear combination of standard rectangular bideterminants. In other words, $F$ can be written in the form

$$F = \sum_{i \in I} b_i \left( U_i \mid U_i' \right), \tag{73}$$

where the $\left[U_i, U_i'\right]$ are standard bitableaux such that the tableaux $U_i'$ are rectangular, and where the $b_i$ are elements of $K$.

For each $i \in I$, we let $g_i$ be the number of rows of the tableau $U_i'$.

Now, let $i \in I$. Then, the tableau $U_i'$ is rectangular and has $g_i$ rows (by the definition of $g_i$). Consequently, this tableau must have shape $\underbrace{(d, \ldots, d)}_{g_i \text{ entries}}$. Hence, this tableau has $dg_i$ cells. Therefore,

$$\text{the bideterminant } \left( U_i \mid U_i' \right) \in P \text{ is homogeneous of degree } dg_i \tag{74}$$

(because the bideterminant $(U \mid U')$ of a bitableau $[U, U']$ always is homogeneous of degree equal to the number of cells of $U'$).

Moreover, the tableau $U_i'$ has shape $\underbrace{(d, \ldots, d)}_{g_i \text{ entries}}$. In other words,

$$\text{the bideterminant } \left( U_i \mid U_i' \right) \text{ has shape } \underbrace{(d, \ldots, d)}_{g_i \text{ entries}}. \tag{75}$$

Now, forget that we fixed $i$. We thus have proven (74) and (75) for each $i \in I$.

Now, let $\pi_t : P \to P$ be the projection of the graded ring $P$ onto its $t$-th homogeneous component. Then, $\pi_t(F) = F$ (since $F$ is homogeneous of degree $t$). But applying the map $\pi_t$ to both sides of the equality (73), we obtain

$$\begin{aligned}
\pi_t(F) &= \pi_t\left(\sum_{i\in I} b_i \left(U_i \mid U_i'\right)\right) = \sum_{i\in I} b_i \pi_t\left(U_i \mid U_i'\right) \\
&= \sum_{\substack{i\in I;\\ dg_i = t}} b_i \underbrace{\pi_t\left(U_i \mid U_i'\right)}_{\substack{=\left(U_i \mid U_i'\right)\\ \text{(since (74) shows that } \left(U_i \mid U_i'\right)\\ \text{is homogeneous of degree } dg_i = t)}} + \sum_{\substack{i\in I;\\ dg_i \neq t}} b_i \underbrace{\pi_t\left(U_i \mid U_i'\right)}_{\substack{=0\\ \text{(since (74) shows that } \left(U_i \mid U_i'\right)\\ \text{is homogeneous of degree } dg_i \neq t)}} \\
&= \sum_{\substack{i\in I;\\ dg_i = t}} b_i \left(U_i \mid U_i'\right) + \underbrace{\sum_{\substack{i\in I;\\ dg_i \neq t}} b_i 0}_{=0} = \sum_{\substack{i\in I;\\ dg_i = t}} b_i \left(U_i \mid U_i'\right) = \sum_{\substack{i\in I;\\ g_i = t/d}} b_i \left(U_i \mid U_i'\right).
\end{aligned}$$

Comparing this with $\pi_t(F) = F$, we obtain

$$F = \sum_{\substack{i\in I;\\ g_i = t/d}} b_i \left(U_i \mid U_i'\right). \tag{76}$$

But for each $i \in I$ satisfying $g_i = t/d$, the shape of the bideterminant $\left(U_i \mid U_i'\right)$ is $\underbrace{(d, \ldots, d)}_{g_i \text{ entries}}$ (by (75)). In other words, for each $i \in I$ satisfying $g_i = t/d$, the shape of the bideterminant $\left(U_i \mid U_i'\right)$ is $\underbrace{(d, \ldots, d)}_{t/d \text{ entries}}$ (since $g_i = t/d$). Thus, (76) shows that $F$ is a linear combination of standard rectangular bideterminants, all of which have the same shape $\underbrace{(d, \ldots, d)}_{t/d \text{ entries}}$. This proves Corollary 5.23. $\square$

# 6. Appendix (Darij Grinberg)

Let me add some remarks on the paper above.

This paper is one of the first studies in characteristic-free invariant theory (i.e., invariant theory over a field $K$ of arbitrary characteristic). What is called an *invariant form* in this paper is more or less the same as (what is nowadays known as) an invariant of the special linear group $\operatorname{SL}(d)$ acting on the coordinate ring $\mathcal{O}\left(\left(K^d\right)^n\right)$, where $\operatorname{SL}(d)$ acts on $K^d$ in the usual way. (This situation is commonly known as "invariants of $\operatorname{SL}(d)$ on $n$ vectors".) An equivalent version of Theorem 5.9 appears already in [13, Theorem 4], but the proof given there is far too brief to be considered complete. Theorem 5.9 also appears in [3, Section 9, Theorem 1], where it is proven more or less in the same way as in the paper above. (Actually, the paper [3] can be regarded as a preliminary version of the above paper; it offers much motivation and neat applications to elementary geometry, but is lacking in mathematical rigor.)

