Function-field symmetric functions: In search of an $\mathbb{F}_q[T]$-combinatorics

Darij Grinberg (UMN)

27 February 2017, Cornell

slides:
http://www.cip.ifi.lmu.de/~grinberg/algebra/cornell-feb17.pdf
preprint (WIP, and currently a mess):
http://www.cip.ifi.lmu.de/~grinberg/algebra/schur-ore.pdf

- The connection between symmetric functions and (big) Witt vectors is due to Cartier around 1970 (vaguely; made explicit by Reutenauer in 1995), and can be used to the benefit of either.
- Modern references: e.g., Hazewinkel's *Witt vectors, part 1* (`arXiv:0804.3888v1`, see also errata), and works of James Borger (mainly `arXiv:0801.1691v6`, as well as `arXiv:math/0407227v1` joint with Wieland).

- The connection between symmetric functions and (big) Witt vectors is due to Cartier around 1970 (vaguely; made explicit by Reutenauer in 1995), and can be used to the benefit of either.
- Modern references: e.g., Hazewinkel's *Witt vectors, part 1* (arXiv:0804.3888v1, see also errata), and works of James Borger (mainly arXiv:0801.1691v6, as well as arXiv:math/0407227v1 joint with Wieland).
- Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. The *(big) Witt vector functor* is a functor W : **CRing** $\rightarrow$ **CRing**, sending any commutative ring A to a new commutative ring $W(A)$ with some extra structure.
- Note that $W(A)$ is a ring, not an A-algebra.

- Let A be a commutative ring.
  We abbreviate a family $(a_k)_{k\in\mathbb{N}_+} \in A^{\mathbb{N}_+}$ as $\mathbf{a}$. Similarly for other letters.
- For each $n \in \mathbb{N}_+$, define a map $w_n : A^{\mathbb{N}_+} \to A$ by

$$w_n(\mathbf{a}) = \sum_{d|n} d a_d^{n/d}.$$

  The map w_n is called the *n-th ghost projection*.
- **Examples:**
  - $w_1 = a_1$.
  - If p is a prime, then $w_p = a_1^p + p a_p$.
  - $w_6 = a_1^6 + 2a_2^3 + 3a_3^2 + 6a_6$.

Definition of Witt vectors, 1: ghost maps

- Let A be a commutative ring.
  We abbreviate a family $(a_k)_{k\in\mathbb{N}_+} \in A^{\mathbb{N}_+}$ as $\mathbf{a}$. Similarly for other letters.
- For each $n \in \mathbb{N}_+$, define a map $w_n : A^{\mathbb{N}_+} \to A$ by

$$w_n(\mathbf{a}) = \sum_{d\mid n} d a_d^{n/d}.$$

  The map w_n is called the *n-th ghost projection*.
- Let $w : A^{\mathbb{N}_+} \to A^{\mathbb{N}_+}$ be the map given by

$$w(\mathbf{a}) = (w_n(\mathbf{a}))_{n\in\mathbb{N}_+}.$$

  We call w the *ghost map*.

- Let A be a commutative ring.
  We abbreviate a family $(a_k)_{k \in \mathbb{N}_+} \in A^{\mathbb{N}_+}$ as $\mathbf{a}$. Similarly for other letters.
- For each $n \in \mathbb{N}_+$, define a map $w_n : A^{\mathbb{N}_+} \to A$ by

$$w_n(\mathbf{a}) = \sum_{d \mid n} d a_d^{n/d}.$$

  The map w_n is called the *n-th ghost projection*.
- Let $w : A^{\mathbb{N}_+} \to A^{\mathbb{N}_+}$ be the map given by

$$w(\mathbf{a}) = (w_n(\mathbf{a}))_{n \in \mathbb{N}_+}.$$

  We call w the *ghost map*.
- This ghost map w is not linear and in general not injective or surjective. **However**, its image turns out to be a subring of $A^{\mathbb{N}_+}$. It is called the *ring of ghost-Witt vectors*.

Definition of Witt vectors, 2: addition

- For example, for any $\mathbf{a}, \mathbf{b} \in A^{\mathbb{N}_+}$, we have $w(\mathbf{a}) + w(\mathbf{b}) = w(\mathbf{c})$ for some $\mathbf{c} \in A^{\mathbb{N}_+}$. How to compute this $\mathbf{c}$?

Definition of Witt vectors, 2: addition

- For example, for any $\mathbf{a}, \mathbf{b} \in A^{\mathbb{N}_+}$, we have $w(\mathbf{a}) + w(\mathbf{b}) = w(\mathbf{c})$ for some $\mathbf{c} \in A^{\mathbb{N}_+}$. How to compute this $\mathbf{c}$?
- Good news:
  - w is injective if A is torsionfree (as $\mathbb{Z}$-module).
  - w is bijective if A is a $\mathbb{Q}$-vector space.

