The Geometry and Topology of Coxeter Groups
Michael W. Davis
Second Edition 2025
Comments and corrections by Darij Grinberg

The following list contains my comments to Michael W. Davis's book *The Geometry and Topology of Coxeter Groups*, specifically its 2nd edition (Springer, 2025). Among these comments are a few corrections (most importantly, the missing condition in Definition 3.2.1), several minor nitpicks, a few reformulations that I made for my own understanding, and a number of alternative proofs that I invented as I was unable to follow some of the existing proofs. In the latter case, I genuinely don't know whether the fault lies with me or the book, as I am new to the subject and come from a combinatorics background.

I have read Chapter 3 (minus Example 3.1.7, the Remark after Theorem 3.3.4, the paragraph after Lemma 3.3.5, and §3.5) and Chapter 4 until Remark 4.8.1, as well as §4.10 until Definition 4.10.3.

N. Errata and comments

1. **Page 23, Example 3.1.4:** In "of all pairs $(x, \varepsilon) \in \mathbf{C}_m \times C_2$", replace "$C_2$" by "$\mathbf{C}_2$".

2. **Page 23, Example 3.1.4:** The equality "$(x, \varepsilon) \cdot (x', \varepsilon') = \left(x^{\varepsilon'} x, \varepsilon\varepsilon'\right)$" should be "$(x, \varepsilon) \cdot (x', \varepsilon') = \left(x^{\varepsilon'} x', \varepsilon\varepsilon'\right)$".

3. **Page 23, Lemma 3.1.5:** After "by distinct elements s and t", add "of order 2".

4. **Page 23, Lemma 3.1.5 (i):** Remove the comma in "the semidirect product,".

5. **Page 26, proof of Lemma 3.1.5 (ii):** In the last sentence, it is worth pointing out that $sps^{-1} = p^{-1}$ is being used to show that φ really is an isomorphism.

6. **Page 26, Example 3.1.7:** "the the real line" $\to$ "the real line".

7. **Page 26, proof of Lemma 3.1.8:** "Extend this to a function" $\to$ "Extend this to a multiplicative function".

8. **Page 27, Definition 3.2.1:** This needs one more condition:

   (e) each $r \in R$ flips at least one edge of Ω.

   This is important (among other things) in order to make Lemma 3.2.5 hold, and even earlier than that, it is used in arguing that R is the set of all conjugates of S. And indeed, without the requirement **(e)**, Lemma 3.2.5 would be false[1].

9. **Page 27:** "the involution wsw^{-1} flips e and since r is the unique such involution" → "the pre-reflection wsw^{-1} flips e and since r is the unique such pre-reflection".

10. **Page 27, (3.1):** I would extend this equality as follows:

$$r_i := w_{i-1}s_iw_{i-1}^{-1} = w_iw_{i-1}^{-1} = (s_1s_2\cdots s_i)(s_1s_2\cdots s_{i-1})^{-1}.$$

These equalities are easy to check, but they are used several times (tacitly) in the text that follows.

11. **Page 28:** "edge path" in a graph means what graph theorists commonly call a walk, not a path. ("Path", to a graph theorist, requires the vertices to be distinct.)

12. **Page 28, after the proof of Lemma 3.2.5:** Here it would be helpful to add "Thus, if (Ω, v_0) is a pre-reflection system for W and if $S = S(v_0)$, then (W, S) is a pre-Coxeter system.".

13. **Page 29:** The combinatorially minded reader who defines graphs as collections of vertices and edges (rather than as topological spaces) should read "midpoints of those edges" as "those edges" here. Similarly, he should read "fixed point set of r" as "fixed point set of r acting on the set of vertices **and** edges of Ω". (Once you get to reflection systems, a pre-reflection r cannot fix any vertices, since Lemma 3.2.12 shows that W acts freely on $\operatorname{Vert}(\Omega)$. Thus, Ω^r consists entirely of edges when (W, S) is a reflection system. But when we are just dealing with a pre-reflection system, both vertices and edges can be fixed.[2])

14. **Page 29:** "if and only r occurs" → "if and only if r occurs".

15. **Page 30, Lemma 3.2.14:** I would replace "$\operatorname{Cay}(W, S)$" by "Ω" here for the sake of familiarity (of course, this makes no real difference, since Ω is isomorphic to $\operatorname{Cay}(W, S)$).

[1] A counterexample: Let Ω be a 6-cycle, visualized as a regular hexagon. Let W be the dihedral group D_6, acting on Ω in the obvious way. Let R be the subset of W that consists of the three reflections across the perpendicular bisectors of the sides of the hexagon (there are 6 sides but only 3 perpendicular bisectors) as well as the 180°-rotation around its center. Then, all four conditions **(a)**, **(b)**, **(c)** and **(d)** of Definition 3.2.1 are satisfied, but **(e)** is not, and Lemma 3.2.5 fails as well (the set S contains only the three perpendicular bisector reflections from R, not the 180°-rotation, and thus only generates a D_3 inside the D_6).

[2] An example is obtained when the graph Ω is a 3-cycle C_3, and W is its symmetry group S_3, with R being the set of all three involutions in S_3. This defines a pre-reflection system (choosing $v_0 \in \operatorname{Vert}(\Omega)$ arbitrarily), but not a reflection system, since each Ω^r contains a fixed vertex **and** a fixed edge.

16. **Page 31, condition (D):** The notion of an "expression" should be defined somewhere (not just "reduced expression"): An *expression* for an element $w \in W$ means a word $\left(t_1, \ldots, t_p\right)$ in S such that $w = t_1 \cdots t_p$.

17. **Page 33, proof of (E) $\Longrightarrow$ (F):** After "Applying Condition (E) to the word $(s_1, \ldots, s_k, t)$", add "(which is a reduced expression of wt)".

18. **Page 33, Definition 3.3.1:** The display
$$m_{st} = \begin{cases} 1, & \text{if } s = t; \\ \geq 2, & \text{otherwise} \end{cases}$$
looks strange (it reads "$m_{st} = \geq 2$" in the "otherwise" case). A better way to write it would be
$$m_{s,t} \begin{cases} = 1, & \text{if } s = t; \\ \geq 2, & \text{otherwise} \end{cases}$$
(or just split it into two separate displays).

19. **Page 34:** Remove the comma in "by the formula, $m_{st} = m(s,t)$".