On the other hand, many more works have appeared since the publication of the paper above. I shall mention only a few of them:

- The paper [9] by DeConcini, Eisenbud and Procesi studies the algebra that we call $P$ in Theorem 5.9 in much more depth. In particular, our Theorem 5.9 is (more or less) [9, Corollary 3.5][263]. Also, [9, Theorem 2.1] combines the above paper's Theorem 3.2 and Corollary 4.8 (and generalizes them to the case where $K$ is an arbitrary commutative ring).

- The booklet [10] by Grosshans, Rota and Stein appears to extend the above considerations to the "superalgebra case" (i.e., forms containing "positively" and "negatively" signed variables). (I have not read the booklet.)

- Swan's preprint [11] gives a different approach to the straightening algorithm. Specifically, the above paper's Corollary 4.8 is covered by [11, Corollary 5.1 and Theorem 5.3] (and it is not hard to derive Theorem 3.2 from [11, Theorem 4.1] as well).

- Procesi's book [12] covers characteristic-free invariant theory in its Chapter 13. In particular, [12, Section 13.4, Theorem] is exactly Corollary 4.8 but using the base ring $\mathbb{Z}$ instead of the field $K$; furthermore, [12, Section 13.6.3] comes rather close to the Theorem 5.9 above.

Let me now briefly discuss the possibility of extending some of the results in the above paper to the more general situation when $K$ is an arbitrary commutative ring (rather than a field).

Assume from now on that $K$ (instead of being a field) is just a commutative ring (with unity).

Then, all claims and proofs made in Section 3 remain valid. (Of course, we need to make obvious modifications, such as replacing the word "vector space" by "module" everywhere.)

Furthermore, all claims and proofs made in Section 4 remain valid, as long as the following modification is made: The claim of Theorem 4.5 (i) must be replaced by "The one-element family $(C(T, T')(T \mid T'))$ is $K$-linearly independent (i.e., if an

[263] The proof of [9, Corollary 3.5] is somewhat similar to our first proof of Theorem 5.9, but is slicker. As an intermediate result, it also characterizes the forms $F \in P$ that are "invariant under lower-unitriangular matrices" (i.e., that satisfy $LF = F$ for every lower-unitriangular $d \times d$-matrix $L \in K^{d \times d}$); namely, these forms are linear combinations of standard bideterminants $(U \mid U')$ such that each row of the tableau $U'$ has the form $u_i u_{i+1} \cdots u_d$ for some $i \in \{1, 2, \ldots, d\}$. (See [9, Theorem 3.3 0)] for this result; but beware that what we call $LF$ for an invertible $d \times d$-matrix $L \in K^{d \times d}$ and a form $F \in P$ corresponds to what [9] would call $\left(L^T\right)^{-1} F$, which is why our lower-triangular matrices correspond to upper-triangular matrices in [9].)

$a \in K$ satisfies $aC(T, T')(T \mid T') = 0$, then $a = 0$).". (The proof of this new claim is exactly the proof of Theorem 4.5 (i) we gave above.)

All claims and proofs made in Subsection 5.1 remain valid (except for the arguments in Remark 5.8, which are unimportant), as long as the following modification is made: The equality (15) must be replaced by "The one-element family $(\phi(T \mid T'))$ is $K$-linearly independent (i.e., if an $a \in K$ satisfies $\phi(T \mid T') = 0$, then $a = 0$).". (The proof of this new claim is exactly the proof of Lemma 5.7 we gave above.)

Much more complicated is the situation with Subsections 5.2, 5.3 and 5.4. Theorem 5.9, to begin with, doesn't even hold when $K$ is a finite field[264]. But merely requiring $K$ to be infinite is not sufficient when $K$ is not a field.[265] Instead, a meaningful way to salvage Theorem 5.9 is to replace the concept of invariant by the concept of an *absolute invariant*:

A form $F \in P$ is said to be an *absolute invariant* if it satisfies $LF = a(L)F$ not just for invertible $d \times d$-matrices $L \in K^{d \times d}$ defined over $K$, but also for invertible $d \times d$-matrices $L \in A^{d \times d}$ defined over any commutative $K$-algebra $A$ (where $a(L)$ is now required to be an element of $A$). In particular, this allows for an invertible