Definition of Witt vectors, 2: addition

- For example, for any $\mathbf{a}, \mathbf{b} \in A^{\mathbb{N}_+}$, we have $w(\mathbf{a}) + w(\mathbf{b}) = w(\mathbf{c})$ for some $\mathbf{c} \in A^{\mathbb{N}_+}$. How to compute this $\mathbf{c}$?
- Good news:
  - w is injective if A is torsionfree (as $\mathbb{Z}$-module).
  - w is bijective if A is a $\mathbb{Q}$-vector space.
- Hence, we can compute $\mathbf{c}$ back from $w(\mathbf{c})$ by recursion (coordinate by coordinate). Miraculously, the denominators vanish.

  Examples:
  - $w_1(\mathbf{c}) = w_1(\mathbf{a}) + w_1(\mathbf{b}) \iff c_1 = a_1 + b_1$.
  - $w_2(\mathbf{c}) = w_2(\mathbf{a}) + w_2(\mathbf{b}) \iff$ $c_1^2 + 2c_2 = (a_1^2 + 2a_2) + (b_1^2 + 2b_2) \overset{\text{naturality}}{\iff}$ $c_2 = a_2 + b_2 + \dfrac{1}{2}\left(a_1^2 + b_1^2 - (a_1 + b_1)^2\right)$, and the RHS is indeed a $\mathbb{Z}$-polynomial.

Definition of Witt vectors, 3: $W(A)$

- Let's make a new ring out of this: We define $W(A)$ to be the ring that equals $A^{\mathbb{N}_+}$ as a set, but whose ring structure is such that $W : \mathbf{CRing} \to \mathbf{CRing}$ is a functor, and w is a natural (in A) ring homomorphism from $W(A)$ to $A^{\mathbb{N}_+}$.

Definition of Witt vectors, 3: $W(A)$

- Let's make a new ring out of this: We define $W(A)$ to be the ring that equals $A^{\mathbb{N}_+}$ as a set, but whose ring structure is such that $W : \mathbf{CRing} \to \mathbf{CRing}$ is a functor, and w is a natural (in A) ring homomorphism from $W(A)$ to $A^{\mathbb{N}_+}$.
- This looks abstract and confusing, but the underlying idea is simple: Define addition on $W(A)$ so that
  $w_n(\mathbf{a} + \mathbf{b}) = w_n(\mathbf{a}) + w_n(\mathbf{b})$ for all n.
  Thus, $\mathbf{a} + \mathbf{b}$ is the $\mathbf{c}$ from last page.
- Functoriality is needed, because there might be several choices for a given A (if A is not torsionfree), but only one consistent choice for all rings A. Functoriality forces us to pick the consistent choice.

- Let's make a new ring out of this: We define $W(A)$ to be the ring that equals $A^{\mathbb{N}_+}$ as a set, but whose ring structure is such that $W : \mathbf{CRing} \to \mathbf{CRing}$ is a functor, and w is a natural (in A) ring homomorphism from $W(A)$ to $A^{\mathbb{N}_+}$.
- This looks abstract and confusing, but the underlying idea is simple: Define addition on $W(A)$ so that
  $w_n(\mathbf{a} + \mathbf{b}) = w_n(\mathbf{a}) + w_n(\mathbf{b})$ for all n.
  Thus, $\mathbf{a} + \mathbf{b}$ is the $\mathbf{c}$ from last page.
- Functoriality is needed, because there might be several choices for a given A (if A is not torsionfree), but only one consistent choice for all rings A. Functoriality forces us to pick the consistent choice.
- If $\mathbf{a} \in W(A)$, then the a_n are called the *Witt coordinates* of $\mathbf{a}$, while the $w_n(\mathbf{a})$ are called the *ghost coordinates* of $\mathbf{a}$.

- The ring $W(A)$ is called the *ring of (big) Witt vectors over* A.
- The functor **CRing** $\to$ **CRing**, $A \mapsto W(A)$ is called the *(big) Witt vector functor*.

- The ring $W(A)$ is called the *ring of (big) Witt vectors over A*.
- The functor **CRing** $\to$ **CRing**, $A \mapsto W(A)$ is called the *(big) Witt vector functor*.
- For any given prime p, there is a canonical quotient $W_p(A)$ of $W(A)$ called the *ring of p-typical Witt vectors of A*. Number theorists usually care about the latter ring. For example, $W_p(\mathbb{F}_p) = \mathbb{Z}_p$ (the p-adics). We have nothing to say about it here.
- $W(A)$ comes with more structure: Frobenius and Verschiebung endomorphisms, a comonad comultiplication map $W(A) \to W(W(A))$, etc.

