

Tales of the descent algebra

Darij Grinberg (Drexel University) joint work with
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Slides of this talk:

- <https://www.cip.ifi.lmu.de/~grinberg/algebra/da2026s.pdf>

CHAPTER 1

The descent algebra: an introduction

References:

- Louis Solomon, *A Mackey formula in the group ring of a Coxeter group*, 1976.
- Franco Saliola, *Hyperplane arrangements and descent algebras*, 2006 (errata).
- A. M. Garsia, C. Reutenauer, *A decomposition of Solomon's descent algebra*, 1989.
- Manfred Schocker, *The descent algebra of the symmetric group*, 2003.
- D. Blessenohl, H. Laue, *Algebraic Combinatorics Related to the Free Lie Algebra*, 1992 (errata).

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- Let $\mathcal{A} := \mathbf{k}[S_n]$ be its group algebra over \mathbf{k} .
- Any permutation $w \in S_n$ is written in *one-line notation* as a bracketed word $[w(1) w(2) \cdots w(n)]$.
- The *descents* of a permutation $w \in S_n$ are the $i \in [n-1]$ that satisfy $w(i) > w(i+1)$.
- The *descent set* of $w \in S_n$ is

$$\text{Des } w := \{\text{descents of } w\} \subseteq [n-1].$$

The descent algebra: definition

- For each $I \subseteq [n - 1]$, we define two elements

$$\mathbf{B}_I = \sum_{\substack{w \in S_n; \\ \text{Des}(w) \subseteq I}} w \quad \text{and} \quad \mathbf{D}_I = \sum_{\substack{w \in S_n; \\ \text{Des}(w) = I}} w$$

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of \mathcal{A} .

- Proposition [2-]:**¹ The families $(\mathbf{B}_I)_{I \subseteq [n-1]}$ and $(\mathbf{D}_I)_{I \subseteq [n-1]}$ are each linearly independent, and are related by inclusion-exclusion:

$$\mathbf{B}_I = \sum_{J \subseteq I} \mathbf{D}_J; \quad \mathbf{D}_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \mathbf{B}_J.$$

¹Difficulty rating following Stanley's EC standards.

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- This span is called the *descent algebra*

$$\mathcal{D} := \mathcal{D}_n := \text{span} \{ \mathbf{B}_I \mid I \subseteq [n-1] \} = \text{span} \{ \mathbf{D}_I \mid I \subseteq [n-1] \}.$$

As a \mathbf{k} -vector space, it has dimension 2^{n-1} (or 1 if $n = 0$).

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- **Solomon Mackey formula (Louis Solomon, 1976) [3+]:**
For any $I, J \subseteq [n - 1]$, we have

$$\mathbf{B}_{\tilde{I}} \mathbf{B}_{\tilde{J}} = \sum_{K \subseteq [n-1]} a_K^{I,J} \mathbf{B}_{\tilde{K}},$$

where $a_K^{I,J} \in \mathbb{N}$ is the # of permutations $w \in S_n$ that are minimal in their double coset $W_I w W_J$ and satisfy $w^{-1} W_I w \cap W_J = W_K$.

Here, “minimal” means “has minimum length (= inversion number) in the double coset”, but it is in fact also the unique minimum in the Bruhat order.

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- **Corollary [1+]:** The descent algebra \mathcal{D} is closed under multiplication, thus really a subalgebra of $\mathcal{A} = \mathbf{k}[S_n]$.
- Solomon proved the analogous fact for arbitrary finite Coxeter groups.

- For $n = 3$, the descent algebra \mathcal{D} has basis $(\mathbf{B}_\emptyset, \mathbf{B}_{\{1\}}, \mathbf{B}_{\{2\}}, \mathbf{B}_{\{1,2\}})$, where (using one-line notation)

$$\mathbf{B}_\emptyset = [123],$$

$$\mathbf{B}_{\{1\}} = [123] + [213] + [312],$$

$$\mathbf{B}_{\{2\}} = [123] + [132] + [231],$$

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- Note that the permutations $\text{id} = [123 \cdots n]$ and $w_0 = [n \cdots 321]$ always lie in \mathcal{D} , since

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- The *Eulerian elements* $\mathbf{E}_k := \sum_{\substack{w \in S_n; \\ |\text{Des } w| = k}} w$ lie in \mathcal{D} as well, and form their own subalgebra, which is furthermore commutative [3+].

