Tales of the descent algebra [talk slides]

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joint work with Ekaterina A. Vassilieva, Sarah Brauner, Patricia Commins and Franco Saliola

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Abstract. The descent algebra (introduced by Solomon in 1976) is a remarkable 2^{n-1}-dimensional subalgebra of the group algebra of the symmetric group S_n, spanned by sums of all permutations with a given descent set. In some aspects, it is similar to the symmetric group algebra, while in others, it is entirely opposite, such as its heavy non-semisimplicity. (It also makes for a natural example of an algebra not isomorphic to its opposite.)

In this talk, I will present two new developments on the descent algebra:

1. A new basis of the symmetric group algebra represents the elements of the descent algebra as triangular matrices with combinatorially meaningful diagonal entries. This gives a new approach to Bidigare's eigenvalue formulas for descent-related shuffles.

   This is joint work with Ekaterina A. Vassilieva.

2. Certain generators of the descent algebra generate right ideals of the group algebra that are Gelfand models of S_n: representations that contain each irreducible exactly once. This leads to a new view of

the classical Gelfand model of Adin, Postnikov and Roichman as well as a new proof of one of the Reiner-Saliola-Welker commutativities; we also resolve recent questions of Lafrenière.

This is joint work with Sarah Brauner, Patricia Commins and Franco Saliola.

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Preprints:

- Darij Grinberg, Ekaterina A. Vassilieva, *The left-to-right minima basis of the group algebra of the symmetric group*, arXiv:2601.02952.
- Sarah Brauner, Patricia Commins, Darij Grinberg, Franco Saliola, *A left ideal Gelfand model for the symmetric group*, `https://www.cip.ifi.lmu.de/~grinberg/algebra/dyadic.pdf`

Slides of this talk:

- `https://www.cip.ifi.lmu.de/~grinberg/algebra/da2026.pdf`

Most results are decorated with approximate difficulty ratings (by the standards of Stanley's EC), such as [2+] or [3–].

1. The descent algebra: an introduction

1.1. Notations

- Let $\mathbf{k}$ be a field of characteristic 0. (Some results hold for $\mathbb{Z}$ as well.)
- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Fix $n \in \mathbb{N}$.
- Consider the symmetric group S_n of the set $[n] := \{1, 2, \ldots, n\}$.
- Let $\mathcal{A} := \mathbf{k}[S_n]$ be its group algebra over $\mathbf{k}$.
- Any permutation $w \in S_n$ is written in one-line notation as a bracketed word $[w(1)\ w(2)\ \cdots\ w(n)]$.
- The *descents* of a permutation $w \in S_n$ are the $i \in [n-1]$ that satisfy $w(i) > w(i+1)$.
- The *descent set* of $w \in S_n$ is

$$\operatorname{Des} w := \{\text{descents of } w\} \subseteq [n-1].$$

1.2. The descent algebra

- For each $I \subseteq [n-1]$, we define two elements

$$\mathbf{B}_I = \sum_{\substack{w \in S_n; \\ \operatorname{Des}(w) \subseteq I}} w \qquad \text{and} \qquad \mathbf{D}_I = \sum_{\substack{w \in S_n; \\ \operatorname{Des}(w) = I}} w$$

  of $\mathcal{A}$.
- **Proposition [2–]:** The families $(\mathbf{B}_I)_{I \subseteq [n-1]}$ and $(\mathbf{D}_I)_{I \subseteq [n-1]}$ are each linearly independent, and are related by inclusion-exclusion:

$$\mathbf{B}_I = \sum_{J \subseteq I} \mathbf{D}_J; \qquad \mathbf{D}_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \mathbf{B}_J.$$

  So they have the same span.
- This span is called the *descent algebra*

  $$\mathcal{D} := \mathcal{D}_n := \operatorname{span}\{\mathbf{B}_I \mid I \subseteq [n-1]\} = \operatorname{span}\{\mathbf{D}_I \mid I \subseteq [n-1]\}.$$

  As a $\mathbf{k}$-vector space, it has dimension 2^{n-1} (or 1 if $n = 0$).

1.3. The Solomon Mackey formula

- Why are we calling it "descent **algebra**"?
- For any $I \subseteq [n-1]$, define W_I to be the *Young subgroup* of S_n corresponding to I. It is the subgroup generated by the simple transpositions $s_i := t_{i,i+1}$ for $i \in I$.
- Let $\widetilde{I} := [n-1] \setminus I$ for each $I \subseteq [n-1]$.
- **Solomon Mackey formula (Louis Solomon, 1976) [3+]:** For any $I, J \subseteq [n-1]$, we have

$$\mathbf{B}_{\widetilde{I}} \mathbf{B}_{\widetilde{J}} = \sum_{K \subseteq [n-1]} a_K^{I,J} \mathbf{B}_{\widetilde{K}},$$

where $a_K^{I,J} \in \mathbb{N}$ is the # of permutations $w \in S_n$ that are minimal in their double coset $W_I w W_J$ and satisfy $w^{-1} W_I w \cap W_J = W_K$.

Here, "minimal" means "has minimum length (= inversion number) in the double coset", but it is in fact also the unique minimum in the Bruhat order.

- **Corollary [1+]:** The descent algebra $\mathcal{D}$ is closed under multiplication, thus really a subalgebra of $\mathcal{A} = \mathbf{k}[S_n]$.
- Solomon proved the analogous fact for arbitrary finite Coxeter groups.