- There are some equivalent ways to define $W(A)$. Let me show two.
- One is the Grothendieck construction using power series (see, again, Hazewinkel, or Rabinoff's arXiv:1409.7445):
- Let $\Lambda(A)$ be the topological ring defined as follows:
  - As topological spaces, $\Lambda(A) = 1 + tA[[t]] = \{\text{power series with constant term } 1\}$.
  - Addition $\widehat{+}$ in $\Lambda(A)$ is multiplication of power series.
  - Multiplication $\widehat{\cdot}$ in $\Lambda(A)$ is given by

$$(1 - at)\widehat{\cdot}(1 - bt) = 1 - abt$$

    (and distributivity and continuity, and naturality in A).

- There are some equivalent ways to define $W(A)$. Let me show two.
- One is the Grothendieck construction using power series (see, again, Hazewinkel, or Rabinoff's arXiv:1409.7445):
- Let $\Lambda(A)$ be the topological ring defined as follows:
  - As topological spaces, $\Lambda(A) = 1 + tA[[t]] = \{\text{power series with constant term } 1\}$.
  - Addition $\widehat{+}$ in $\Lambda(A)$ is multiplication of power series.
  - Multiplication $\widehat{\cdot}$ in $\Lambda(A)$ is given by

$$(1 - at)\widehat{\cdot}(1 - bt) = 1 - abt$$

    (and distributivity and continuity, and naturality in A).
- Canonical ring isomorphism

$$W(A) \to \Lambda(A), \qquad \mathbf{a} \mapsto \prod_{n=1}^{\infty} (1 - a_n t^n) .$$

- Here is another: Let Λ be the Hopf algebra of symmetric functions over $\mathbb{Z}$. (No direct relation to $\Lambda(A)$; just traditional notations clashing.)
- Define ring $\operatorname{Alg}(\Lambda, A)$ as follows:
  - As set, $\operatorname{Alg}(\Lambda, A) = \{\text{algebra homomorphisms } \Lambda \to A\}$.
  - Addition = convolution.
  - Multiplication = convolution using the *second comultiplication* on Λ (= Kronecker comultiplication = Hall dual of Kronecker multiplication).

- Here is another: Let Λ be the Hopf algebra of symmetric functions over $\mathbb{Z}$. (No direct relation to $\Lambda(A)$; just traditional notations clashing.)
- Define ring $\operatorname{Alg}(\Lambda, A)$ as follows:
  - As set, $\operatorname{Alg}(\Lambda, A) = \{\text{algebra homomorphisms } \Lambda \to A\}$.
  - Addition = convolution.
  - Multiplication = convolution using the *second comultiplication* on Λ (= Kronecker comultiplication = Hall dual of Kronecker multiplication).
- The elements of $\operatorname{Alg}(\Lambda, A)$ are known as *characters* of Λ (as in Aguiar-Bergeron-Sottile) or *virtual alphabets* (to the Lascoux school) or as *specializations of symmetric functions* (as in Stanley's EC2).

Avatars of Witt vectors, 2: Characters of Λ (virtual alphabets)

- Here is another: Let Λ be the Hopf algebra of symmetric functions over $\mathbb{Z}$. (No direct relation to $\Lambda(A)$; just traditional notations clashing.)
- Define ring $\operatorname{Alg}(\Lambda, A)$ as follows:
  - As set, $\operatorname{Alg}(\Lambda, A) = \{\text{algebra homomorphisms } \Lambda \to A\}$.
  - Addition = convolution.
  - Multiplication = convolution using the *second comultiplication* on Λ (= Kronecker comultiplication = Hall dual of Kronecker multiplication).
- There is a unique family $(w_n)_{n \in \mathbb{N}_+}$ of symmetric functions satisfying $p_n = \sum_{d \mid n} d w_d^{n/d}$ for all $n \in \mathbb{N}_+$. (Equivalently, it is determined by $h_n = \sum_{\lambda \vdash n} w_\lambda$, where $w_\lambda = w_{\lambda_1} w_{\lambda_2} \cdots$.) These are called the *Witt coordinates*.
- We have a ring isomorphism

$$\operatorname{Alg}(\Lambda, A) \to W(A), \qquad f \mapsto (f(w_n))_{n \in \mathbb{N}_+}.$$

Avatars of Witt vectors, 2: Characters of Λ, cont'd

- There is a unique family $(w_n)_{n\in\mathbb{N}_+}$ of symmetric functions satisfying $p_n = \sum_{d|n} d w_d^{n/d}$ for all $n \in \mathbb{N}_+$. (Equivalently, it is determined by $h_n = \sum_{\lambda\vdash n} w_\lambda$, where $w_\lambda = w_{\lambda_1} w_{\lambda_2} \cdots$.) These generate Λ as a ring, are called the *Witt coordinates*, and were first introduced in 1995 by Reutenauer.
- We have a ring isomorphism

$$\mathrm{Alg}(\Lambda, A) \to W(A), \qquad f \mapsto (f(w_n))_{n\in\mathbb{N}_+}.$$