20. **Page 34:** In the middle of the page, before "The next theorem is the main result", I would add "Recall from Example 3.2.4 that any pre-Coxeter system (W, S) becomes a pre-reflection system $\Omega = \operatorname{Cay}(W, S)$ where we define $v_0 = 1$ and let R be the set of all elements of W that are conjugate to an element of S.".

21. **Page 34, Remark after Theorem 3.3.4:** "denote dihedral subgroup" $\rightarrow$ "denote the dihedral subgroup".

22. **Page 35, Lemma 3.3.5 (i):** After "depends only on w", add "and r".

23. **Page 35, Lemma 3.3.5 (ii):** Remove the comma after "There is a homomorphism".

24. **Page 35, proof of Lemma 3.3.5:** After "Let $\mathbf{s} = (s_1, \ldots, s_k)$", add "$\in S^k$".

25. **Pages 35–36, proof of Lemma 3.3.5:** Every "$n(\mathbf{s}, r)$" here should actually be an "$n(r, \mathbf{s})$".

26. **Page 35, proof of Lemma 3.3.5:** "The case $k = 1$ is trivial" $\rightarrow$ "The case $k = 0$ is trivial".

27. **Page 35, proof of Lemma 3.3.5:** "Suppose $k > 1$" $\rightarrow$ "Suppose $k > 0$".

28. **Page 35, proof of Lemma 3.3.5:** In "$\Phi(\mathbf{s}) = \left(\Phi(\mathbf{s}'), u^{-1} s_k u\right)$", there is a little abuse of notation: The right hand side is not to be understood as a pair consisting of $\Phi(\mathbf{s}')$ and $u^{-1} s_k u$, but rather as the $(k-1)$-tuple $\Phi(\mathbf{s}')$ with an extra entry $u^{-1} s_k u$ inserted into it at its end.

29. **Page 36, proof of Lemma 3.3.5:** After "the map $\mathbf{s} \to \phi_{\mathbf{s}}$ descends to a homomorphism from W to the group of permutations of $R \times \{\pm 1\}$", add "defined by

$$W \to \{\text{permutations of } R \times \{\pm 1\}\},$$
$$w \mapsto \phi_{\mathbf{s}} \text{ where } \mathbf{s} = (s_1, s_2, \ldots, s_k) \text{ is a word in } S \text{ such that } w = s_k s_{k-1} \cdots s_1$$

". (The important part here is that we want $w = s_k s_{k-1} \cdots s_1$, not $w = s_1 s_2 \cdots s_k$.)

30. **Page 36, proof of Proposition 3.3.6:** I think this proof could be worded more clearly as follows: "Lemma 3.3.5 (i) shows that the condition (ii) of Lemma 3.2.9 is satisfied for each $r \in R$. Thus, condition (i) is also satisfied for each $r \in R$; but this means that (Ω, v_0) is a reflection system. Finally, the claim about $\widehat{R}_w$ follows from the parenthetical statement in Lemma 3.2.9 condition (ii)."

31. **Page 36, Definition 3.4.1:** The word "subword" means a contiguous subword here, unlike in Lemma 3.2.6. It would probably better to call it a "factor" (as combinatorialists usually do).

32. **Page 37, proof of Theorem 3.4.2:** "(**s** and **u** begin with the same letter as do $\mathbf{u}'$ and **t**)" could be reworded less ambiguously as follows: "(**s** and **u** begin with the same letter, and so do $\mathbf{u}'$ and **t**)".

33. **Page 37, proof of Theorem 3.4.2:** Remove the comma before "for a t in front".

34. **Page 38, proof of Theorem 3.3.4:** It would be helpful to explain why "the Coxeter system associated to the Coxeter matrix of (W, S)" exists in the first place. Its construction has been given in §3.3 (in the last paragraph of page 33), but it has not been shown there that it is a Coxeter system. So let us make up for this now: The generators $\widetilde{s} \in \widetilde{S}$ are involutions in $\widetilde{W}$, since the set $\mathcal{R}$ of relations imposed on them include the relations $(\widetilde{s}\widetilde{s})^{m_{ss}} = (\widetilde{s}\widetilde{s})^1 = \widetilde{s}\widetilde{s}$. Moreover, these generators $\widetilde{s}$ differ from each other and from the identity of $\widetilde{W}$, because the same holds for their images under the canonical projection $\widetilde{W} \to W$. Thus, these generators $\widetilde{s}$ have order 2 in $\widetilde{W}$. Finally, these generators $\widetilde{s}$ generate $\widetilde{W}$ by the definition of $\widetilde{W}$. Hence, $\left(\widetilde{W}, \widetilde{S}\right)$ is a pre-Coxeter system. Since the canonical epimorphism $\widetilde{\widetilde{W}} \to \widetilde{W}$, $\widetilde{\widetilde{s}} \mapsto \widetilde{s}$ is an isomorphism (indeed, $\widetilde{\widetilde{W}}$ is just an isomorphic copy of $\widetilde{W}$ with the generators s renamed as $\widetilde{s}$, because if $s, t \in S$ are two elements, then the order $m_{\widetilde{s}\widetilde{t}}$ of $\widetilde{s}\widetilde{t}$ in $\widetilde{W}$ always equals the order m_{st} of st in W [3]), this entails that $\left(\widetilde{W}, \widetilde{S}\right)$ is a Coxeter system, qed.

[3] *Proof:* Let $s, t \in S$. We must show that $m_{\widetilde{s}\widetilde{t}} = m_{st}$. Recall that the canonical projection $\widetilde{W} \to W$

35. **Page 41, proof of Proposition 4.1.1:** I would replace "from Theorem 3.4.2" by "from Theorem 3.4.2 (ii)" here just to be a bit more specific.

36. **Page 42, proof of Corollary 4.1.5:** I would replace "By Proposition 4.1.1" by "By Corollary 4.1.2, we have $S(w) \subset T$, so that" here.

37. **Page 42, proof of Theorem 4.1.6:** I would replace "$l_T(tw) < l_T(w)$" and "$l_S(tw) < l_S(w)$" by "$l_T(tw) \neq l_T(w) + 1$" and "$l_S(tw) \neq l_T(w) + 1$", respectively, in order to match the wording of the Exchange Condition more closely.

38. **Page 43, proof of Lemma 4.2.1:** "(iv) follows from (ii) and (iii)" should probably be "(iv) follows from (i) and (iii)".