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- There is a bijection

$$D : \text{Comp}_n \rightarrow \{\text{subsets of } [n-1]\},$$
$$(\alpha_1, \alpha_2, \dots, \alpha_k) \mapsto \{\alpha_1 + \alpha_2 + \dots + \alpha_i \mid i \in [k-1]\}$$

(the “*partial sum encoding*” of a composition). For example,

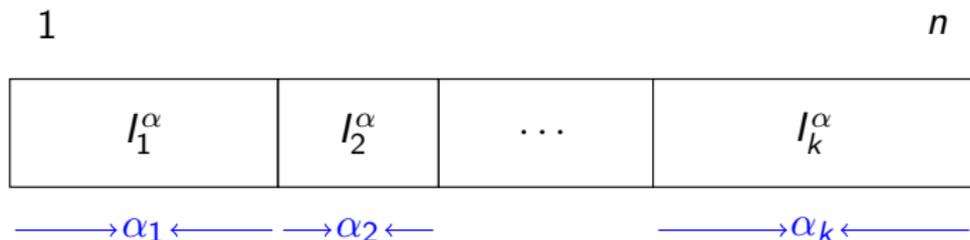
$$D(2, 4, 1, 3) = \{2, 6, 7\}.$$

Compositions, 2

- For any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \text{Comp}_n$, we can break up the interval $[n]$ into k intervals $I_1^\alpha, I_2^\alpha, \dots, I_k^\alpha$ of lengths $\alpha_1, \alpha_2, \dots, \alpha_k$; explicitly:

$$I_j^\alpha := [\alpha_{j-1} + 1, \alpha_j] \quad \text{for all } j \in [k].$$

Visually:



For instance, if $\alpha = (3, 1, 4) \in \text{Comp}_8$, then

$$I_1^\alpha = \{1, 2, 3\}, \quad I_2^\alpha = \{4\}, \quad I_3^\alpha = \{5, 6, 7, 8\}.$$

- For any $\alpha \in \text{Comp}_n$, set

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- So $(\mathbf{B}_\alpha)_{\alpha \in \text{Comp}_n}$ is a basis of \mathcal{D} , just reindexing the basis $(\mathbf{B}_I)_{I \subseteq [n-1]}$.

The Solomon Mackey formula: composition form

- **Solomon Mackey formula in composition form (A. M. Garsia, J. Remmel, 1985) [3]:** For any compositions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_\ell)$ of n , we have

$$\mathbf{B}_\alpha \mathbf{B}_\beta = \sum_{\substack{M \in \mathbb{N}^{\ell \times k} \text{ has} \\ \text{row sums } \beta_1, \beta_2, \dots, \beta_\ell \\ \text{and column sums } \alpha_1, \alpha_2, \dots, \alpha_k}} \mathbf{B}_{\text{read } M},$$

where $\text{read } M$ is the reading word of M (= the concatenation of the rows of M from top to bottom) with the zeroes removed. Visually:

	α_1	α_2	\cdots	α_k
	\downarrow	\downarrow		\downarrow
$\beta_1 \rightarrow$	r	e	a	d
$\beta_2 \rightarrow$	i	n	g	w
\vdots	o	r	d	\cdots
$\beta_\ell \rightarrow$				

- **Example:** To compute $\mathbf{B}_{(1,n-1)}\mathbf{B}_{(n-1,1)}$, solve the nonogram

$$\begin{array}{cc} 1 & n-1 \\ \downarrow & \downarrow \\ n-1 \rightarrow & \begin{array}{|c|c|} \hline x & y \\ \hline z & w \\ \hline \end{array} \end{array} \quad (\text{for } x, y, z, w \in \mathbb{N}).$$

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Get

$$\mathbf{B}_{(1,n-1)} \mathbf{B}_{(n-1,1)} = \begin{pmatrix} 0 & n-1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & n-2 \\ 0 & 1 \end{pmatrix}$$

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- Let \mathcal{Z} be the center of \mathcal{A} . As a \mathbf{k} -algebra, $\mathcal{Z} \cong \prod_{\lambda \vdash n} \mathbf{k}$, with each factor corresponding to a Specht module (irrep) of S_n .