1.4. Examples

- For $n = 3$, the descent algebra $\mathcal{D}$ has basis $\left(\mathbf{B}_\varnothing, \mathbf{B}_{\{1\}}, \mathbf{B}_{\{2\}}, \mathbf{B}_{\{1,2\}}\right)$, where (using one-line notation)

$$\begin{aligned}
\mathbf{B}_\varnothing &= [123], \\
\mathbf{B}_{\{1\}} &= [123] + [213] + [312], \\
\mathbf{B}_{\{2\}} &= [123] + [132] + [231], \\
\mathbf{B}_{\{1,2\}} &= [123] + [132] + [213] + [231] + [312] + [321].
\end{aligned}$$

- Note that the permutations $\mathrm{id} = [123 \cdots n]$ and $w_0 = [n \cdots 321]$ always lie in $\mathcal{D}$, since

$$\mathrm{id} = \mathbf{B}_\varnothing = \mathbf{D}_\varnothing \qquad \text{and} \qquad w_0 = \mathbf{D}_{[n-1]}.$$

- The *Eulerian elements* $\mathbf{E}_k := \sum_{\substack{w \in S_n; \\ |\operatorname{Des} w| = k}} w$ lie in $\mathcal{D}$ as well, and form their own subalgebra, which is furthermore commutative [3+].

1.5. Compositions

- A *composition* of n means a tuple of positive integers summing to n. For instance, $(2, 1, 5)$ is a composition of 8.
- Let Comp_n be the set of all compositions of n.
- There is a bijection

$$D : \operatorname{Comp}_n \to \{\text{subsets of } [n-1]\},$$
$$(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in [k-1]\}$$

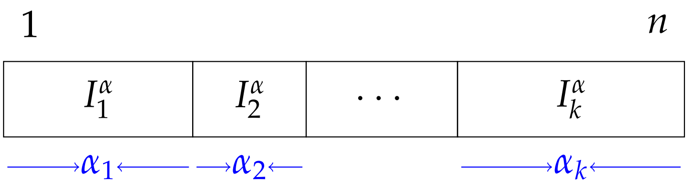
(the "*partial sum encoding*" of a composition). For example,

$$D(2, 4, 1, 3) = \{2, 6, 7\}.$$

- For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \operatorname{Comp}_n$, we can break up the interval $[n]$ into k intervals $I_1^\alpha, I_2^\alpha, \ldots, I_k^\alpha$ of lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$; explicitly:

$$I_j^\alpha := [\alpha_{j-1} + 1, \ \alpha_j] \qquad \text{for all } j \in [k].$$

Visually:



For instance, if $\alpha = (3, 1, 4) \in \operatorname{Comp}_8$, then

$$I_1^\alpha = \{1, 2, 3\}, \qquad I_2^\alpha = \{4\}, \qquad I_3^\alpha = \{5, 6, 7, 8\}.$$

- For any $\alpha \in \operatorname{Comp}_n$, set

$$\mathbf{B}_\alpha := \mathbf{B}_{D(\alpha)} = \sum_{\substack{w \in S_n \\ \text{increases on each } I_j^\alpha}} w.$$

- So $(\mathbf{B}_\alpha)_{\alpha \in \operatorname{Comp}_n}$ is a basis of $\mathcal{D}$, just reindexing the basis $(\mathbf{B}_I)_{I \subseteq [n-1]}$.

- **Solomon Mackey formula in composition form (A. M. Garsia, J. Remmel, 1985) [3]:** For any compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ of n, we have

$$\mathbf{B}_\alpha \mathbf{B}_\beta = \sum_{\substack{M \in \mathbb{N}^{\ell \times k} \text{ has} \\ \text{row sums } \beta_1, \beta_2, \ldots, \beta_\ell \\ \text{and column sums } \alpha_1, \alpha_2, \ldots, \alpha_k}} \mathbf{B}_{\operatorname{read} M},$$

where $\operatorname{read} M$ is the reading word of M (that is, the concatenation of the rows of M from top to bottom) with the zeroes removed.

Visually:

$$\begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{array}{c} \beta_1 \to \\ \beta_2 \to \\ \vdots \\ \beta_\ell \to \end{array} & \begin{array}{|c|c|c|c|} \hline \mathrm{r} & \mathrm{e} & \mathrm{a} & \mathrm{d} \\ \hline \mathrm{i} & \mathrm{n} & \mathrm{g} & \mathrm{w} \\ \hline \mathrm{o} & \mathrm{r} & \mathrm{d} & \cdots \\ \hline & & & \\ \hline \end{array} \end{array}$$

- **Example:** To compute $\mathbf{B}_{(1,n-1)} \mathbf{B}_{(n-1,1)}$, solve the nonogram

$$\begin{array}{cc} & \begin{array}{cc} 1 & n-1 \\ \downarrow & \downarrow \end{array} \\ \begin{array}{c} n-1 \to \\ 1 \to \end{array} & \begin{array}{|c|c|} \hline x & y \\ \hline z & w \\ \hline \end{array} \end{array} \qquad (\text{for } x, y, z, w \in \mathbb{N}).$$

Get

$$\mathbf{B}_{(1,n-1)} \; \mathbf{B}_{(n-1,1)} = \underbrace{\mathbf{B}_{(n-1,1)}}_{\begin{pmatrix} 0 & n-1 \\ 1 & 0 \end{pmatrix}} + \underbrace{\mathbf{B}_{(1,n-2,1)}}_{\begin{pmatrix} 1 & n-2 \\ 0 & 1 \end{pmatrix}}.$$

1.6. Some properties of the descent algebra

- Recall that $\mathcal{A} = \mathbf{k}[S_n]$ is semisimple (Maschke). But $\mathcal{D}$ is anything but!
- Let $\mathcal{Z}$ be the center of $\mathcal{A}$. As a $\mathbf{k}$-algebra, $\mathcal{Z} \cong \prod_{\lambda \vdash n} \mathbf{k}$, with each factor corresponding to a Specht module (irrep) of S_n.