39. **Page 43, proof of Lemma 4.2.2:** "this to new path" → "this new path".

40. **Page 44, proof of Lemma 4.2.3:** "gallery" should be "edge path" (galleries have not been defined yet).

41. **Page 44, end of §4.2:** For later use, let me add another lemma here:

    Lemma 4.2.4. Let $x, y \in W$ and $t \in R$ satisfy $l(xt) > l(x)$ and $l(ty) > l(y)$. Then, $l(xty) > l(xy)$.

    Proof of Lemma 4.2.4. Assume the contrary. Thus, $l(xty) \leq l(xy)$, so that $l(xty) < l(xy)$ (since Remark 3.2.18 ensures that $l(xty)$ and $l(xy)$ have different parities (because $l(t)$ is odd), and thus cannot be equal).

    Set $r := xtx^{-1} \in R$. Thus, $rx = xt$, so that $l(rxy) = l(xty) < l(xy)$. Hence, Lemma 4.2.2 (applied to $w = xy$) yields that

    $$r \in R(1, xy) \subseteq R(1, x) \cup R(x, xy)$$

    (since Lemma 4.2.1 (ii) shows that $R(1, xy)$ is the symmetric difference of $R(1, x)$ and $R(x, xy)$). But $rx = xt$ also leads to $l(rx) = l(xt) > l(x)$, so that we don't have $l(x) > l(rx)$. Thus, $r \notin R(1, x)$ (since Lemma 4.2.2 shows that $r \in R(1, x)$ if and only if $l(x) > l(rx)$). Combining this with $r \in R(1, x) \cup R(x, xy)$, we obtain $r \in R(x, xy) = xR(1, y)x^{-1}$ (by Lemma 4.2.1 (i)). In view of $r = xtx^{-1}$, we can rewrite this as $xtx^{-1} \in xR(1, y)x^{-1}$. In other words, $t \in R(1, y)$. By Lemma 4.2.2 (applied to t and y instead of r and w), this means that $l(y) > l(ty)$. But this contradicts $l(ty) > l(y)$. This contradiction completes our proof of Lemma 4.2.4.

---

sends $\widetilde{s}\widetilde{t}$ to st. Thus, if $m_{st} = \infty$, then $m_{\widetilde{s}\widetilde{t}} = \infty$ as well, and thus $m_{\widetilde{s}\widetilde{t}} = m_{st}$ is obvious in this case. Hence, we can WLOG assume that $m_{st} \neq \infty$ from now on.

Then, $m_{\widetilde{s}\widetilde{t}} \mid m_{st}$ because the relation $(\widetilde{s}\widetilde{t})^{m_{st}} \in \mathcal{R}$ forces $(\widetilde{s}\widetilde{t})^{m_{st}}$ to be 1 in $\widetilde{W}$. Conversely, $m_{st} \mid m_{\widetilde{s}\widetilde{t}}$ because the canonical projection $\widetilde{W} \to W$ sends $\widetilde{s}\widetilde{t}$ to st. Combining these two divisibilities, we obtain $m_{\widetilde{s}\widetilde{t}} = m_{st}$, qed.

42. **Page 44, Lemma 4.3.1:** Remove the comma after "in the double coset" and the comma after "written in the form".

43. **Page 44, proof of Lemma 4.3.1:** "Hence, one of the deleted letters must occur in **s** and the other in **s**′.": This is true for the first deletion step, but might not be literally true for the following ones, since the sequences **s** and **s**′ may lose their reducedness after the first step (or at least I don't see why they necessarily remain reduced). However, fortunately, you don't need this claim in its full force; all you need is that no letters are ever deleted from **u**.

44. **Page 44, proof of Lemma 4.3.1:** Replace "W_T'" by "$W_{T'}$" (on the third-to-last line of the proof).

45. **Page 44, Definition 4.3.2:** Replace "W_T'" by "$W_{T'}$".

46. **Page 44, after Definition 4.3.2:** It might be worth reminding the reader that both left and right cosets are instance of double cosets: $W_T a = W_T a W_\varnothing$ and $a W_T = W_\varnothing a W_T$ for any $a \in W$.

47. **Page 44, proof of Lemma 4.3.3:** The equalities "$l(tw) = l(w) + l(t)$" and "$l(tw) = l(ta) + l(u)$" are somewhat unjustified, seeing that the equality $l(w') = l(a) + l(w) + l(a')$ in Lemma 4.3.1 is stated only for **one** factorization $w' = awa'$ of w', not for **every** such factorization.

    However, this can be fixed: At the end of Lemma 4.3.3 (i), insert the additional claim "Moreover, $l(w') = l(a) + l(w)$.". This additional claim follows from Lemma 4.3.1. But now, the equalities "$l(tw) = l(w) + l(t)$" and "$l(tw) = l(ta) + l(u)$" in the proof of Lemma 4.3.3 (ii) follow from this additional claim (because the factorization $w' = aw$ in Lemma 4.3.3 (i) is unique, and thus the additional claim holds for every such factorization).

48. **Page 45, proof of Lemma 4.3.3:** "an element T" should be "an element of T satisfying $l(at) < l(a)$".

49. **Page 45, after the proof of Lemma 4.3.3:** For later use, let me state two corollaries of Lemma 4.3.3 and Lemma 4.3.1 here:

    Corollary 4.3.4. Suppose $T \subset S$.

    (a) If $m \in W$ is $(T, \varnothing)$-reduced, then $l(um) = l(u) + l(m)$ for each $u \in W_T$.

    (b) If $m \in W$ is $(\varnothing, T)$-reduced, then $l(mu) = l(u) + l(m)$ for each $u \in W_T$.

    (c) Suppose $U \subset S$. If $m \in W$ is simultaneously $(U, \varnothing)$-reduced and $(\varnothing, T)$-reduced, then $l\left(m^{-1}um\right) = l(u)$ for each $u \in W_U \cap mW_Tm^{-1}$.

Proof of Corollary 4.3.4. **(a)** Let $m \in W$ be $(T, \varnothing)$-reduced. Thus, m is the unique shortest element in the double coset $W_T m W_\varnothing = W_T m$. Hence, m is the only $(T, \varnothing)$-reduced element in the coset $W_T m$.