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- **Theorem (still Louis Solomon, 1976) [3–].** There is a surjective \mathbf{k} -algebra morphism

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- **Corollary [2–].** Each irreducible representation of \mathcal{D} is 1-dimensional.
- **Fun fact.** $\mathcal{D} \not\cong \mathcal{D}^{\text{op}}$ as algebras for $n = 5$.

- Nowadays, the Solomon Mackey formula is usually proved using the face algebra of the braid arrangement (combinatorially: a monoid of set compositions): see [Franco Saliola's notes \(errata\)](#) or [Blessenohl/Laue](#) (Proposition 4.3). The proof was found by [Patrick Bidigare](#). Further developments by Ken Brown (*Semigroups, rings and Markov chains* and *Semigroup methods*).

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- [Garsia and Reutenauer 1989](#) construct idem-/nilpotent bases in \mathcal{D} and relate it to the free Lie algebra.

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- The algebra \mathcal{D} is isomorphic to NSym_n under the *internal (Kronecker) product*. This is a lift of the Kronecker product of symmetric functions.
- For Hopf algebraists, NSym consists of universal endomorphisms of a graded connected cocommutative Hopf algebra. Internal product = composition.
- Generalization to non-cocommutative Hopf algebras: [arXiv:2401.14648](#). But this does not live in $\mathbf{k}[S_n]$ or anything like that.

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- And there is **much more**.

CHAPTER 2

The LRM-basis of the symmetric group algebra

References:

- Darij Grinberg, Ekaterina A. Vassilieva, *The left-to-right minima basis of the group algebra of the symmetric group*, arXiv:2601.02952.

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Can we construct one explicitly?

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Can we construct one explicitly?

- It is reasonable to build it from a basis of \mathcal{A} that contains (as subfamilies) bases of all the right ideals $\mathbf{B}_I \mathcal{A} = \mathbf{B}_\alpha \mathcal{A}$.
- First step: understand these right ideals.

- All of this chapter is joint work with Ekaterina A. Vassilieva ([arXiv:2601.02952](https://arxiv.org/abs/2601.02952)).

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- For each composition $\alpha \in \text{Comp}_n$, there is a right ideal $\mathbf{B}_\alpha \mathcal{A}$ of \mathcal{A} . But some of these are equal:

$$\mathbf{B}_{(2,3)} \mathcal{A} = \mathbf{B}_{(3,2)} \mathcal{A} \quad \text{because } \mathbf{B}_{(2,3)} = \mathbf{B}_{(3,2)} [45123].$$

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More generally:

- **Proposition [2–].** If two compositions $\alpha, \beta \in \text{Comp}_n$ differ only in the order of their entries, then $\mathbf{B}_\alpha \mathcal{A} = \mathbf{B}_\beta \mathcal{A}$.

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- For any composition α , let $\tilde{\alpha}$ be the partition obtained by sorting the parts of α (in decreasing order). So we just showed:
- **Proposition [2–].** If two compositions $\alpha, \beta \in \text{Comp}_n$ satisfy $\tilde{\alpha} = \tilde{\beta}$, then $\mathbf{B}_\alpha \mathcal{A} = \mathbf{B}_\beta \mathcal{A}$.

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$$\mathbf{B}_{(2,3)} \mathcal{A} = \mathbf{B}_{(3,2)} \mathcal{A} \quad \text{because } \mathbf{B}_{(2,3)} = \mathbf{B}_{(3,2)} [45123].$$

More generally:

- **Proposition [2–].** If two compositions $\alpha, \beta \in \text{Comp}_n$ differ only in the order of their entries, then $\mathbf{B}_\alpha \mathcal{A} = \mathbf{B}_\beta \mathcal{A}$.
- For any composition α , let $\tilde{\alpha}$ be the partition obtained by sorting the parts of α (in decreasing order). So we just showed:
- **Proposition [2–].** If two compositions $\alpha, \beta \in \text{Comp}_n$ satisfy $\tilde{\alpha} = \tilde{\beta}$, then $\mathbf{B}_\alpha \mathcal{A} = \mathbf{B}_\beta \mathcal{A}$.
- Thus the right ideals $\mathbf{B}_\alpha \mathcal{A}$ only depend on the partitions $\tilde{\alpha}$.