- **Theorem (still Louis Solomon, 1976) [3–].** There is a surjective $\mathbf{k}$-algebra morphism

$$\mathcal{D} \to \mathcal{Z},$$
$$\mathbf{B}_\alpha \mapsto (\text{character of Young module for } \alpha).$$

- **Theorem (still Louis Solomon, 1976) [3].** The kernel of this morphism is nilpotent! Thus it is the radical of $\mathcal{D}$, while the semisimple quotient of $\mathcal{D}$ is $\mathcal{Z} \cong \prod_{\lambda \vdash n} \mathbf{k}$.

- **Corollary [2–].** Each irreducible representation of $\mathcal{D}$ is 1-dimensional.

- **Fun fact.** The algebras $\mathcal{D}$ and $\mathcal{D}^{\text{op}}$ are not isomorphic for $n = 5$.

1.7. More on the history of the descent algebra

- Nowadays, the Solomon Mackey formula is usually proved using the face algebra of the braid arrangement (combinatorially: a monoid of set compositions): see Franco Saliola's notes (errata) or Blessenohl/Laue (Proposition 4.3). The proof was found by Patrick Bidigare.

  Further developments by Ken Brown (*Semigroups, rings and Markov chains* and *Semigroup methods*).

- Garsia and Reutenauer 1989 construct idem-/nilpotent bases in $\mathcal{D}$ and relate it to the free Lie algebra.

- $\mathcal{D}$ is wild in characteristic p, but some things are known (e.g., Erdmann and Schocker 2004).

- The Ext-quiver of $\mathcal{D}$ is known. See Schocker 2003 for survey on all of the above.

- The algebra $\mathcal{D}$ is isomorphic to NSym_n under the *internal (Kronecker) product*. This is a lift of the Kronecker product of symmetric functions.

- For Hopf algebraists, NSym consists of universal endomorphisms of a graded connected cocommutative Hopf algebra. Internal product = composition.

- Generalization to non-cocommutative Hopf algebras: arXiv:2401.14648. But this does not live in $\mathbf{k}[S_n]$ or anything like that.

- And there is much more.

2. The LRM basis of the symmetric group algebra

2.1. The question

- Clearly, $\mathcal{A}$ is a left $\mathcal{D}$-module (since $\mathcal{D} \subseteq \mathcal{A}$). But all irreps of $\mathcal{D}$ are 1-dimensional. So $\mathcal{A}$ has a Jordan-Hölder flag: a filtration by left $\mathcal{D}$-submodules of dimensions $0, 1, 2, \ldots, n!$.

  Can we construct one explicitly?

- It is reasonable to build it from a basis of $\mathcal{A}$ that contains (as subfamilies) bases of all the right ideals $\mathbf{B}_I \mathcal{A} = \mathbf{B}_\alpha \mathcal{A}$.

- First step: understand these right ideals.

2.2. The left ideals $\mathbf{B}_\alpha \mathcal{A}$

- **All of this chapter is joint work with Ekaterina A. Vassilieva (arXiv:2601.02952).**

- For each composition $\alpha \in \operatorname{Comp}_n$, there is a right ideal $\mathbf{B}_\alpha \mathcal{A}$ of $\mathcal{A}$. But some of these are equal:

  $$\mathbf{B}_{(2,3)} \mathcal{A} = \mathbf{B}_{(3,2)} \mathcal{A} \qquad \text{because } \mathbf{B}_{(2,3)} = \mathbf{B}_{(3,2)} [45123] .$$

  More generally:

- **Proposition [2–].** If two compositions $\alpha, \beta \in \operatorname{Comp}_n$ differ only in the order of their entries, then $\mathbf{B}_\alpha \mathcal{A} = \mathbf{B}_\beta \mathcal{A}$.

- For any composition α, let $\widetilde{\alpha}$ be the partition obtained by sorting the parts of α (in decreasing order). So we just showed:

- **Proposition [2–].** If two compositions $\alpha, \beta \in \operatorname{Comp}_n$ satisfy $\widetilde{\alpha} = \widetilde{\beta}$, then $\mathbf{B}_\alpha \mathcal{A} = \mathbf{B}_\beta \mathcal{A}$.

- Thus the right ideals $\mathbf{B}_\alpha \mathcal{A}$ only depend on the partitions $\widetilde{\alpha}$.

- A partition $\lambda \vdash n$ is said to *refine* a partition $\mu \vdash n$ if we can obtain μ from λ by merging parts (i.e., if there exists a map f such that each j satisfies $\mu_j = \sum_{i;\ f(i)=j} \lambda_i$).

  We write $\lambda \preceq_\pi \mu$ for this (π for "partition"). Not the same as refinement of compositions!

- **Proposition [2+].** If two compositions $\alpha, \beta \in \operatorname{Comp}_n$ satisfy $\widetilde{\alpha} \preceq_\pi \widetilde{\beta}$, then $\mathbf{B}_\alpha \mathcal{A} \subseteq \mathbf{B}_\beta \mathcal{A}$.
- **Remark.** Over $\mathbb{Q}$, we also have $\mathbf{B}_\alpha \mathcal{D} \subseteq \mathbf{B}_\beta \mathcal{D}$, but our above results hold even over $\mathbb{Z}$.
- So the right ideals $\mathbf{B}_\alpha \mathcal{A}$ form a "poset-indexed filtration" of $\mathcal{A}$ (indexed by the partitions of n, ordered by $\preceq_\pi$).
- **Proposition [2].** Each $\mathbf{B}_\alpha \mathcal{A}$ is closed under left $\mathcal{D}$-action, i.e., is a $(\mathcal{D}, \mathcal{A})$-subbimodule of $\mathcal{A}$.
- So our "poset-indexed filtration" of $\mathcal{A}$ is a filtration by left $\mathcal{D}$-modules, not just by right ideals. How does $\mathcal{D}$ act on its subquotients?
- **Proposition [2].** Fix $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be two compositions of n. Let $\eta_\beta(\alpha)$ be the number of functions $f : [\ell] \to [k]$ such that for every $j \in [k]$ one has $\alpha_j = \sum\limits_{i;\ f(i)=j} \beta_i$. Then left multiplication by $\mathbf{B}_\alpha$ acts on the quotient $\mathbf{B}_\beta \mathcal{A} / \sum\limits_{\widetilde{\gamma} \prec_\pi \widetilde{\beta}} \mathbf{B}_\gamma \mathcal{A}$ as multiplication by the scalar $\eta_\beta(\alpha)$.
- So we have a "poset-indexed filtration" of $\mathcal{A}$ by $(\mathcal{D}, \mathcal{A})$-subbimodules, and $\mathcal{D}$ acts by scalars on the subquotients.
- Thus there must be a basis of $\mathcal{A}$ that restricts to bases of these subbimodules. In fact, many such bases. Let's find one!