Avatars of Witt vectors, 2: Characters of Λ, cont'd

- There is a unique family $(w_n)_{n\in\mathbb{N}_+}$ of symmetric functions satisfying $p_n = \sum_{d|n} d w_d^{n/d}$ for all $n \in \mathbb{N}_+$. (Equivalently, it is determined by $h_n = \sum_{\lambda\vdash n} w_\lambda$, where $w_\lambda = w_{\lambda_1} w_{\lambda_2} \cdots$.) These generate Λ as a ring, are called the *Witt coordinates*, and were first introduced in 1995 by Reutenauer.
- We have a ring isomorphism

$$\mathrm{Alg}(\Lambda, A) \to W(A), \qquad f \mapsto (f(w_n))_{n\in\mathbb{N}_+}.$$

- We also have a ring homomorphism (isomorphism when A is a $\mathbb{Q}$-algebra)

$$\mathrm{Alg}(\Lambda, A) \to A^{\mathbb{N}_+}, \qquad f \mapsto (f(p_n))_{n\in\mathbb{N}_+}.$$

These form a commutative diagram

$$\begin{array}{ccc} \mathrm{Alg}(\Lambda, A) & \xrightarrow{\cong} & W(A) \\ & \searrow & \downarrow{\scriptstyle w} \\ & & A^{\mathbb{N}_+} \end{array}$$

Reconstructing Λ from $W = \mathrm{Alg}(\Lambda, -)$

- This also works in reverse: We can reconstruct Λ from the functor W, as its representing object. Namely:
  - The functor $\mathrm{Forget} \circ W : \mathbf{CRing} \to \mathbf{Set}$ determines Λ as a ring (by Yoneda).
  - The functor $\mathrm{Forget} \circ W : \mathbf{CRing} \to \mathbf{Ab}$ (additive group of $W(A)$) determines Λ as a Hopf algebra.
  - The functor $W : \mathbf{CRing} \to \mathbf{CRing}$ determines Λ as a Hopf algebra equipped with a second comultiplication.
  - The comonad structure on W additionally determines plethysm on Λ.

Reconstructing Λ from $W = \mathrm{Alg}(\Lambda, -)$

- This also works in reverse: We can reconstruct Λ from the functor W, as its representing object. Namely:
  - The functor $\mathrm{Forget} \circ W : \mathbf{CRing} \to \mathbf{Set}$ determines Λ as a ring (by Yoneda).
  - The functor $\mathrm{Forget} \circ W : \mathbf{CRing} \to \mathbf{Ab}$ (additive group of $W(A)$) determines Λ as a Hopf algebra.
  - The functor $W : \mathbf{CRing} \to \mathbf{CRing}$ determines Λ as a Hopf algebra equipped with a second comultiplication.
  - The comonad structure on W additionally determines plethysm on Λ.
- Thus, if symmetric functions hadn't been around, Witt vectors would have let us rediscover them.

The ghost-Witt integrality theorem (aka Dwork lemma), 1

- Assume you don't know about $\Lambda(A)$ or Λ. How would you go about proving that the Witt vector functor W exists?

The ghost-Witt integrality theorem (aka Dwork lemma), 1

- Assume you don't know about $\Lambda(A)$ or Λ. How would you go about proving that the Witt vector functor W exists?
  In other words, why do the denominators (e.g., in the computation of $\mathbf{c}$ satisfying $w(\mathbf{a}) + w(\mathbf{b}) = w(\mathbf{c})$) "miraculously" vanish?

The ghost-Witt integrality theorem (aka Dwork lemma), 1

- Assume you don't know about $\Lambda(A)$ or Λ. How would you go about proving that the Witt vector functor W exists?
  In other words, why do the denominators (e.g., in the computation of $\mathbf{c}$ satisfying $w(\mathbf{a}) + w(\mathbf{b}) = w(\mathbf{c})$) "miraculously" vanish?
- This is a consequence of the *ghost-Witt integrality theorem*, also known (in parts) as *Dwork's lemma*. I shall state a (more or less) maximalist version of it; only the $\mathcal{C} \iff \mathcal{E}$ part is actually needed.