- A partition $\lambda \vdash n$ is said to *refine* a partition $\mu \vdash n$ if we can obtain μ from λ by merging parts (i.e., if there exists a map f such that each j satisfies $\mu_j = \sum_{i; f(i)=j} \lambda_i$).

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- So the right ideals $\mathbf{B}_\alpha \mathcal{A}$ form a “poset-indexed filtration” of \mathcal{A} (indexed by the partitions of n , ordered by \preceq_π).
- **Proposition [2].** Each $\mathbf{B}_\alpha \mathcal{A}$ is closed under left \mathcal{D} -action, i.e., is a $(\mathcal{D}, \mathcal{A})$ -subbimodule of \mathcal{A} .

- So our “poset-indexed filtration” of \mathcal{A} is a filtration by left \mathcal{D} -modules, not just by right ideals. How does \mathcal{D} act on its subquotients?

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- **Proposition [2].** Fix $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_\ell)$ be two compositions of n . Let $\eta_\beta(\alpha)$ be the number of “ways how $\tilde{\beta}$ refines $\tilde{\alpha}$ ” (that is, of functions $f : [\ell] \rightarrow [k]$ such that each $j \in [k]$ satisfies $\alpha_j = \sum_{i: f(i)=j} \beta_i$).

Then left multiplication by \mathbf{B}_α acts on the quotient $\mathbf{B}_\beta \mathcal{A} / \sum_{\tilde{\gamma} \prec_\pi \tilde{\beta}} \mathbf{B}_\gamma \mathcal{A}$ as multiplication by the scalar $\eta_\beta(\alpha)$.

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- So we have a “poset-indexed filtration” of \mathcal{A} by $(\mathcal{D}, \mathcal{A})$ -subbimodules, and \mathcal{D} acts by scalars on the subquotients.

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- So we have a “poset-indexed filtration” of \mathcal{A} by $(\mathcal{D}, \mathcal{A})$ -subbimodules, and \mathcal{D} acts by scalars on the subquotients.
 - Thus there must be a basis of \mathcal{A} that restricts to bases of these subbimodules. In fact, many such bases. Let’s find one!

- For any permutation $w \in S_n$, the *left-to-right minima* of w are the numbers that are smaller than everything before them in the one-line notation of w . Formally, they are the $w(i)$ such that $w(1), w(2), \dots, w(i-1)$ are all $> w(i)$.

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Furthermore,

- we let $\text{LRM } w \subseteq [n]$ be the set of all left-to-right minima of w (including 1 and $w(1)$);
- we let $\text{LRM}' w \subseteq [n-1]$ be the set of $k-1$ for all $k \in \text{LRM } w$ distinct from 1;
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- **Example:** If $n = 9$ and $w = [946283517]$, then

$$\text{LRM } w = \{9, 4, 2, 1\};$$

$$\text{LRM}' w = \{8, 3, 1\};$$

$$\text{cLRM}' w = (1, 2, 5, 1).$$

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- **Exercise [2]:** For every $w \in S_n$, show

$$\text{LRM}(w^{-1}) = w^{-1}(\text{LRM } w).$$

I.e., the LRMs of w^{-1} are the **positions** of the LRMs of w .

- For any permutation $w \in S_n$, the *left-to-right minima* of w are the numbers that are smaller than everything before them in the one-line notation of w . Formally, they are the $w(i)$ such that $w(1), w(2), \dots, w(i-1)$ are all $> w(i)$.
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- Know Foata's fundamental transformation \mathcal{F} ? The partition $\text{cLRM}' w$ (obtained by sorting $\text{cLRM}' w$ in decreasing order) is the cycle type of $\mathcal{F}(w^{-1})$.

- **Theorem [3–]:** The family

$$\left(\mathbf{B}_{\text{LRM}'(w)} w \right)_{w \in S_n} = \left(\mathbf{B}_{\text{cLRM}'(w)} w \right)_{w \in S_n}$$

is a basis of \mathcal{A} . We call it the *LRM-basis*.