Let $u \in W_T$. Lemma 4.3.3 (i) (applied to $w' = um$) shows that um can be written uniquely in the form $um = aw$, where $a \in W_T$ and w is $(T, \varnothing)$-reduced, and moreover, that we have $l(um) = l(a) + l(w)$ (this follows from the "Moreover" claim that I have inserted into Lemma 4.3.3 (i) in one of my comments above). But this factorization $um = aw$ must satisfy $w = m$ (since

$$w = \underbrace{a^{-1}}_{\substack{\in W_T \\ (\text{since } a \in W_T)}} \underbrace{aw}_{=um} \in \underbrace{W_T u}_{\substack{=W_T \\ (\text{since } u \in W_T)}} m = W_T m$$

shows that w is a $(T, \varnothing)$-reduced element in the coset $W_T m$; but we know that m is the only $(T, \varnothing)$-reduced element in the coset $W_T m$) and thus $a = u$ (since $um = a \underbrace{w}_{=m} = am$ and thus $u = a$). Hence, the equality $l(um) = l(a) + l(w)$ shown above can be rewritten as $l(um) = l(u) + l(m)$. This proves part (a).

(b) This is analogous to part (a) using left instead of right cosets. (Or apply part (a) to m^{-1} and u^{-1} instead of m and u, and recall that taking inverses in W turns left cosets into right cosets and vice versa.)

(c) Let $m \in W$ be simultaneously $(U, \varnothing)$-reduced and $(\varnothing, T)$-reduced. Let $u \in W_U \cap m W_T m^{-1}$. Thus, $u \in W_U$ and $u \in m W_T m^{-1}$, so that $m^{-1} u m \in W_T$. Thus, part (b) (applied to $m^{-1} u m$ instead of u) shows that $l\left(m m^{-1} u m\right) = l\left(m^{-1} u m\right) + l(m)$, so that

$$l\left(m^{-1} u m\right) + l(m) = l\left(\underbrace{m m^{-1} u m}_{=um}\right) = l(um) = l(u) + l(m)$$

(by part (a), applied to U instead of T). Subtracting $l(m)$ from this, we obtain $l\left(m^{-1} u m\right) = l(u)$. This proves part (c).

Corollary 4.3.5. Let $T \subset S$ and $T' \subset S$ and $w \in S$.

(a) The element w is (T, T')-reduced if and only if each $s \in T$ satisfies $l(sw) > l(w)$ and each $s \in T'$ satisfies $l(ws) > l(w)$.

(b) The element w is $(T, \varnothing)$-reduced if and only if each $s \in T$ satisfies $l(sw) > l(w)$.

(c) The element w is $(\varnothing, T')$-reduced if and only if each $s \in T'$ satisfies $l(ws) > l(w)$.

(d) The element w is (T, T')-reduced if and only if it is both $(T, \varnothing)$-reduced and $(\varnothing, T')$-reduced.

Proof of Corollary 4.3.5. **(a)** The "only if" direction is clear (since both sw and ws belong to the double coset $W_T w W_{T'}$ and are distinct from w). It remains to prove the "if" direction. So let us assume that each $s \in T$ satisfies $l(sw) > l(w)$ and each $s \in T'$ satisfies $l(ws) > l(w)$. We must show that w is (T, T')-reduced.

Assume the contrary. Let m be the unique element of minimum length in the double coset $W_T w W_{T'}$ (this is unique by Lemma 4.3.1). Then, m is (T, T')-reduced. Thus, $w \neq m$ (since we assumed that w is not (T, T')-reduced). But we have $m \in W_T w W_{T'}$ and thus $W_T m W_{T'} = W_T w W_{T'}$, so that $w \in W_T m W_{T'}$. Hence, Lemma 4.3.1 (applied to m and w instead of w and w') yields that w can be written in the form $w = ama'$ with $a \in W_T$ and $a' \in W_{T'}$ and $l(w) = l(a) + l(m) + l(a')$. Consider these a and a'.

We must have $a \neq 1$ or $a' \neq 1$ (since otherwise, $w = ama'$ would rewrite as $w = m$, which would contradict $w \neq m$). We WLOG assume that $a \neq 1$ (since the case $a' \neq 1$ is analogous). Thus, a has a reduced expression $(s_1, s_2, \ldots, s_k)$ with $k > 0$ and $s_1, s_2, \ldots, s_k \in T$ (since $a \in W_T$). Consider this reduced expression. Then, $a = s_1 s_2 \cdots s_k$, so that $s_1 a = s_2 s_3 \cdots s_k$ and therefore $l(s_1 a) \leq k - 1 = l(a) - 1$. Furthermore, from $w = ama'$, we obtain

$$\begin{aligned} & l(s_1 w) \\ & = l(s_1 a m a') \\ & \leq \underbrace{l(s_1 a)}_{\leq l(a)-1} + l(m) + l(a') \qquad \text{(since } l(uv) \leq l(u) + l(v) \text{ for all } u, v \in W) \\ & \leq \underbrace{l(a) + l(m) + l(a')}_{= l(w)} - 1 = l(w) - 1 < l(w). \end{aligned}$$

But we have assumed that each $s \in T$ satisfies $l(sw) > l(w)$. Hence, $l(s_1 w) > l(w)$ (since $s_1 \in T$), contradicting $l(s_1 w) < l(w)$. This contradiction shows that our assumption is wrong, at least in the case when $a \neq 1$. As we said, the case $a' \neq 1$ is analogous (but now we must fix a reduced expression $(s_1, s_2, \ldots, s_k)$ of a' and argue that $l(a' s_k) < l(a') - 1$ and $l(w s_k) < l(w)$, which contradicts the assumption that each $s \in T'$ satisfies $l(ws) > l(w)$). Hence, we always have a contradiction, and this completes the proof of Corollary 4.3.6 (a).

(b) This follows from part (a), applied to $T' = \varnothing$.

(c) This follows from part (a), applied to $T = \varnothing$.

(d) This follows from parts (a), (b) and (c).

50. **Page 45, (4.3):** A "}" bracket is missing after "$l(sw) > l(w)$".

51. **Page 45, Remark 4.4.1:** After "Then $r = wsw^{-1}$ for some $s \in S$", add "and $w \in W$".

52. **Page 45, Remark 4.4.1:** It would be helpful to refer to Lemma 4.2.2 and Proposition 3.3.6 for the reasons why these claims are true.