- **Ghost-Witt integrality theorem.**
  Let A be a commutative ring. For every $n \in \mathbb{N}_+$, let $\varphi_n : A \to A$ be an endomorphism of the ring A. Assume that:
  - We have $\varphi_p(a) \equiv a^p \mod pA$ for every $a \in A$ and every prime p.
  - We have $\varphi_1 = \mathrm{id}$, and we have $\varphi_n \circ \varphi_m = \varphi_{nm}$ for every $n, m \in \mathbb{N}_+$. (Thus, $n \mapsto \varphi_n$ is an action of the multiplicative monoid $\mathbb{N}_+$ on A by ring endomorphisms.)

- **Ghost-Witt integrality theorem.**
  Let A be a commutative ring. For every $n \in \mathbb{N}_+$, let $\varphi_n : A \to A$ be an endomorphism of the ring A. Assume that:
  - We have $\varphi_p(a) \equiv a^p \mod pA$ for every $a \in A$ and every prime p.
  - We have $\varphi_1 = \mathrm{id}$, and we have $\varphi_n \circ \varphi_m = \varphi_{nm}$ for every $n, m \in \mathbb{N}_+$. (Thus, $n \mapsto \varphi_n$ is an action of the multiplicative monoid $\mathbb{N}_+$ on A by ring endomorphisms.)

  [For a stupid example, let $A = \mathbb{Z}$ and $\varphi_n = \mathrm{id}$.
  For an example that is actually useful to Witt vectors, let A be a polynomial ring over $\mathbb{Z}$, and let φ_n send each indeterminate to its n-th power.]

- **Ghost-Witt integrality theorem.**
  Let A be a commutative ring. For every $n \in \mathbb{N}_+$, let $\varphi_n : A \to A$ be an endomorphism of the ring A. Assume that:
  - We have $\varphi_p(a) \equiv a^p \bmod pA$ for every $a \in A$ and every prime p.
  - We have $\varphi_1 = \mathrm{id}$, and we have $\varphi_n \circ \varphi_m = \varphi_{nm}$ for every $n, m \in \mathbb{N}_+$. (Thus, $n \mapsto \varphi_n$ is an action of the multiplicative monoid $\mathbb{N}_+$ on A by ring endomorphisms.)

  [For a stupid example, let $A = \mathbb{Z}$ and $\varphi_n = \mathrm{id}$.
  For an example that is actually useful to Witt vectors, let A be a polynomial ring over $\mathbb{Z}$, and let φ_n send each indeterminate to its n-th power.]
- Let $\mathbf{b} = (b_n)_{n \in \mathbb{N}_+} \in A^{\mathbb{N}_+}$ be a sequence of elements of A.
  Then, the following assertions are equivalent: [continued on next page]

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:
  - $\mathcal{C}$**:** Every $n \in \mathbb{N}_+$ and every prime divisor p of n satisfy
  $$\varphi_p\left(b_{n/p}\right) \equiv b_n \operatorname{mod} p^{v_p(n)} A$$
  (where $v_p(n)$ is the multiplicity of p in the factorization of n).

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:
  $\mathcal{C}$: Every $n \in \mathbb{N}_+$ and every prime divisor p of n satisfy

  $$\varphi_p\left(b_{n/p}\right) \equiv b_n \bmod p^{v_p(n)} A$$

  (where $v_p(n)$ is the multiplicity of p in the factorization of n).
  $\mathcal{D}$: There exists a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}_+} \in A^{\mathbb{N}_+}$ of elements of A such that

  $$b_n = \sum_{d \mid n} d x_d^{n/d} = w_n(\mathbf{x}) \text{ for every } n \in \mathbb{N}_+.$$

  In other words, $\mathbf{x}$ belongs to the image of the ghost map w.

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:

  $\mathcal{C}$: Every $n \in \mathbb{N}_+$ and every prime divisor p of n satisfy

  $$\varphi_p\left(b_{n/p}\right) \equiv b_n \operatorname{mod} p^{v_p(n)} A$$

  (where $v_p(n)$ is the multiplicity of p in the factorization of n).

  $\mathcal{D}$: There exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}_+} \in A^{\mathbb{N}_+}$ of elements of A such that

  $$b_n = \sum_{d \mid n} d x_d^{n/d} = w_n(\mathbf{x}) \quad \text{for every } n \in \mathbb{N}_+.$$

  In other words, $\mathbf{x}$ belongs to the image of the ghost map w.