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- **Example:** For $n = 3$, this basis consists of the six elements

$$\mathbf{B}_{\text{LRM}'([123])} [123] = \mathbf{B}_{(3)} [123] = [123];$$

$$\mathbf{B}_{\text{LRM}'([132])} [132] = \mathbf{B}_{(3)} [132] = [132];$$

$$\mathbf{B}_{\text{LRM}'([213])} [213] = \mathbf{B}_{(1,2)} [213] = [213] + [123] + [132];$$

$$\mathbf{B}_{\text{LRM}'([231])} [231] = \mathbf{B}_{(1,2)} [231] = [231] + [132] + [123];$$

$$\mathbf{B}_{\text{LRM}'([312])} [312] = \mathbf{B}_{(2,1)} [312] = [312] + [213] + [123];$$

$$\begin{aligned} \mathbf{B}_{\text{LRM}'([321])} [321] &= \mathbf{B}_{(1,1,1)} [321] \\ &= [321] + [312] + [231] + [213] + [132] + [123]. \end{aligned}$$

- **Theorem [3+]:** This LRM-basis restricts to bases of all the right ideals $\mathbf{B}_\alpha \mathcal{A}$. Namely,

$$\mathbf{B}_\alpha \mathcal{A} = \text{span} \{ \mathbf{B}_{\text{LRM}'(w)} w \mid \widetilde{\text{cLRM}'(w)} \preceq_\pi \tilde{\alpha} \}$$

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- **Proof idea:** Let \mathcal{A} act on the n -th degree component of the free algebra in n variables x_1, x_2, \dots, x_n from the right by permuting positions (not entries!). For each $\beta = (\beta_1, \beta_2, \dots, \beta_k) \in \text{Comp}_n$, let \mathbf{V}_β be a product of nested commutators of lengths $\beta_1, \beta_2, \dots, \beta_k$. Show that $\mathbf{V}_\beta \mathbf{B}_\gamma = 0$ whenever $\tilde{\beta} \not\preceq_\pi \tilde{\gamma}$, and compute $\mathbf{V}_\beta \mathbf{B}_\gamma$ in the case $\ell(\beta) = \ell(\gamma)$. The crucial tool is a formula of Garsia and Reutenauer (Theorem 2.1 in [their 1989 paper](#), or Lemma 9.33 in [Reutenauer's *Free Lie algebras*](#)). Finish with leading term arguments.

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- **Question:** Easier proof?

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$$L(\mathbf{a}) : \mathcal{A} \rightarrow \mathcal{A},$$
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- In general, their eigenvalues are a mess. However, if $\mathbf{a} \in \mathcal{D}$, then they all lie in \mathbf{k} , and have explicit expressions:

- **Corollary [2].** Let $\mathbf{a} = \sum_{\alpha \in \text{Comp}_n} \lambda_\alpha \mathbf{B}_\alpha$ (with $\lambda_\alpha \in \mathbf{k}$) be any element of \mathcal{D} . Then, the eigenvalues of the map $L(\mathbf{a})$ (with their algebraic multiplicities) are the numbers $\sum_{\alpha \in \text{Comp}_n} \lambda_\alpha \eta_{\text{cLRM}'(w)}(\alpha)$, where w ranges over S_n , and where the integers $\eta_\beta(\alpha)$ were defined before. Moreover, the same holds for $R(\mathbf{a})$.

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- This corollary is responsible for many descriptions of spectra of shuffles (uniform BHR-shuffles such as the Tsetlin library).

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- Proof.** Order the LRM-basis so that each of the $(\mathcal{D}, \mathcal{A})$ -subbimodules $\mathbf{B}_\alpha \mathcal{A}$ is spanned by the first so-and-so many basis vectors. Then, the map $L(\mathbf{a})$ is represented by an upper-triangular matrix, and its diagonal entries are the scalars by which it acts on the subquotients of the $\mathbf{B}_\alpha \mathcal{A}$ -filtration; these are the $\sum_{\alpha \in \text{Comp}_n} \lambda_\alpha \eta_\beta(\alpha)$. Thus the claim about $L(\mathbf{a})$ follows. As for $R(\mathbf{a})$, it always has the same eigenvalues as $L(\mathbf{a})$, since \mathcal{A} is semisimple (or Frobenius).

- **Remark.** If we acted on \mathcal{D} instead of \mathcal{A} , then $L(\mathbf{a})$ and $R(\mathbf{a})$ would not have the same eigenvalues!
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- Bidigare ((4.6) in [his thesis](#)) states the above formulas in a different way. To move between the two, recall Foata's fundamental transformation, which shows that any partition $\lambda \vdash n$ satisfies

$$\begin{aligned} & \left(\# \text{ of } w \in S_n \text{ such that } \widetilde{\text{cLRM}}'(w) = \lambda \right) \\ & = \left(\# \text{ of } w \in S_n \text{ with cycle type } \lambda \right). \end{aligned}$$

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- Cycles come out naturally from Bidigare's approach; LRMs from ours.