53. **Page 45:** "the *initial* and *final* elements of the γ" should be "the *initial* and *final* elements of the gallery γ".

54. **Page 46, proof of Proposition 4.4.2:** In case (a), replace the equality "$sw = w_1 \cdots w_{i-1} w_{i+1} \cdots w_k$" by "$sw = s_1 \cdots s_{i-1} s_{i+1} \cdots s_k$".

55. **Page 46, proof of Proposition 4.4.2:** In case (b), after "$l(sw) = l(w') > l(sw') = l(w)$", I would add "and thus $l(sw) = l(w) + 1$".

56. **Page 46, proof of Lemma 4.5.3:** This is correct, but it's missing the most important part: What is the relation between these properties of R's and minimal galleries? What you are tacitly using here is the fact that a gallery is minimal if and only if it intersects each wall Ω^r at most once. (This is easy to show: If the gallery starts at the vertex 1, then this is a restatement of the "if and only if" part of Lemma 3.2.14. In the general case, we can apply w_0^{-1} to this gallery to reduce it to the previous case, where w_0 is the first vertex of the gallery.) This fact is used so often that it should probably be stated explicitly.

57. **Page 47, proof of Lemma 4.5.7:** "chamber in U" should be "element of U". (Chambers have not been defined yet.)

58. **Page 47, proof of Lemma 4.5.7:** I don't understand why "The first wall Ω^r that it crosses must separate v from U".

    Here is how I would prove $D = U$ instead (sadly my proof is much more involved):

    Assume the contrary. Since $U \subset D$ is clear (by definition of D), we thus must have $D \not\subset U$. Hence, there exists some $d \in D - U$. Pick such a d. There clearly exists a gallery from d to some vertex in U (here, we need $U \neq \varnothing$; but we can easily assume this to be WLOG this case, since the case $U = \varnothing$ is easy to handle[4]). Let γ be such a gallery **of minimum length**. This gallery has at least two vertices (since it starts at $d \notin U$ but ends at a vertex in U). Let u be the last vertex and v the second-to-last vertex of this gallery. Then, $v \notin U$ (otherwise, we could get an even shorter gallery from d to some vertex in U by removing the last vertex from γ) but $u \in U$ (by definition of γ). The graph Ω has an edge $\{u, v\}$ (since v and u are the last two vertices of the gallery γ). Also, $u \in U \subset D$.

    Note that γ is a minimal gallery (since it has minimum length). Hence, any convex set that contains its initial element d and its final element u must

[4] To be very pedantic, Lemma 4.5.7 does not hold in the very degenerate case when W is the trivial group and the subset is empty, since there are no half-spaces to intersect. But we leave this silly case aside.

also contain all its intermediate elements. In particular, any such set must contain v. But D is such a set (indeed, D is an intersection of half-spaces, hence convex, and furthermore contains d and u). Hence, we conclude that $v \in D$.

Let $r \in R$ be the reflection that flips the edge $\{u, v\}$ of Ω. Then, the vertices u and v belong to different half-spaces of the wall Ω^r. Let us call these half-spaces H_u and H_v, so that $u \in H_u$ and $v \in H_v$ and $W = H_u \sqcup H_v$. If the halfspace H_u contained U as a subset, then we would have $D \subset H_u$ (since D is the intersection of all half-spaces that contain U as a subset) and thus $v \in D \subset H_u$, which would contradict $v \in H_v$ (since H_u and H_v are disjoint). Hence, H_u does not contain U as a subset. In other words, there exists some $w \in U - H_u$. Consider this w. Then, $w \notin H_u$. Combined with $u \in H_u$, this shows that the vertices w and u lie on different sides of the wall Ω^r.

Now, let $l(\delta)$ denote the number of edges in a gallery δ; we call this the *length* of δ. The graph Ω is bipartite (since Remark 3.2.18 constructs a function $\varepsilon : W \to \{\pm 1\}$ such that any two adjacent vertices p, q of Ω satisfy $\varepsilon(p) = -\varepsilon(q)$). Hence, every closed gallery (i.e., closed edge path of Ω) has even length.

Let α be a minimal gallery from u to w, and let β be a minimal gallery from w to v. Concatenating these two galleries α and β as well as the edge $\{u, v\}$, we obtain a closed gallery from u to u having length $l(\alpha) + l(\beta) + 1$. Since every closed gallery has even length, this entails that $l(\alpha) + l(\beta) + 1$ is even. Hence, $l(\alpha) \not\equiv l(\beta) \bmod 2$.

However, we can obtain an edge path α' from u to w by concatenating the edge $\{u, v\}$ with the gallery β (from w to v) walked in reverse. This edge path α' has its initial element u belong to U and its final element w belong to U as well (since $w \in U - H_u$), but its intermediate element v does not belong to U (since $v \notin U$). Since U is a convex subset, this entails that this edge path α' cannot be a minimal gallery (because a minimal gallery starting and ending inside the convex subset U would force all its intermediate elements to belong to U as well). Thus, its length $l(\alpha') = 1 + l(\beta)$ must be **larger** than the length of a minimal gallery from u to w, which is $l(\alpha)$. That is, $1 + l(\beta) > l(\alpha)$. In other words, $l(\beta) \geq l(\alpha)$.

On the other hand, we can obtain an edge path β' from w to v by concatenating the gallery α (from u to w) walked in reverse with the edge $\{u, v\}$. This edge path β' crosses the wall Ω^r twice (since the gallery α crosses this wall (because the vertices w and u lie on different sides of the wall Ω^r), but the edge $\{u, v\}$ also crosses this wall); hence it cannot be a minimal gallery (since a minimal gallery cannot cross any wall more than once). Thus, its length $l(\beta') = l(\alpha) + 1$ must be **larger** than the length of a minimal gallery from w to v, which is $l(\beta)$. That is, $l(\alpha) + 1 > l(\beta)$. In other words, $l(\alpha) \geq l(\beta)$.

Combining the inequalities $l(\alpha) \geq l(\beta)$ and $l(\beta) \geq l(\alpha)$, we obtain $l(\alpha) = l(\beta)$. But this contradicts $l(\alpha) \not\equiv l(\beta) \bmod 2$. This contradiction completes our proof.