  $\mathcal{E}$: There exists a sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}_+} \in A^{\mathbb{N}_+}$ of elements of A such that

  $$b_n = \sum_{d \mid n} d \varphi_{n/d}(y_d) \quad \text{for every } n \in \mathbb{N}_+.$$

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$: Every $n \in \mathbb{N}_+$ satisfies

$$\sum_{d \mid n} \mu(d) \varphi_d \left(b_{n/d}\right) \in nA.$$

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$: Every $n \in \mathbb{N}_+$ satisfies

$$\sum_{d \mid n} \mu(d) \varphi_d \left(b_{n/d}\right) \in nA.$$

  $\mathcal{G}$: Every $n \in \mathbb{N}_+$ satisfies

$$\sum_{d \mid n} \phi(d) \varphi_d \left(b_{n/d}\right) \in nA.$$

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$: Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d \mid n} \mu(d) \varphi_d \left(b_{n/d}\right) \in nA.$$

  $\mathcal{G}$: Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d \mid n} \phi(d) \varphi_d \left(b_{n/d}\right) \in nA.$$

  $\mathcal{J}$: There exists a ring homomorphism from the ring Λ to A which sends p_n (the n-th power sum symmetric function) to b_n for every $n \in \mathbb{N}_+$.

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$: Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d \mid n} \mu(d) \varphi_d\left(b_{n/d}\right) \in nA.$$

  $\mathcal{G}$: Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d \mid n} \phi(d) \varphi_d\left(b_{n/d}\right) \in nA.$$

  $\mathcal{J}$: There exists a ring homomorphism from the ring Λ to A which sends p_n (the n-th power sum symmetric function) to b_n for every $n \in \mathbb{N}_+$.

- Note that this theorem has various neat consequences, like the famous necklace divisibility $n \mid \sum_{d \mid n} \mu(d) q^{n/d}$ for $n \in \mathbb{N}_+$ and $q \in \mathbb{Z}$. (And various generalizations.)

Now to something completely different...

- Fix a prime power q.
- There is a famous analogy between the elements of $\mathbb{Z}$ and the elements of $\mathbb{F}_q[T]$. (This is related to q-enumeration, the lore of the field with 1 element, etc.)
  All that matters to us is that
  - **positive** integers in $\mathbb{Z}$ correspond to **monic** polynomials in $\mathbb{F}_q[T]$;
  - **primes** in $\mathbb{Z}$ correspond to **irreducible** monic polynomials in $\mathbb{F}_q[T]$.

Now to something completely different...

- Fix a prime power q.
- There is a famous analogy between the elements of $\mathbb{Z}$ and the elements of $\mathbb{F}_q[T]$. (This is related to q-enumeration, the lore of the field with 1 element, etc.)
  All that matters to us is that
  - **positive** integers in $\mathbb{Z}$ correspond to **monic** polynomials in $\mathbb{F}_q[T]$;
  - **primes** in $\mathbb{Z}$ correspond to **irreducible** monic polynomials in $\mathbb{F}_q[T]$.
- Let $\mathbb{F}_q[T]_+$ be the set of all **monic** polynomials in $\mathbb{F}_q[T]$.

$\mathbb{Z}$ and $\mathbb{F}_q[T]$: a tale of two rings

Now to something completely different...

- Fix a prime power q.
- There is a famous analogy between the elements of $\mathbb{Z}$ and the elements of $\mathbb{F}_q[T]$. (This is related to q-enumeration, the lore of the field with 1 element, etc.)
  All that matters to us is that
  - **positive** integers in $\mathbb{Z}$ correspond to **monic** polynomials in $\mathbb{F}_q[T]$;
  - **primes** in $\mathbb{Z}$ correspond to **irreducible** monic polynomials in $\mathbb{F}_q[T]$.
- Let $\mathbb{F}_q[T]_+$ be the set of all **monic** polynomials in $\mathbb{F}_q[T]$.
- Let's define an analogue of (big) Witt vectors for $\mathbb{F}_q[T]$ instead of $\mathbb{Z}$.

Definition of $\mathbb{F}_q[T]$-Witt vectors, 1: ghost maps

- Let A be a commutative $\mathbb{F}_q[T]$-algebra.
  We abbreviate a family $(a_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ as $\mathbf{a}$.
- For each $N \in \mathbb{F}_q[T]_+$, define a map $w_N : A^{\mathbb{F}_q[T]_+} \to A$ by

$$w_N(\mathbf{a}) = \sum_{D \mid N} D a_D^{q^{\deg(N/D)}},$$

  where the sum is over all **monic** divisors D of N.
- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

$$w(\mathbf{a}) = (w_N(\mathbf{a}))_{N \in \mathbb{F}_q[T]_+}.$$

Definition of $\mathbb{F}_q[T]$-Witt vectors, 1: ghost maps

- Let A be a commutative $\mathbb{F}_q[T]$-algebra.
  We abbreviate a family $(a_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ as $\mathbf{a}$.
- For each $N \in \mathbb{F}_q[T]_+$, define a map $w_N : A^{\mathbb{F}_q[T]_+} \to A$ by

$$w_N(\mathbf{a}) = \sum_{D \mid N} D a_D^{q^{\deg(N/D)}},$$

  where the sum is over all **monic** divisors D of N.
- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

$$w(\mathbf{a}) = (w_N(\mathbf{a}))_{N \in \mathbb{F}_q[T]_+}.$$

- This "ghost map" w is $\mathbb{F}_q$-linear, but not $\mathbb{F}_q[T]$-linear.