2.3. Left-to-right minima

- For any permutation $w \in S_n$, the *left-to-right minima* of w are the numbers that are smaller than everything before them in the one-line notation of w. Formally, they are the $w(i)$ such that $w(1), w(2), \ldots, w(i-1)$ are all $> w(i)$.

  Furthermore,

  - we let $\operatorname{LRM} w \subseteq [n]$ be the set of all left-to-right minima of w (including 1 and $w(1)$);
  - we let $\operatorname{LRM}' w \subseteq [n-1]$ be the set of $k-1$ for all $k \in \operatorname{LRM} w$ distinct from 1;
  - we let $\operatorname{cLRM}' w$ denote the composition $\alpha \in \operatorname{Comp}_n$ such that $D(\alpha) = \operatorname{LRM}' w$.

- **Example:** If $n = 9$ and $w = [946283517]$, then

$$\begin{aligned} \operatorname{LRM} w &= \{9,4,2,1\}; \\ \operatorname{LRM}' w &= \{8,3,1\}; \\ \operatorname{cLRM}' w &= (1,2,5,1). \end{aligned}$$

- **Exercise [2]:** For every $w \in S_n$, show $\operatorname{LRM}\left(w^{-1}\right) = w^{-1}(\operatorname{LRM} w)$.

  That is, the LRMs of w^{-1} are the **positions** of the LRMs of w.

- Know Foata's fundamental transformation $\mathcal{F}$? The partition $\widetilde{\operatorname{cLRM}'}\, w$ (obtained by sorting $\operatorname{cLRM}' w$ in decreasing order) is the cycle type of $\mathcal{F}\left(w^{-1}\right)$.

2.4. The LRM-basis of $\mathcal{A}$

- **Theorem [3–]:** The family

$$\left(\mathbf{B}_{\operatorname{LRM}'(w)}\, w\right)_{w \in S_n} = \left(\mathbf{B}_{\operatorname{cLRM}'(w)}\, w\right)_{w \in S_n}$$

  is a basis of $\mathcal{A}$. We call it the *LRM-basis*.

- **Proof idea:** Show that $\mathbf{B}_{\operatorname{LRM}'(w)}\, w = w + (\text{lex-smaller permutations})$.

- **Example:** For $n = 3$, this basis consists of the six elements

$$\begin{aligned} \mathbf{B}_{\operatorname{LRM}'([123])}\ [123] &= \mathbf{B}_{(3)}\ [123] = [123]; \\ \mathbf{B}_{\operatorname{LRM}'([132])}\ [132] &= \mathbf{B}_{(3)}\ [132] = [132]; \\ \mathbf{B}_{\operatorname{LRM}'([213])}\ [213] &= \mathbf{B}_{(1,2)}\ [213] = [213] + [123] + [132]; \\ \mathbf{B}_{\operatorname{LRM}'([231])}\ [231] &= \mathbf{B}_{(1,2)}\ [231] = [231] + [132] + [123]; \\ \mathbf{B}_{\operatorname{LRM}'([312])}\ [312] &= \mathbf{B}_{(2,1)}\ [312] = [312] + [213] + [123]; \\ \mathbf{B}_{\operatorname{LRM}'([321])}\ [321] &= \mathbf{B}_{(1,1,1)}\ [321] \\ &= [321] + [312] + [231] + [213] + [132] + [123]. \end{aligned}$$

- **Theorem [3+]:** This LRM-basis restricts to bases of all the right ideals $\mathbf{B}_\alpha \mathcal{A}$. Namely,

$$\mathbf{B}_\alpha \mathcal{A} = \operatorname{span}\left\{\mathbf{B}_{\operatorname{LRM}'(w)}\, w \mid \widetilde{\operatorname{cLRM}'}(w) \preceq_\pi \widetilde{\alpha}\right\}$$

  for each $\alpha \in \operatorname{Comp}_n$.

- **Proof idea:** Let $\mathcal{A}$ act on the n-th degree component of the free algebra in n variables $x_1, x_2, \ldots, x_n$ from the right by permuting positions (not entries!). For each $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \operatorname{Comp}_n$, let $\mathbf{V}_\beta$ be a product of nested commutators of lengths $\beta_1, \beta_2, \ldots, \beta_k$. Show that $\mathbf{V}_\beta \mathbf{B}_\gamma = 0$ whenever $\widetilde{\beta} \not\leq_\pi \widetilde{\gamma}$, and compute $\mathbf{V}_\beta \mathbf{B}_\gamma$ in the case $\ell(\beta) = \ell(\gamma)$. The crucial tool is a formula of Garsia and Reutenauer (Theorem 2.1 in their 1989 paper, or Lemma 9.33 in Reutenauer's *Free Lie algebras*). Finish with leading term arguments.

- **Alternative proof idea:** Get the dimensions of the $\mathbf{B}_\alpha \mathcal{A}$ from Bidigare's 1997 thesis, then argue only $\supseteq$ part. Needs some tweaks to work over $\mathbb{Z}$.