59. **Page 47, Lemma 4.5.9:** I don't understand how this follows from Example 4.5.8. However, this is a particular case of the claim that "D is the intersection of the $A_r(D)$" made in §4.9 (on page 53), obtained by letting $X = T$ (so that $R' = R \cap W_T$ and $Z(X) = \Omega(T)$) and letting D be the chamber (i.e., connected component of $Z(X) = \Omega(T)$) that contains the identity vertex. (Indeed, the $A_r(D)$ in this case are precisely the A_s for $s \in T$, and thus their intersection is the sector A_T.)

60. **Page 47, proof of Proposition 4.5.10:** In "By Lemma 4.2.3, conjugation by w maps $R \cap W_T$ to $R \cap W_U$", I don't see any need for Lemma 4.2.3.

61. **Page 47, proof of Proposition 4.5.10:** "hence, it takes" should be "hence, w takes" (the action of w on Ω is not by conjugation).

62. **Pages 47–48, proof of Proposition 4.5.10:** Here is a simpler proof of Proposition 4.5.10:

    Proof of Proposition 4.5.10. By assumption, $W_U = wW_Tw^{-1}$ for some $w \in W$. Consider this w. Thus, $W_Uw = wW_T$. Hence, the set $W_Uw = wW_T$ is simultaneously a left coset, a right coset and a double coset. Thus, by Lemma 4.3.1, it has a unique element m of minimum length. This element m is both $(U, \varnothing)$-reduced (since it has minimum length in $W_Uw = W_UwW_\varnothing$) and $(\varnothing, T)$-reduced (since it has minimum length in $wW_T = W_\varnothing wW_T$).

    From $m \in wW_T$, we obtain $mW_T = wW_T = W_Uw = W_Um$ (since $m \in W_Uw$), so that $mW_Tm^{-1} = W_U$. Thus, $W_U \cap mW_Tm^{-1} = W_U \cap W_U = W_U$.

    Now, let $s \in T$. Then, $s \in T \subset W_T$, so that $ms \in mW_T = W_Um$. In other words, $ms = um$ for some $u \in W_U$. Consider this u. From $ms = um$, we obtain $m^{-1}um = s$ and $u = msm^{-1}$.

    Since m is simultaneously $(U, \varnothing)$-reduced and $(\varnothing, T)$-reduced, and since $u \in W_U = W_U \cap mW_Tm^{-1}$, we obtain $l\left(m^{-1}um\right) = l(u)$ by Corollary 4.3.4 (c). In view of $m^{-1}um = s$, we can rewrite this as $l(s) = l(u)$. Hence, $l(u) = l(s) = 1$ (since $s \in T \subset S$). Therefore, $u \in S$, so that $u \in W_U \cap S = U$ (by Corollary 4.1.3). In view of $u = msm^{-1}$, we can rewrite this as $msm^{-1} \in U$.

    Forget that we fixed s. We thus have shown that $msm^{-1} \in U$ for each $s \in T$. In other words, $mTm^{-1} \subset U$. Similarly, we can see that $m^{-1}Um \subset T$, so that $U \subset mTm^{-1}$. Combining this with $mTm^{-1} \subset U$, we find $mTm^{-1} = U$. This proves Proposition 4.5.10.

63. **Page 48, §4.6:** Why do you call w_0 the "*element of longest length*" and not just the "*longest element*"? After all, you use the latter notation for maximum-length elements of cosets.

64. **Page 49, Lemma 4.6.1:** This contains all the basic properties of w_0. Should it really be just a "Lemma"?

65. **Page 49, proof of Lemma 4.6.1:** Remove the comma after "Rewriting (a) as".

66. **Page 49, proof of Lemma 4.6.2:** Remove the comma after "to obtain".

67. **Page 49, proof of Lemma 4.6.2:** When you say "Hence, condition (a) of Lemma 4.6.1 holds", you are applying your (previously proved) claim to a sequence $\mathbf{s} = \left(s_1, s_2, \ldots, s_{l(u)}\right)$ that is a reduced expression of $u \in W$ (this claim is applicable here, since any initial subsequence of a reduced expression is again a reduced expression, and this property is preserved if you write it in reverse order). The claim then shows that w_0 has a reduced expression beginning with $\mathbf{s}$ (and thus ending with a reduced expression of $u^{-1}w_0$); this entails $l(w_0) = l(u) + l\left(u^{-1}w_0\right)$, hence showing that condition (a) of Lemma 4.6.1 holds indeed.

68. **Page 50, proofs of Lemma 4.7.2 and of Lemma 4.7.3:** I struggle to understand the proofs of Lemma 4.7.2 and of Lemma 4.7.3 (i), so let me rewrite them for my own clarity.

    Let $w \in W$. We shall show that

    (i) the subgroup $W_{\mathrm{In}(w)}$ is finite;

    (ii) we have $l(wu) = l(w) - l(u)$ for each $u \in W_{\mathrm{In}(w)}$;

    (iii) the element w is the longest element of $wW_{\mathrm{In}(w)}$.

    We begin by setting $T := \mathrm{In}(w) \subset S$. Let m be the shortest element of the left coset wW_T (this is unique by Definition 4.3.2, since wW_T is the double coset $W_\varnothing w W_T$). In other words, m is the unique $(\varnothing, T)$-reduced element in this left coset wW_T. Of course, $m^{-1}w \in W_T$ because $m \in wW_T$. Hence, Corollary 4.3.4 (b)[5] (applied to $u = m^{-1}w$) yields

    $$l\left(mm^{-1}w\right) = l\left(m^{-1}w\right) + l(m).$$

    In other words,

    $$l(w) = l\left(m^{-1}w\right) + l(m) \qquad (1)$$

[5]This corollary was stated in one of my comments above.

(since $mm^{-1}w = w$). However, if $s \in T$ is arbitrary, then we have $m^{-1}ws \in W_T$ (since $m^{-1}w \in W_T$ and $s \in T \subset W_T$) and

$$l\left(\underbrace{ws}_{=mm^{-1}ws}\right) = l\left(mm^{-1}ws\right) = l\left(m^{-1}ws\right) + l\left(m\right)$$

(by Corollary 4.3.4 (b), applied to $u = m^{-1}ws$ because $m^{-1}ws \in W_T$) and thus

$$l\left(m^{-1}ws\right) = \underbrace{l\left(ws\right)}_{\substack{<l(w) \\ (\text{since } s\in T=\operatorname{In}(w))}} - l\left(m\right) < l\left(w\right) - l\left(m\right) = l\left(m^{-1}w\right)$$

(by (1)). Therefore, Lemma 4.6.2 (applied to T, W_T and $m^{-1}w$ instead of S, W and w_0) shows that W_T is finite and that $m^{-1}w$ is its element of longest length (since $m^{-1}w \in W_T$). In particular, W_T is finite; this proves claim **(i)** (since $T = \operatorname{In}\left(w\right)$).