- Geometric multiplicities and semisimplicity (\iff diagonalizability) questions are subtler.

Theorem. Let $a \in \mathcal{D}$ be a linear combination of the \mathbf{B}_α with **nonnegative** coefficients. Then:

- (a) (Brown, 2000) The element a (that is, any action of a on \mathcal{D} or on \mathcal{A} or on any left or right \mathcal{D} -module) is diagonalizable.
- (b) (G. and Parlett, 2025) The same holds for the elements aw_0 and w_0a , where w_0 is the permutation $[n \cdots 321] \in S_n$.

This, of course, is all in characteristic 0.

- **Question 1.** \mathcal{A} has a filtration with 1-dimensional subquotients not just as a left \mathcal{D} -module, but also as a $(\mathcal{D}, \mathcal{D})$ -bimodule. Is there a good basis for the latter structure, too? (So both $L(\mathbf{a})$ and $R(\mathbf{a})$ act triangularly when $\mathbf{a} \in \mathcal{D}$.) General reasoning says “yes”, but we don’t have a combinatorial construction or even a canonical choice.

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- **Question 3.** What about the other types? What are left-to-right minima?

CHAPTER 3

The dyadic shuffles and the dyadic Gelfand model

References:

- Sarah Brauner, Patricia Commins, Darij Grinberg, Franco Saliola, *A left ideal Gelfand model for the symmetric group*, working draft.

[https:](https://www.cip.ifi.lmu.de/~grinberg/algebra/dyadic.pdf)

[//www.cip.ifi.lmu.de/~grinberg/algebra/dyadic.pdf](https://www.cip.ifi.lmu.de/~grinberg/algebra/dyadic.pdf)

- The *antipode* of $\mathcal{A} = \mathbf{k}[S_n]$ is the \mathbf{k} -linear map

$$\begin{aligned} \mathcal{A} &\rightarrow \mathcal{A}, \\ w &\mapsto w^{-1} \quad \text{for all } w \in S_n. \end{aligned}$$

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- **Basic properties [2–]:** The antipode is an anti-involution of \mathcal{A} : that is, all $a, b \in \mathcal{A}$ satisfy

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- The antipode of the descent algebra \mathcal{D} is the “anti-descent algebra” \mathcal{D}^* . It does not commute with \mathcal{D} .

Defining the S_α

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- In general, S_α don't commute and don't have integer eigenvalues (e.g., $S_{(3,2)}$ has some irrational eigenvalues). But Reiner, Saliola and Welker found another subfamily that do:

- **Theorem (Reiner, Saliola, Welker, 2011) [4–]:** The elements $\mathbf{S}_{(2^k, 1^{n-2k})}$ for all $k \leq n/2$ (where $2^k = \underbrace{2, 2, \dots, 2}_{k \text{ times}}$ and $1^{n-2k} = \underbrace{1, 1, \dots, 1}_{n-2k \text{ times}}$) commute. Moreover, each of them acts diagonalizably (from left and right) on \mathcal{A} with integer eigenvalues.

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The second commuting subfamily

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- **Question:** What are their eigenvalues?
- We will give a very partial answer. (Formulas with \pm signs are easy to get; but the eigenvalues are nonnegative integers, and yet no combinatorial expression is known.)
But we will learn a lot more about the dyadic shuffles.

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- For instance, the permutation $[24513] \in S_5$ increases on exactly four 2-matchings, namely
 - on $\{\{1, 2\}, \{4, 5\}\}$ since **24513**;
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- We have $\mathcal{S}_{3,1} = 3 [123] + 2 [132] + 2 [213] + [231] + [312]$.
- We have $\mathcal{S}_{n,k} = 0$ for all $k > n/2$, for lack of k -matchings.