- Yes: all the results in this chapter (except for one remark) hold over $\mathbb{Z}$ and over any commutative ring.

- **Question:** Easier proof?

2.5. Eigenvalues recovered

- Using the LRM-basis, we can recover Bidigare's formulas for the eigenvalues of an element of $\mathcal{D}$ acting on $\mathcal{A}$:

- For any $a \in \mathcal{A}$, we define the $\mathbf{k}$-linear maps

$$\begin{aligned} L(\mathbf{a}) : \mathcal{A} &\to \mathcal{A}, \\ \mathbf{x} &\mapsto \mathbf{ax} \end{aligned}$$

  and

$$\begin{aligned} R(\mathbf{a}) : \mathcal{A} &\to \mathcal{A}, \\ \mathbf{x} &\mapsto \mathbf{xa}. \end{aligned}$$

  These are the *left* and *right multiplication* by $\mathbf{a}$.

- In general, their eigenvalues are a mess. However, if $\mathbf{a} \in \mathcal{D}$, then they all lie in $\mathbf{k}$, and have explicit expressions:

- **Corollary [2].** Let $\mathbf{a} = \sum\limits_{\alpha \in \operatorname{Comp}_n} \lambda_\alpha \mathbf{B}_\alpha$ (with $\lambda_\alpha \in \mathbf{k}$) be any element of $\mathcal{D}$. Then, the eigenvalues of the map $L(\mathbf{a})$ (with their algebraic multiplicities) are the numbers $\sum\limits_{\alpha \in \operatorname{Comp}_n} \lambda_\alpha \eta_{\operatorname{cLRM}'(w)}(\alpha)$, where w ranges over S_n, and where the integers $\eta_\beta(\alpha)$ were defined before.

Moreover, the same holds for $R(\mathbf{a})$.

- This corollary is responsible for many descriptions of spectra of shuffles (uniform BHR-shuffles such as the Tsetlin library).

- **Proof.** Order the LRM-basis so that each of the $(\mathcal{D}, \mathcal{A})$-subbimodules $\mathbf{B}_\alpha \mathcal{A}$ is spanned by the first so-and-so many basis vectors.

  Then, the map $L(\mathbf{a})$ is represented by an upper-triangular matrix, and its diagonal entries are the scalars by which it acts on the subquotients of the $\mathbf{B}_\alpha \mathcal{A}$-filtration; these are the $\sum\limits_{\alpha \in \operatorname{Comp}_n} \lambda_\alpha \eta_\beta(\alpha)$.

  Thus the claim about $L(\mathbf{a})$ follows. As for $R(\mathbf{a})$, it always has the same eigenvalues as $L(\mathbf{a})$, since $\mathcal{A}$ is semisimple (or Frobenius).

- **Remark.** If we acted on $\mathcal{D}$ instead of $\mathcal{A}$, then $L(\mathbf{a})$ and $R(\mathbf{a})$ would not have the same eigenvalues!

  (At least not counting with multiplicities; the sets are always the same.)

- Bidigare ((4.6) in his thesis) states the above formulas in a different way. To move between the two, recall Foata's fundamental transformation, which shows that any partition $\lambda \vdash n$ satisfies

$$\left(\text{\# of } w \in S_n \text{ such that } \widetilde{\text{cLRM}'}(w) = \lambda\right)$$
$$= (\text{\# of } w \in S_n \text{ with cycle type } \lambda).$$

- Cycles come out naturally from Bidigare's approach; LRMs from ours.

- Geometric multiplicities and semisimplicity ($\Longleftrightarrow$ diagonalizability) questions are subtler.

  Theorem. Let $a \in \mathcal{D}$ be a linear combination of the $\mathbf{B}_\alpha$ with **nonnegative** coefficients. Then:

  (a) (Brown, 2000) The element a (that is, any action of a on $\mathcal{D}$ or on $\mathcal{A}$ or on any left or right $\mathcal{D}$-module) is diagonalizable.

  (b) (G. and Parlett, 2025) The same holds for the elements aw_0 and $w_0 a$, where w_0 is the permutation $[n \cdots 321] \in S_n$.

  This, of course, is all in characteristic 0.

2.6. Questions

- **Question 1.** $\mathcal{A}$ has a filtration with 1-dimensional subquotients not just as a left $\mathcal{D}$-module, but also as a $(\mathcal{D}, \mathcal{D})$-bimodule. Is there a good basis for the latter structure, too? (So both $L(\mathbf{a})$ and $R(\mathbf{a})$ act triangularly when $\mathbf{a} \in \mathcal{D}$.)

  General reasoning says "yes", but we don't have a combinatorial construction or even a canonical choice.

- **Question 2 (downstream from 1).** What are the eigenvalues of $L(\mathbf{a}) + R(\mathbf{b})$ when $\mathbf{a}, \mathbf{b} \in \mathcal{D}$?

- **Question 3.** What about the other types? What are left-to-right minima?

3. The dyadic shuffles and the dyadic Gelfand model

3.1. Antipodes and symmetrization

- The *antipode* of $\mathcal{A} = \mathbf{k}[S_n]$ is the $\mathbf{k}$-linear map
$$\mathcal{A} \to \mathcal{A}, \qquad w \mapsto w^{-1} \qquad \text{for all } w \in S_n.$$
The image of some $a \in \mathcal{A}$ under the antipode will be denoted a^* here.

- **Basic properties [2–]:** The antipode is an anti-involution of $\mathcal{A}$: that is, all $a, b \in \mathcal{A}$ satisfy
$$a^{**} = a \qquad \text{and} \qquad (ab)^* = b^* a^*.$$

- The antipode of the descent algebra $\mathcal{D}$ is the "anti-descent algebra" $\mathcal{D}^*$. It does not commute with $\mathcal{D}$.