Definition of $\mathbb{F}_q[T]$-Witt vectors, 2: $W_q(A)$

- Let's make a new $\mathbb{F}_q[T]$-algebra out of this: We define $W_q(A)$ to be the $\mathbb{F}_q[T]$-algebra
  - that equals $A^{\mathbb{F}_q[T]_+}$ as a set, but
  - which is functorial in A (that is, we are really defining a functor $W_q : \mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{CRing}_{\mathbb{F}_q[T]}$, where $\mathbf{CRing}_R$ is the category of commutative R-algebras), and
  - whose $\mathbb{F}_q[T]$-algebra structure is such that w is a natural (in A) homomorphism of $\mathbb{F}_q[T]$-algebras from $W_q(A)$ to $A^{\mathbb{F}_q[T]_+}$.

Definition of $\mathbb{F}_q[T]$-Witt vectors, 2: $W_q(A)$, cont'd

- **Example:** The addition in $W_q(A)$ is the same as in $A^{\mathbb{F}_q[T]_+}$ (since w is $\mathbb{F}_q$-linear, and so $W_q(A) = A^{\mathbb{F}_q[T]_+}$ as $\mathbb{F}_q$-modules), so this would be boring.

- **Example:** The addition in $W_q(A)$ is the same as in $A^{\mathbb{F}_q[T]_+}$ (since w is $\mathbb{F}_q$-linear, and so $W_q(A) = A^{\mathbb{F}_q[T]_+}$ as $\mathbb{F}_q$-modules), so this would be boring. Instead, let's set $\mathbf{c} = T\mathbf{a}$ in $W_q(A)$, and compute $w_\pi(\mathbf{c})$ for an **irreducible** π.

- **Example:** The addition in $W_q(A)$ is the same as in $A^{\mathbb{F}_q[T]_+}$ (since w is $\mathbb{F}_q$-linear, and so $W_q(A) = A^{\mathbb{F}_q[T]_+}$ as $\mathbb{F}_q$-modules), so this would be boring. Instead, let's set $\mathbf{c} = T\mathbf{a}$ in $W_q(A)$, and compute $w_\pi(\mathbf{c})$ for an **irreducible** π. Start with $c_1 = Ta_1$, which is easy to check.

- **Example:** The addition in $W_q(A)$ is the same as in $A^{\mathbb{F}_q[T]_+}$ (since w is $\mathbb{F}_q$-linear, and so $W_q(A) = A^{\mathbb{F}_q[T]_+}$ as $\mathbb{F}_q$-modules), so this would be boring. Instead, let's set $\mathbf{c} = T\mathbf{a}$ in $W_q(A)$, and compute $w_\pi(\mathbf{c})$ for an **irreducible** π.
  Start with $c_1 = Ta_1$, which is easy to check.
  $$w_\pi(\mathbf{c}) = Tw_\pi(\mathbf{a})$$
  $$\Longleftrightarrow c_1^{q^{\deg \pi}} + \pi c_\pi = Ta_1^{q^{\deg \pi}} + T\pi a_\pi$$
  $$\overset{c_1 = Ta_1}{\Longleftrightarrow} (Ta_1)^{q^{\deg \pi}} + \pi c_\pi = Ta_1^{q^{\deg \pi}} + T\pi a_\pi$$
  $$\Longleftrightarrow \pi c_\pi = T\pi a_\pi - \left(T^{q^{\deg \pi}} - T\right) a_1^{q^{\deg \pi}}$$
  $$\overset{\text{naturality}}{\Longleftrightarrow} c_\pi = Ta_\pi - \frac{T^{q^{\deg \pi}} - T}{\pi} a_1^{q^{\deg \pi}}.$$
  The fraction on the RHS is a polynomial due to a known fact from Galois theory (namely:
  $$T^{q^k} - T = \prod_{\gamma \in \mathbb{F}_q[T]_+ \text{ irreducible; } \deg \gamma | k} \gamma).$$

Definition of $\mathbb{F}_q[T]$-Witt vectors, 3: coda

- There is also a second construction of $W_q(A)$, using Carlitz polynomials, yielding an isomorphic $\mathbb{F}_q[T]$-algebra. (See the preprint.)

- Can we find anything similar to the two avatars of $W(A)$?

Avatars of $\mathbb{F}_q[T]$-Witt vectors?

- Can we find anything similar to the two avatars of $W(A)$?
- Power series? This appears to require a notion of power series where the exponents are polynomials in $\mathbb{F}_q[T]$. Product ill-defined due to lack of actual "positivity". Seems too much to wish...