But we have also shown that $m^{-1}w$ is the element of longest length in W_T. Thus, we can apply Lemma 4.6.1 to T, W_T and $m^{-1}w$ instead of S, W and w_0. As a consequence, we see that $m^{-1}w$ is an involution (by Lemma 4.6.1 (iii)), and that each $u \in W_T$ satisfies

$$l\left(m^{-1}w\right) = l\left(u\right) + l\left(u^{-1}m^{-1}w\right) \qquad (2)$$

(by Lemma 4.6.1 (a)). Note that $\left(m^{-1}w\right)^{-1} = m^{-1}w$ (since $m^{-1}w$ is an involution).

Now, for each $u \in W_T$, we have

$$\begin{aligned}
l(wu) &= l\left(mm^{-1}wu\right) \qquad \left(\text{since } wu = mm^{-1}wu\right) \\
&= \underbrace{l\left(m^{-1}wu\right)}_{\substack{=l\left(\left(m^{-1}wu\right)^{-1}\right) \\ (\text{since } l(p)=l\left(p^{-1}\right) \\ \text{for any } p\in W)}} + l(m) \\
&\qquad \left(\begin{array}{c}\text{by Corollary 4.3.4 (b), applied to } u = m^{-1}wu \\ (\text{since } m^{-1}w \in W_T \text{ and } u \in W_T \text{ entail } m^{-1}wu \in W_T)\end{array}\right) \\
&= l\left(\left(m^{-1}wu\right)^{-1}\right) + l(m) = \underbrace{l\left(u^{-1}m^{-1}w\right)}_{\substack{=l\left(m^{-1}w\right)-l(u) \\ (\text{by (2)})}} + l(m) \\
&\qquad \left(\text{since } \left(m^{-1}wu\right)^{-1} = u^{-1}\underbrace{\left(m^{-1}w\right)^{-1}}_{=m^{-1}w} = u^{-1}m^{-1}w\right) \\
&= l\left(m^{-1}w\right) - l(u) + l(m) = \underbrace{l\left(m^{-1}w\right) + l(m)}_{\substack{=l(w) \\ (\text{by (1)})}} - l(u) = l(w) - l(u).
\end{aligned}$$

This proves our claim **(ii)** (since $T = \operatorname{In}(w)$).

Finally, claim **(iii)** follows from claim **(ii)**, because the latter claim shows that each $u \in W_{\operatorname{In}(w)}$ satisfies $l(wu) = l(w) - l(u) \leq l(w)$.

69. **Page 50, Lemma 4.7.6:** The comma in "Let $s \in S - T$," should be a period.

70. **Page 51, proof of Lemma 4.7.6:** "By Lemma 4.6.1 (i)" should be "By Lemma 4.7.3 (i)".

71. **Page 51, proof of Lemma 4.7.5:** "By Lemma 4.6.1 (ii)" should probably be "By Lemma 4.6.1 (b)".

72. **Page 51, Remark 4.8.1:** Why is $wB \not\subset sB_s$?

73. **Page 53, §4.9:** The claim that "D is the intersection of the $A_r(D)$" is not completely obvious, so let me prove it:

    Clearly, $D \subset A_r(D)$ for each $r \in R'$, since $A_r(D)$ is the vertex set of a component of $\Omega - \Omega^r$ that contains D. Thus, D is a subset of the intersection of the $A_r(D)$ over all $r \in R'$. It remains to prove the converse inclusion – i.e., that the latter intersection is a subset of D.

So let u be an element of this intersection. We must show that $u \in D$. Choose any $v \in D$ (this exists since D is a path component and thus is nonempty). Note that both u and v belong to the intersection of the $A_r(D)$ over all $r \in R'$ (since D is a subset of this intersection). In other words, both u and v belong to $A_r(D)$ for each $r \in R'$.

Pick a minimal gallery γ from u to v. This minimal gallery γ cannot cross any of the walls Ω^r for $r \in R'$ (because if it crossed such a wall, then it would put u and v on different sides of this wall, i.e., in two different components of $\Omega - \Omega^r$; but this would contradict the fact that both u and v lie in the same component $A_r(D)$ of $\Omega - \Omega^r$, as we saw in the previous paragraph). Hence, γ is an edge path of the graph $\Omega - \bigcup_{r \in R'} \Omega^r = Z(X)$. Thus, u and v are connected by a path of $Z(X)$. In other words, u and v belong to the same chamber (since the chambers are the path components of $Z(X)$). Since v belongs to the chamber D, this shows that u must belong to the chamber D as well. In other words, $u \in D$, qed.

74. **Page 54, Theorem 4.9.2:** A closing parenthesis should be inserted at the end of "for $(W', S'(D), \{A_s(D)\}_{s \in S'(D)}$".

75. **Page 55:** "Here [D] denotes the vertex" → "Here $[D]$ denotes the vertex".

76. **Page 56, proof of Lemma 4.10.1:** Once again, I struggle to understand this proof (why is it that "For any $w \in W$, wA_T is a sector for wW_Tw^{-1}"?), so let me write down my own.

Proof of Lemma 4.10.1. We begin with part (ii):

(ii) Let $w \in G_T$. Then, $wA_T = A_T$ by the definition of G_T. Hence, $w = w \underbrace{1}_{\in A_T} \in wA_T = A_T$, so that w is $(T, \varnothing)$-reduced (since A_T is the set of all $(T, \varnothing)$-reduced elements of W). Furthermore, $ww^{-1} = 1 \in A_T = wA_T$, thus $w^{-1} \in A_T$, so that w^{-1} is $(T, \varnothing)$-reduced. In other words, w is $(\varnothing, T)$-reduced (since taking inverses of elements of W preserves length but turns left cosets into right cosets and vice versa). Since w is both $(T, \varnothing)$-reduced and $(\varnothing, T)$-reduced, we conclude by Corollary 4.3.5 (d) (see one of my other comments above) that w is (T, T)-reduced. This proves part (ii).