[264]Indeed, let $K$ be the finite field $\mathbb{F}_q$ for some prime power $q$. Set $n = 1$. Then, the ring $P$ is the polynomial ring over $K$ in the indeterminates $(x_1 \mid u_j)$ for all $1 \leq j \leq d$. Let us denote these indeterminates by $X_j$; thus, $P = K[X_1, X_2, \ldots, X_d]$. As before, the group of all invertible $d \times d$-matrices $L \in K^{d \times d}$ acts on $P$ by algebra endomorphisms. If the conclusion of Theorem 5.9 were to hold for our field $K$, then, for any $d \geq 2$, the only invariant forms in $P$ would be constants (since the only standard rectangular bideterminant in this setting is the empty one $(\ \mid\ )$, because $d \geq 2 > 1 = n$ ensures that a rectangular bitableau cannot be standard). But this is not the case: For each $d$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ of nonnegative integer, the determinant

$$A_\alpha = \det \begin{pmatrix} X_1^{q^{\alpha_1}} & X_2^{q^{\alpha_1}} & \cdots & X_d^{q^{\alpha_1}} \\ X_1^{q^{\alpha_2}} & X_2^{q^{\alpha_2}} & \cdots & X_d^{q^{\alpha_2}} \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{q^{\alpha_d}} & X_2^{q^{\alpha_d}} & \cdots & X_d^{q^{\alpha_d}} \end{pmatrix}$$

is an invariant in the sense of the above paper (more precisely, it has the property that every $d \times d$-matrix $L \in K^{d \times d}$ satisfies $LA_\alpha = (\det L)\, A_\alpha$; see [16, 7th Variation] for this). If the entries of the $d$-tuple $\alpha$ are distinct (and if $d \geq 1$), then $A_\alpha$ is not a constant. Thus, the requirement that $K$ is infinite in Theorem 5.9 cannot be dropped!

The phenomenon of these nonconstant invariants over finite fields is interesting in its own right; it has been studied already by Dickson in 1911. We can find nonconstant forms in $F$ that are not only invariant in the above sense, but actually satisfy the stronger condition that each invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies $LF = F$ (not just $LF = a(L)F$ for some scalar $a(L)$). Indeed, such invariants can be constructed from the above determinants $A_\alpha$; namely, if we let $\delta$ be the $d$-tuple $(d-1, d-2, \ldots, 0)$, then it can be shown that $A_\alpha / A_\delta$ is a polynomial for every $d$-tuple $\alpha$, and this polynomial $A_\alpha / A_\delta$ satisfies the stronger condition (i.e., if we set $F = A_\alpha / A_\delta$, then each invertible $d \times d$-matrix $L \in K^{d \times d}$ satisfies $LF = F$). Polynomials satisfying this condition have been characterized by Dickson (see, e.g., [15, Theorem 1.2] or [14, Theorem A]).

[265]Counterexamples can easily be obtained when $K$ is the (infinite) ring $\mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q \times \cdots$ for a prime power $q$.

$d \times d$-matrix whose entries are distinct indeterminates. This makes the $LF = a(L)F$ condition stronger, and as one can see, this added strength suffices to make Theorem 5.9 valid for every commutative ring $K$:

**Theorem 6.1** (the first fundamental theorem of invariant theory)**.** Let $K$ be a commutative ring. A form in $P$ is an absolute invariant iff it is a linear combination of standard rectangular bideterminants.

The first proof of Theorem 5.9 given above actually can be extended (with some additional work) to prove Theorem 6.1[266]. (The second proof can probably not be extended, at least not easily.)

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[266] The exact place where we need the form to be an absolute invariant (instead of just being invariant) is the definition of the matrix $L$ that acts on $P$ by (31) and (32). In the original proof of Theorem 5.9, it was sufficient to let $c$ be a nonzero scalar in $K$. However, if $K$ is not an infinite field, there may not be enough nonzero scalars to make such a definition useful. (In particular, we might not be able to derive the equality (37) from knowing that every nonzero scalar $c \in K$ satisfies (35).) Thus, instead of letting $c$ be a nonzero scalar in $K$, we have to set $c$ to be the indeterminate $\mathbf{t}$ in the univariate polynomial ring $K[\mathbf{t}]$. The matrix $L$ will thus be a matrix in $(K[\mathbf{t}])^{d\times d}$, not in $K^{d\times d}$. Assuming that $F$ is an absolute invariant, we shall then be able to deduce (37) from (35) quickly: Every two elements $s$ and $t$ satisfy $\alpha_s c^{b_j^{(s)}} = \alpha_s a(L) = \alpha_s c^{b_j^{(t)}}$; this rewrites as $\alpha_s \mathbf{t}^{b_j^{(s)}} = \alpha_s \mathbf{t}^{b_j^{(t)}}$; but since $\alpha_s$ is nonzero and $\mathbf{t}$ is an indeterminate, this can only happen if $b_j^{(s)} = b_j^{(t)}$.

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[267] *Correction:* The references below have been inserted by me. Also, I have taken the liberty to fix two typos in the references [7] and [8] above, as well as include URLs, DOIs and ISBNs whenever I could find them.

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