- Proposition (Reiner, Saliola, Welker, 2011) [2+]:** Let $k \leq n/2$. If $\alpha \in \text{Comp}_n$ is a composition consisting of k many 2's and $n - 2k$ many 1's (for example, $(2^k, 1^{n-2k})$), then

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- It turns out that something stronger holds:

- **Theorem [3+]:** Consider the composition

$$\alpha := \begin{cases} (2^{n/2}), & \text{if } n \text{ is even;} \\ (2^{(n-1)/2}, 1), & \text{if } n \text{ is odd} \end{cases} \in \text{Comp}_n.$$

Then, the left ideal $\mathcal{A}\mathbf{B}_\alpha^*$ (or, equivalently, the right ideal $\mathbf{B}_\alpha\mathcal{A}$) of \mathcal{A} is a *Gelfand model* of S_n : that is, viewed as representation of S_n , it is isomorphic to the direct sum of all irreps (= irreducible representations)!

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- **Proof idea:**

- Gelfand model: see below.
- All $\mathcal{S}_{n,k}$ lie in the ideal: follows from the $\mathbf{B}_\alpha\mathcal{A} \subseteq \mathbf{B}_\beta\mathcal{A}$ result in the previous chapter.

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- To recover the commutativity and nice eigenvalues of $\mathcal{S}_{n,k}$ from this theorem, we need to study general properties of Gelfand models – and, more generally, of multiplicity-free representations.

- What follows are general facts about multiplicity-free representations of S_n . Some of them generalize to all finite groups G , or even to all semisimple \mathbf{k} -algebras.
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- Except for fact 1, I have never seen them before, but I can't believe they are new!
- A representation V of S_n is called *multiplicity-free* if it contains no two isomorphic irreducible subrepresentations.

- **Theorems [2+ to 3-]:** Let J be a left ideal of \mathcal{A} that is multiplicity-free as representation of S_n . Then:
 - 1 The endomorphism ring $\text{End}_{\mathcal{A}} J$ is commutative.
 - 2 We have $J[J, J] = 0$ (where $[U, V]$ means the commutator space $\text{span} \{uv - vu \mid u \in U \text{ and } v \in V\}$).
 - 3 We have $[J^*J, J^*J] = 0$.
More generally, $[K^*J, K^*J] = 0$ if K is a further multiplicity-free left ideal of \mathcal{A} .
 - 4 We have $v^*w = w^*v$ for all $v, w \in J$.
 - 5 We have $x^* = x$ for each $x \in J^*J$.
 - 6 Each $a \in J$ acts (from left and right) on \mathcal{A} with all eigenvalues in \mathbf{k} .
 - 7 Each irrep \mathcal{S}^λ of S_n satisfies $\dim(J^*\mathcal{S}^\lambda) \leq 1$.
 - 8 If \mathbf{k} is an ordered field, then the subspace J^*J of \mathcal{A} is a nonunital \mathbf{k} -subalgebra that is isomorphic to the direct product \mathbf{k}^r of r copies of \mathbf{k} , where r is the # of irreps in J . (So it is unital, but not with the unity of \mathcal{A} .)

- **Proof ideas:** Representation basics (Maschke, Schur); geometric irreducibility of irreps (true for S_n over any characteristic-0 field); existence of nondegenerate symmetric invariant bilinear form on each irrep of S_n .

Why \mathcal{AB}_α^* is a Gelfand model, 1

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$$\mathcal{S}^\lambda = \text{span}(\text{all polytabloids } \mathbf{e}_T \text{ of shape } \lambda)$$

for all the partitions $\lambda \vdash n$. (A *polytabloid* is a column-antisymmetrized tabloid. A *tabloid* is a row-equivalence class of tableaux. A *tableau* must have the entries $1, 2, \dots, n$, each exactly once.)

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- Thus a representation V of S_n is a Gelfand model if and only if

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- **Proposition [2+]**. For any left ideal V of \mathcal{A} , we have

$$\text{Hom}_{\mathcal{A}}(\mathcal{S}^\lambda, V) \cong V^* \mathcal{S}^\lambda \quad \text{as vector spaces.}$$

- Thus, in order to prove that $\mathcal{A}\mathbf{B}_\alpha^*$ is multiplicity-free, it suffices to show that $\dim((\mathcal{A}\mathbf{B}_\alpha^*)^* \mathcal{S}^\lambda) = 1$ for all $\lambda \vdash n$. That is, we must show that $\dim(\mathbf{B}_\alpha \mathcal{S}^\lambda) = 1$ for all $\lambda \vdash n$.