- Still, for all $\alpha \in \operatorname{Comp}_n$, we can define the "symmetrized B-basis element"
$$\mathbf{S}_\alpha := \mathbf{B}_\alpha \mathbf{B}_\alpha^*.$$

- These do not span a subalgebra. But:

- **Theorem (Reiner, Saliola, Welker 2011; Dieker, Saliola 2015; Lafrenière 2019; Axelrod-Freed, Brauner, Chiang, Commins, Lang 2024; Brauner, Commins, G, Saliola 2025) [4]:** The elements $\mathbf{S}_{\left(k,1^{n-k}\right)}$ for all $k \in [n]$ (where 1^{n-k} denotes $\underbrace{1,1,\ldots,1}_{n-k \text{ times}}$) commute. Moreover, each of them acts diagonalizably (from left and right) on $\mathcal{A}$ with integer eigenvalues.

  These are the *k-random-to-random shuffles* and have been the subject of a prior talk.

- In general, $\mathbf{S}_\alpha$ don't commute and don't have integer eigenvalues (e.g., $\mathbf{S}_{(3,2)}$ has some irrational eigenvalues). However, Reiner, Saliola and Welker found another subfamily that do:

- **Theorem (Reiner, Saliola, Welker, 2011) [4–]:** The elements $\mathbf{S}_{\left(2^k,1^{n-2k}\right)}$ for all $k \leq n/2$ (where $2^k = \underbrace{2,2,\ldots,2}_{k \text{ times}}$ and $1^{n-2k} =$

$\underbrace{1,1,\ldots,1}_{n-2k \text{ times}}$) commute. Moreover, each of them acts diagonalizably (from left and right) on $\mathcal{A}$ with integer eigenvalues.

- The two theorems look similar, but are anything but! The proofs have no ideas in common.
- We refer to the $\mathbf{S}_{(2^k,1^{n-2k})}$ (up to a scalar factor) as the *dyadic shuffles*.
- **Question:** What are their eigenvalues?
- We will give a very partial answer. (Formulas with $\pm$ signs are easy to get; but the eigenvalues are nonnegative integers, and yet no combinatorial expression is known.)

3.2. The dyadic shuffles

- **All of this chapter is joint work with Sarah Brauner, Patricia Commins and Franco Saliola (partial draft available at https://www.cip.ifi.lmu.de/~grinberg/algebra/dyadic.pdf).**
- The easiest way to define the dyadic shuffles is different.
- An *edge* will mean a 2-element subset of $[n]$.
- A permutation $w \in S_n$ is said to *increase* on an edge $\{i < j\}$ if and only if $w(i) < w(j)$.
- A set of k disjoint edges will be called a *k-matching*.
- A permutation $w \in S_n$ is said to *increase* on a k-matching M if w increases on each edge $P \in M$.
- For instance, the permutation $[24513] \in S_5$ increases on exactly four 2-matchings, namely

$$\begin{aligned} &\text{on } \{\{1,2\},\{4,5\}\} \text{ since } 24513; \\ &\text{on } \{\{1,3\},\{4,5\}\} \text{ since } 24513; \\ &\text{on } \{\{2,3\},\{4,5\}\} \text{ since } 24513; \\ &\text{on } \{\{1,5\},\{2,3\}\} \text{ since } 24513. \end{aligned}$$

- Given a permutation $w \in S_n$ and an integer $k \geq 0$, we let

$$\operatorname{incmat}_k w := (\# \text{ of } k\text{-matchings on which } w \text{ increases}).$$

- For any $k \in \mathbb{N}$, we define the *dyadic shuffle*

$$\mathcal{S}_{n,k} := \sum_{w \in S_n} \operatorname{incmat}_k(w)\, w \in \mathcal{A}.$$

- **Examples:**
  - We have $\mathcal{S}_{n,0} = \sum_{w \in S_n} w$.
  - We have $\mathcal{S}_{3,1} = 3[123] + 2[132] + 2[213] + [231] + [312]$.
  - We have $\mathcal{S}_{n,k} = 0$ for all $k > n/2$, for lack of k-matchings.

- **Proposition (Reiner, Saliola, Welker, 2011) [2+]:** Let $k \leq n/2$. If $\alpha \in \operatorname{Comp}_n$ is a composition consisting of k many 2's and $n - 2k$ many 1's (for example, $(2^k, 1^{n-2k})$), then

$$\mathcal{S}_{n,k} = \frac{1}{k!\,(n-2k)!} \underbrace{\mathbf{S}_\alpha}_{=\mathbf{B}_\alpha \mathbf{B}_\alpha^*} \in \underbrace{\mathbf{B}_\alpha \mathcal{A}}_{\substack{\text{our favorite} \\ \text{right ideal}}} \cap \underbrace{\mathcal{A}\mathbf{B}_\alpha^*}_{\text{its antipode}} .$$

- We can now restate the result of Reiner, Saliola and Welker:

- **Theorem (Reiner, Saliola, Welker, 2011) [4–]:** The elements $\mathcal{S}_{n,k}$ for all $k \geq 0$ commute. Moreover, each of them acts diagonalizably (from left and right) on $\mathcal{A}$ with integer eigenvalues.

3.3. The Gelfand model

- It turns out that something stronger holds:

- **Theorem [3+]:** Consider the composition

$$\alpha = \begin{cases} \left(2^{n/2}\right), & \text{if } n \text{ is even;} \\ \left(2^{(n-1)/2}, 1\right), & \text{if } n \text{ is odd} \end{cases} \in \operatorname{Comp}_n .$$

  Then, the left ideal $\mathcal{A}\mathbf{B}_\alpha^*$ (or, equivalently, the right ideal $\mathbf{B}_\alpha \mathcal{A}$) of $\mathcal{A}$ is a *Gelfand model* of S_n: that is, viewed as representation of S_n, it is isomorphic to the direct sum of all irreps (= irreducible representations)!