(iii) As you point out, this follows from (ii), since the only (T, T)-reduced (or even just $(T, \varnothing)$-reduced) element of W_T is 1.

(i) We must prove that $mW_Tm^{-1} = W_T$ for each $m \in G_T$. It suffices to prove the inclusion $mW_Tm^{-1} \subset W_T$ for each $m \in G_T$ (since G_T is a subgroup, so that we can then apply this inclusion to m^{-1} instead of m and obtain $m^{-1}W_Tm \subset W_T$, that is, $W_T \subset mW_Tm^{-1}$, which we can then combine with $mW_Tm^{-1} \subset W_T$ to obtain the desired equality $mW_Tm^{-1} = W_T$).

So let $m \in G_T$. We must prove that $mW_Tm^{-1} \subset W_T$. Since conjugation by m is a group automorphism, it will suffice to show that $mtm^{-1} \in W_T$ for each $t \in T$. So let us do this.

Fix $t \in T$. By part (ii), the element $m \in G_T$ is (T, T)-reduced, thus both $(T, \varnothing)$-reduced and $(\varnothing, T)$-reduced.

We have $m \in G_T$, so that $mA_T = A_T$ (by the definition of G_T). But $t \in T$ entails $W_T t = W_T$ and thus $t \notin A_T$ (since the coset $W_T t = W_T$ contains the shorter element 1). Thus, $mt \notin mA_T = A_T$. Hence, mt is not $(T, \varnothing)$-reduced. Thus, there exists some $s \in T$ such that $l(smt) \leq l(mt)$ (by Corollary 4.3.5 (b), applied to $w = mt$). Consider this s. We have $l(sm) = l(m) + 1$ (by Corollary 4.3.4 (a), since m is $(T, \varnothing)$-reduced and $s \in T \subset W_T$) and $l(mt) = l(m) + 1$ (by Corollary 4.3.4 (b), since m is $(\varnothing, T)$-reduced and $t \in T \subset W_T$). Hence, the Folding Condition (F) entails that either $l(smt) = l(m) + 2$ or $smt = m$. Since $l(smt) \leq l(mt) = l(m) + 1$, we cannot have $l(smt) = l(m) + 2$, and thus we must have $smt = m$. Thus, $sm = mt$, so that $mtm^{-1} = s \in T \subset W_T$, just as we wanted to prove. Thus the proof of part (i) is complete.

(iv) Let $w \in N(W_T)$. Thus, $wW_T = W_T w$.

Let me set $U := T$. Thus, the equality $wW_T = W_T w$ can be rewritten as $wW_T = W_U w$. Hence, $W_U = wW_T w^{-1}$. This shows that we are in the exact same situation as in my above proof of Proposition 4.5.10. As in this latter proof, let m be the unique element of minimum length in the coset $W_U w = wW_T$. As was shown in the proof of Proposition 4.5.10, this element m is both $(U, \varnothing)$-reduced and $(\varnothing, T)$-reduced and satisfies $mTm^{-1} = U$. In particular, m is $(U, \varnothing)$-reduced; in other words, $m \in A_U$.

Now we shall show that $mA_T \subset A_U$. For this purpose, let $a \in A_T$. We want to show that $ma \in A_U$. In other words, we want to show that ma is $(U, \varnothing)$-reduced (since A_U is the set of all $(U, \varnothing)$-reduced elements of W). By Corollary 4.3.5 (b), this only requires us to show that each $u \in U$ satisfies $l(uma) > l(ma)$. So let $u \in U$. Then, $m^{-1}um \in m^{-1}Um = T$ (since $mTm^{-1} = U$). Set $t := m^{-1}um$; thus, $mt = um$ and $t = m^{-1}um \in T$. But a is $(T, \varnothing)$-reduced (since $a \in A_T$), and thus we have $l(ta) > l(a)$ (by Corollary 4.3.5 (b), since $t \in T$). Furthermore, m is $(\varnothing, T)$-reduced; thus Corollary 4.3.5 (a) yields $l(mt) > l(m)$ (since $t \in T$).

Thus, we know that $t \in T \subset S \subset R$ and $l(mt) > l(m)$ and $l(ta) > l(a)$. Hence, Lemma 4.2.4 (applied to $x = m$ and $y = a$) yields $l(mta) > l(ma)$. In view of $mt = um$, this rewrites as $l(uma) > l(ma)$.

Forget that we fixed u. We thus have shown that each $u \in U$ satisfies $l(uma) > l(ma)$. By Corollary 4.3.5 (b), this means that ma is $(U, \varnothing)$-reduced. In other words, $ma \in A_U$.

Forget that we fixed a. We thus have shown that $ma \in A_U$ for each $a \in A_T$. In other words, $mA_T \subset A_U$.

Recall that taking the inverse of an element of W preserves its length, but changes left cosets into right cosets. Thus, since m is $(U, \varnothing)$-reduced, its inverse m^{-1} is $(\varnothing, U)$-reduced. Moreover, from $mTm^{-1} = U$, we obtain $m^{-1}U\left(m^{-1}\right)^{-1} = T$. Thus, just as we have shown that $mA_T \subset A_U$, we can prove that $m^{-1}A_U \subset A_T$ (we just need to apply the same arguments to m^{-1}, U and T instead of m, T and U). In other words, $A_U \subset mA_T$. Combining this with $mA_T = A_U$.

But $m \in wW_T = W_Tw$. Hence, there exists some $u \in W_T$ that satisfies $w = um$ and therefore also satisfies $wA_T = u\underbrace{mA_T}_{=A_U} = uA_U = uA_T$ (since $U = T$). This proves the existence part of Lemma 4.10.1 (iv). It remains to prove the uniqueness part. But it is easy: If $u \in W_T$ satisfies $wA_T = uA_T$, then

$$\begin{aligned} w = w1 &\in wA_T \qquad\qquad (\text{since } 1 \in A_T) \\ &= uA_T, \end{aligned}$$

so that w can be factored as $w = uq$ for some $q \in A_T$; but Lemma 4.3.3 (i) shows that such a factorization of w is unique (since $q \in A_T$ is just saying that q is $(T, \varnothing)$-reduced), and thus u is unique as well. This proves the uniqueness of u, and thus finishes the proof of Lemma 4.10.1 (iv).