The common-submatrix Laplace expansion

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`https://cseweb.ucsd.edu/~gill/CILASite/Resources/CSM_Laplace.pdf`

version of 5 May 2015

Errata and addenda by Darij Grinberg (version of March 21, 2018)

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I will refer to the results appearing in "The common-submatrix Laplace expansion" by the numbers under which they appear in this preprint (specifically, in its version of 5 May 2015, available from `https://cseweb.ucsd.edu/~gill/CILASite/Resources/CSM_Laplace.pdf`).

1. Errata

- **page 2:** You write: "We abbreviate $\sum_{x \in S} f(x)$ as $\sum_{S} f(x)$". I find this abbreviation confusing, since it makes the "$\sum_{S}$" symbol ambiguous: does it mean a sum over all **subsets** S of something, or does it mean a sum over all **elements** of S (and in the latter case, which variable is the summation index?)? For example, in Lemma 2.1, the summation sign $\sum_{J}$ has the former meaning, while the summation signs $\sum_{F \cup I}$ and $\sum_{G \cup J}$ have the latter meaning. Wouldn't it be better to abbreviate $\sum_{x \in S} f(x)$ as $f(S)$ instead? As far as I can tell, $\rho_S^{\Lambda}(T)$ (for a subset T of Λ) is not otherwise specified, so this wouldn't create ambiguity.

- **page 2:** From what I understand, your concept of "ordered set partition" here allows for empty blocks (e.g., in Example 3.3, the block F will be empty when $n = 2$). This would be good to point out, since it flies in the face of standard usage of this word (even in your "*Matrix Canonical Forms*", you require the blocks to be nonempty).

- **page 2, Theorem 1.4:** I'd replace "$|\Phi| = |\Lambda| = n$" by "$|\Phi| = |\Lambda|$". The letter n is not used in Theorem 1.4 itself, and it is best left unspoiled, since later Theorem 1.4 is applied to a situation in which $|\Phi| = |\Lambda|$ is (usually) larger than n.

- **page 2:** Remove the comma in "Theorem 1.4, is a classical result".

- **page 3, Definition 2.1:** I'd replace "indexed by $\underline{n}_G^+$, $G \subseteq \underline{n}$, $|G| = |F|$" by "indexed by $\underline{n}_G^+$ for any $G \subseteq \underline{n}$ satisfying $|G| = |F|$". Your commas evoke the false impression that G is also being indexed over.

- **page 5, Example 2.11:** After (2.14), I'd replace the "Note that" by "It is easy to observe that

$$\widetilde{A}\left(F^{+} \mid G^{+}\right) = A, \qquad \widetilde{A}\left[F^{+} \mid G^{+}\right] = A\left[F \mid G\right], \tag{1}$$

$$\widetilde{A}\left[F^{+} \mid G^{+}\right) = \Theta, \qquad \widetilde{A}\left[I \mid G^{+}\right] = \Theta, \tag{2}$$

$$\widetilde{A}\left[I' \mid G^{+}\right] = A\left[I' \mid G\right]. \tag{3}$$

  The third of these five equalities shows that $\widetilde{A}$ is a block-triangular matrix up to permuting rows and columns, and so we get".

  I'd also suggest citing the five equations above in the proof of Lemma 2.26. They are really the crux of why $\widetilde{A}$ is useful.

- **page 5, Example 2.11:** In the display after "Note that", a closing parenthesis is missing after "$\det(\widetilde{A}\left(F^{+} \mid G^{+}\right)$".

- **page 6, Remark 2.18:** A closing parenthesis too much in "$\overrightarrow{A}\left[F^{+} \cup I' \mid L'\right]$) contains a zero column".

- **page 6, proof of Lemma 2.22:** I suggest replacing "$f\left(x\right)$" by "$g_J\left(x\right)$" by "f" and "g_J" throughout the proof. There is no variable x on which these f and g_J would depend.

- **page 6, proof of Lemma 2.22:** "From remark 2.18" $\rightarrow$ "From (2.19) and (2.20)" (for more clarity).

- **page 6, proof of Lemma 2.22:** On the last two lines of this proof, the "(x)"es should be in the exponents. So, not "$(-1)^{\sum_{F\cup I} \rho^{n}_{I\cup I'}}\left(x\right)$" but "$(-1)^{\sum_{F\cup I} \rho^{n}_{I\cup I'}(x)}$", and similarly for the other expression.

- **page 6, Lemma 2.26:** A closing parenthesis is missing after "$A\left[F \mid G\right]$" in (2.27).

- **page 7, proof of Lemma 2.26:** As mentioned above, I'd wish to see a precise reference to the five equations (1), (2) and (3) here, particularly because the "we find only one term" argument is a consequence of $\widetilde{A}\left[F^{+} \mid G^{+}\right) = \Theta$.

- **page 7, proof of Lemma 2.26:** In "$\sum\limits_{G^{+}} \rho^{\Phi}\left(x\right) = |G| + \sum\limits_{G} \rho^{\Phi}\left(x\right)$", replace both "$\Phi$"s by "$\Lambda$"s.

- **page 7, proof of Lemma 2.26:** On the last two lines of this proof, the "(x)"es should be in the exponents. So, not "$(-1)^{\sum_{F} \rho^{n}_{I\cup I'}}\left(x\right)$" but "$(-1)^{\sum_{F} \rho^{n}_{I\cup I'}(x)}$", and similarly for the other expression.

- **page 7, proof of Lemma 2.31:** Add a missing parenthesis at the end of the first display.

- **page 7, proof of Lemma 2.31:** In the second display, add a missing parenthesis after "$\det(A\,[F \mid G]$".
- **page 8, Example 3.1:** By "$\widehat{A}$", do you mean "$\overrightarrow{A}$"?