  All the $\mathcal{S}_{n,k}$ lie in this ideal.

- **Proof idea:**
  - Gelfand model: see below.

  - All $\mathcal{S}_{n,k}$ lie in the ideal: follows from the $\mathbf{B}_\alpha \mathcal{A} \subseteq \mathbf{B}_\beta \mathcal{A}$ result in the previous chapter.

- **Corollary [2]:** The dimension of this ideal is

$$\sum_{\lambda \vdash n} \dim \left(\mathcal{S}^\lambda \right) = (\# \text{ of involutions in } S_n) .$$

  This also follows from the previous chapter.

- To recover the commutativity and nice eigenvalues of $\mathcal{S}_{n,k}$ from this theorem, we need to study general properties of Gelfand models – and, more generally, of multiplicity-free representations.

3.4. Multiplicity-free representations

- What follows are general facts about multiplicity-free representations of S_n. Some of them generalize to all finite groups G, or even to all semisimple $\mathbf{k}$-algebras.

- Except for fact 1, I have never seen them before, but I can't believe they are new!

- A representation V of S_n is called *multiplicity-free* if it contains no two isomorphic irreducible subrepresentations.

- **Theorems [2+ to 3–]:** Let J be a left ideal of $\mathcal{A}$ that is multiplicity-free as representation of S_n. Then:

  1. The endomorphism ring $\operatorname{End}_{\mathcal{A}} J$ is commutative.
  2. We have $J[J, J] = 0$ (where $[U, V]$ means the commutator space span $\{uv - vu \mid u \in U \text{ and } v \in V\}$).
  3. We have $[J^* J, \ J^* J] = 0$.

     More generally, $[K^* J, \ K^* J] = 0$ if K is a further multiplicity-free left ideal of $\mathcal{A}$.
  4. We have $v^* w = w^* v$ for all $v, w \in J$.
  5. We have $x^* = x$ for each $x \in J^* J$.
  6. Each $a \in J$ acts (from left and right) on $\mathcal{A}$ with all eigenvalues in $\mathbf{k}$.
  7. Each irrep $\mathcal{S}^\lambda$ of S_n satisfies $\dim \left(J^* \mathcal{S}^\lambda \right) \leq 1$.

8. If $\mathbf{k}$ is an ordered field, then the subspace J^*J of $\mathcal{A}$ is a nonunital $\mathbf{k}$-subalgebra that is isomorphic to the direct product $\mathbf{k}^r$ of r copies of $\mathbf{k}$, where r is the number of irreps in J. (So it is unital, but not with the unity of $\mathcal{A}$.)

- **Proof ideas:** Representation basics (Maschke, Schur); geometric irreducibility of irreps (true for S_n over any characteristic-0 field); existence of nondegenerate symmetric invariant bilinear form on each irrep of S_n.

3.5. Why $\mathcal{A}\mathbf{B}_\alpha^*$ is a Gelfand model

- Let α be the composition $\left(2^{n/2}\right)$ or $\left(2^{(n-1)/2}, 1\right)$ as above. Why is the left ideal $\mathcal{A}\mathbf{B}_\alpha^*$ a Gelfand model of S_n ?
- The irreps of S_n are the *Specht modules*

$$\mathcal{S}^\lambda = \operatorname{span}\left(\text{all polytabloids } \mathbf{e}_T \text{ of shape } \lambda\right)$$

  for all the partitions $\lambda \vdash n$. (A *polytabloid* is a column-antisymmetrized tabloid. A *tabloid* is a row-equivalence class of tableaux. A *tableau* must have the entries $1, 2, \ldots, n$, each exactly once.)
- Thus a representation V of S_n is a Gelfand model if and only if

$$\dim \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, V\right) = 1 \qquad \text{for all } \lambda \vdash n$$

  (since $\dim \operatorname{End}_{\mathcal{A}}\left(\mathcal{S}^\lambda\right) = 1$ for each λ).
- **Proposition [2+].** For any left ideal V of $\mathcal{A}$, we have

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, V\right) \cong V^* \mathcal{S}^\lambda \qquad \text{as vector spaces.}$$

- Thus, in order to prove that $\mathcal{A}\mathbf{B}_\alpha^*$ is multiplicity-free, it suffices to show that $\dim\left(\left(\mathcal{A}\mathbf{B}_\alpha^*\right)^* \mathcal{S}^\lambda\right) = 1$ for all $\lambda \vdash n$. That is, we must show that $\dim\left(\mathbf{B}_\alpha \mathcal{S}^\lambda\right) = 1$ for all $\lambda \vdash n$.
- **Reasonable proof idea:** Find the one vector that spans $\mathbf{B}_\alpha \mathcal{S}^\lambda$.
- And indeed, there is a nice one:

$$\mathbf{v}_\lambda := \sum_{\substack{S \text{ is a column-standard} \\ \text{tableau of shape } \lambda}} \mathbf{e}_S.$$

  ("*Column-standard*" means that the entries increase down columns, but nothing is known about the rows.)

- **Proposition [3+?]:** For any $\lambda \vdash n$, the vector $\mathbf{v}_\lambda$ is nonzero and spans the vector space $\mathbf{B}_\alpha \mathcal{S}^\lambda$.
- Once this is proved, Gelfandness follows.
- The nonzeroness of $\mathbf{v}_\lambda$ is not too hard (either using leading terms or using positivity on the determinantal avatar).
- That $\mathbf{v}_\lambda \in \mathbf{B}_\alpha \mathcal{S}^\lambda$ follows from an explicit factorization, related to the Pfaffian of the "1 above the diagonal, -1 below it" alternating matrix.
- The spanning is hard! We would expect a direct proof of $\mathbf{B}_\alpha \mathbf{e}_T = (\text{scalar}) \cdot \mathbf{v}_\lambda$ for all T. But we don't know how!
- Instead, we show that $\mathcal{A}\mathbf{B}_\alpha$ has the right dimension to be a Gelfand model. By what we proved above, it contains one as a subrepresentation, so it must be it.
- This is very roundabout and uncombinatorial!