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- Once this is proved, Gelfandness follows.

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- This is very roundabout and uncombinatorial!
Question. How to prove $\mathbf{B}_\alpha \mathbf{e}_T = (\text{scalar}) \cdot \mathbf{v}_\lambda$ directly?

Why \mathcal{A}_{α}^* is a Gelfand model, 4: looking for a nice proof

- **Restatement of the question:** Choose $2k$ distinct cells $c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k$ in the Young diagram of a partition $\lambda \vdash n$. Prove that

$$\sum_{\substack{T \text{ is a tableau of shape } \lambda; \\ T(c_i) < T(d_i) \text{ for each } i}} \mathbf{e}_T = (\text{scalar}) \cdot \mathbf{v}_{\lambda} \quad \text{in } \mathcal{S}^{\lambda}.$$

- Some **examples** (each $\{c_m, d_m\}$ is colored in some color):

c_1	c_2	c_3	
	d_3	d_2	
d_1			

c_1	c_2	c_3	d_4
	d_3	d_2	
d_1	c_4		

c_1	c_2	c_3	d_4
	d_3	d_2	
d_1	c_4	d_5	
c_5			

c_1	c_2	c_3	d_4
c_5	d_3	d_2	
d_1	c_4	d_5	

Exercise. Do you see why the first one gives 0 ?

Why \mathcal{AB}_{α}^* is a Gelfand model, 4: looking for a nice proof

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Exercise. Do you see why the first one gives 0 ?

- **Bonus question:** If $k < \sum_{i \geq 1} \lfloor \lambda_i^t / 2 \rfloor$, then the left hand side is 0.

So what about the eigenvalues?

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- As a consequence of the above, each element of \mathcal{AB}_α^* acts on each irrep \mathcal{S}^λ of S_n as a matrix of rank ≤ 1 .
- So it has at most one nonzero eigenvalue, which is its character value and thus $\in \mathbf{k}$.
- For general reasons, the eigenvalues of $\mathcal{S}_{n,k} \in \mathcal{AB}_\alpha^*$ are nonnegative integers. **Can we find them?**

- We only have an answer for λ of hook shape:
- **Theorem [4].** Let λ be the hook-shaped partition $(n - \ell, 1^\ell)$, where $0 \leq \ell < n$. Let $k \in \mathbb{N}$.

Then (on the understanding that $p! := \infty$ for any $p < 0$, and that a fraction with ∞ in its denominator is 0):

- 1 If ℓ is even, then the only nonzero eigenvalue of $\mathcal{S}_{n,k}$ on \mathcal{S}^λ is

$$\frac{n! (\ell/2)! (n - \ell)!}{(\ell + 1)! (n - 2k)! (k - \ell/2)! 2^{2k - \ell}}.$$

- 2 If ℓ is odd, then the only nonzero eigenvalue of $\mathcal{S}_{n,k}$ on \mathcal{S}^λ is

$$\frac{(n + 1)! ((\ell + 1)/2)! (n - \ell - 1)!}{(\ell + 2)! (n - 2k)! (k - (\ell + 1)/2)! 2^{2k - \ell - 1}}.$$

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- The proof is surprisingly hard and uses various deep properties of the seminormal form.

- For other shapes λ , we have recursions connecting the eigenvalues of $\mathcal{S}_{n,k}$ and $\mathcal{S}_{n,k-1}$. These are obtained from the...
Nice recursion [3]: For any $k \geq 0$, we have

$$\begin{aligned} \binom{n-2(k-1)}{2} \mathcal{S}_{n,k-1} &= \mathcal{S}_{n,k} (\mathcal{B}_n - (n-2k)) \\ &= (\mathcal{B}_n^* - (n-2k)) \mathcal{S}_{n,k}, \end{aligned}$$

where

$$\mathcal{B}_n = \mathbf{B}_{(n-1,1)}^* = \sum_{j=1}^n \text{cyc}_{n,n-1,n-2,\dots,j}$$

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- Remark [2+].** We have $w_0 \mathcal{S}_{n,k} = \mathcal{S}_{n,k} w_0$ for each $k \geq 0$.

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- **Aguado, Araujo 2006** construct one using harmonic polynomials.
Some similarities with ours, but exact relation unclear.

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