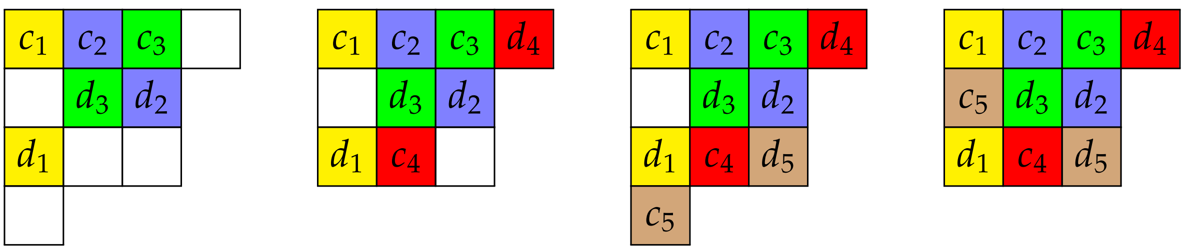
  Question. How to prove $\mathbf{B}_\alpha \mathbf{e}_T = (\text{scalar}) \cdot \mathbf{v}_\lambda$ directly?

  Restatement: Choose $2k$ distinct cells $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k$ in the Young diagram of a partition $\lambda \vdash n$. Prove that

$$\sum_{\substack{T \text{ is a tableau of shape } \lambda; \\ T(c_i) < T(d_i) \text{ for each } i}} \mathbf{e}_T = (\text{scalar}) \cdot \mathbf{v}_\lambda \qquad \text{in } \mathcal{S}^\lambda.$$

  Bonus question: If $k < \sum_{i \geq 1} \lfloor \lambda_i^t / 2 \rfloor$, then the left hand side is 0.

- Some examples (each $\{c_m, d_m\}$ is colored in some color):



  Exercise. Do you see why the first one gives 0 ?

3.6. So what about the eigenvalues?

- As a consequence of the above, each element of $\mathcal{A}\mathbf{B}_\alpha^*$ acts on each irrep $\mathcal{S}^\lambda$ of S_n as a matrix of rank ≤ 1.

- So it has at most one nonzero eigenvalue, which is its character value and thus $\in \mathbf{k}$.

- For general reasons, the eigenvalues of $\mathcal{S}_{n,k} \in \mathcal{A}\mathbf{B}_\alpha^*$ are nonnegative integers. **Can we find them?**

- We only have an answer for λ of hook shape:

- **Theorem [4].** Let λ be the hook-shaped partition $\left(n-\ell, 1^\ell\right)$, where $0 \leq \ell < n$. Let $k \in \mathbb{N}$.

  Then (on the understanding that $p! := \infty$ for any $p < 0$, and that a fraction with ∞ in its denominator is 0):

  1. If ℓ is even, then the only nonzero eigenvalue of $\mathcal{S}_{n,k}$ on $\mathcal{S}^\lambda$ is
  $$\frac{n!\,(\ell/2)!\,(n-\ell)!}{(\ell+1)!\,(n-2k)!\,(k-\ell/2)!2^{2k-\ell}}.$$

  2. If ℓ is odd, then the only nonzero eigenvalue of $\mathcal{S}_{n,k}$ on $\mathcal{S}^\lambda$ is
  $$\frac{(n+1)!\,((\ell+1)/2)!\,(n-\ell-1)!}{(\ell+2)!\,(n-2k)!\,(k-(\ell+1)/2)!2^{2k-\ell-1}}.$$

- The proof is surprisingly hard and uses various deep properties of the seminormal form.

- For other shapes λ, we have recursions connecting the eigenvalues of $\mathcal{S}_{n,k}$ and $\mathcal{S}_{n,k-1}$. These are obtained from the...

  Nice recursion [3]: For any $k \geq 0$, we have
  $$\binom{n-2(k-1)}{2}\mathcal{S}_{n,k-1} = \mathcal{S}_{n,k}\left(\mathcal{B}_n - (n-2k)\right) = \left(\mathcal{B}_n^* - (n-2k)\right)\mathcal{S}_{n,k},$$
  where
  $$\mathcal{B}_n = \mathbf{B}_{(n-1,1)}^* = \sum_{j=1}^{n} \operatorname{cyc}_{n,n-1,n-2,\ldots,j}$$
  (and where $\mathcal{S}_{n,-1} := 0$).

- **Remark [2+].** We have $w_0\mathcal{S}_{n,k} = \mathcal{S}_{n,k}w_0$ for each $k \geq 0$.

3.7. Other Gelfand models

- $\mathcal{A}\mathbf{B}^*_\alpha$ is not the first Gelfand model of S_n to be found.
- Easy to construct others using Young symmetrizers.
- Kodiyalam, Verma 2004 and Adin, Postnikov, Roichman 2007 construct one using involutions.

  It is the associated graded module of $\mathcal{A}\mathbf{B}^*_\alpha$, equipped with the filtration

  $$\mathcal{A}\mathbf{B}^*_{(1^n)} \subseteq \mathcal{A}\mathbf{B}^*_{(2,1^{n-2})} \subseteq \mathcal{A}\mathbf{B}^*_{(2,2,1^{n-4})} \subseteq \cdots .$$

- Aguado, Araujo 2006 construct one using harmonic polynomials.

  Some similarities with ours, but exact relation unclear.

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