- Can we find anything similar to the two avatars of $W(A)$?
- Power series? This appears to require a notion of power series where the exponents are polynomials in $\mathbb{F}_q[T]$. Product ill-defined due to lack of actual "positivity". Seems too much to wish...
- $\mathrm{Alg}(\Lambda, A)$? Well, we can try brute force: Remember how Λ was reconstructed from W, and do something similar to "reconstruct" a representing object from W_q. We'll come back to this shortly.

Surprise: $\mathcal{F}$-modules, 1

- First, a surprise...

- First, a surprise...
- We aren't using the whole $\mathbb{F}_q[T]$-algebra structure on A ! (This is unlike the $\mathbb{Z}$-case, where it seems that we use the commutative ring A in full.)

- Let $\mathcal{F}$ be the noncommutative ring

$$\mathbb{F}_q \langle F, T \mid FT = T^q F \rangle .$$

This is an $\mathbb{F}_q$-vector space with basis $\left(T^i F^j\right)_{(i,j)\in\mathbb{N}^2}$, and is an Ore polynomial ring. It shares many properties of usual univariate polynomials (see papers of Ore).

Surprise: $\mathcal{F}$-modules, 2

- Let $\mathcal{F}$ be the noncommutative ring
$$\mathbb{F}_q \langle F, T \mid FT = T^q F \rangle .$$
This is an $\mathbb{F}_q$-vector space with basis $\left(T^i F^j\right)_{(i,j)\in\mathbb{N}^2}$, and is an Ore polynomial ring. It shares many properties of usual univariate polynomials (see papers of Ore).
- Actually,
$$\mathcal{F} \cong \left(\mathbb{F}_q [T] [X]_{q-\mathrm{lin}}, +, \circ\right),$$
where $\mathbb{F}_q [T] [X]_{q-\mathrm{lin}}$ are the polynomials in X over $\mathbb{F}_q [T]$ where X occurs only with exponents q^k, and where $\circ$ is composition of polynomials.

- Let $\mathcal{F}$ be the noncommutative ring

$$\mathbb{F}_q \langle F, T \mid FT = T^q F \rangle .$$

This is an $\mathbb{F}_q$-vector space with basis $\left(T^i F^j\right)_{(i,j)\in\mathbb{N}^2}$, and is an Ore polynomial ring. It shares many properties of usual univariate polynomials (see papers of Ore).

- What matters to us:
Each commutative $\mathbb{F}_q [T]$-algebra canonically becomes a (left) $\mathcal{F}$-module by having
  - T act as multiplication by T, and
  - F act as the Frobenius (i.e., taking q-th powers).

Thus, we have a functor $\mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{Mod}_{\mathcal{F}}$.

Surprise: $\mathcal{F}$-modules, 2

- Let $\mathcal{F}$ be the noncommutative ring

$$\mathbb{F}_q \langle F, T \mid FT = T^q F \rangle .$$

  This is an $\mathbb{F}_q$-vector space with basis $\left(T^i F^j\right)_{(i,j)\in\mathbb{N}^2}$, and is an Ore polynomial ring. It shares many properties of usual univariate polynomials (see papers of Ore).
- What matters to us:
  Each commutative $\mathbb{F}_q[T]$-algebra canonically becomes a (left) $\mathcal{F}$-module by having
  - T act as multiplication by T, and
  - F act as the Frobenius (i.e., taking q-th powers).

  Thus, we have a functor $\mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{Mod}_{\mathcal{F}}$.
- There are other sources of $\mathcal{F}$-modules too (cf. Jacobson on "commutative restricted Lie algebras").

$\mathbb{F}_q[T]$-Witt vectors of an $\mathcal{F}$-module

- Let A be a (left) $\mathcal{F}$-module.
  We abbreviate a family $(a_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ as $\mathbf{a}$.
- For each $N \in \mathbb{F}_q[T]_+$, define a map $w_N : A^{\mathbb{F}_q[T]_+} \to A$ by

$$w_N(\mathbf{a}) = \sum_{D \mid N} D F^{\deg(N/D)} a_D,$$

  where the sum is over all **monic** divisors D of N.
- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

$$w(\mathbf{a}) = (w_N(\mathbf{a}))_{N \in \mathbb{F}_q[T]_+}.$$

$\mathbb{F}_q[T]$-Witt vectors of an $\mathcal{F}$-module

- Let A be a (left) $\mathcal{F}$-module.
  We abbreviate a family $(a_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ as $\mathbf{a}$.
- For each $N \in \mathbb{F}_q[T]_+$, define a map $w_N : A^{\mathbb{F}_q[T]_+} \to A$ by

$$w_N(\mathbf{a}) = \sum_{D|N} DF^{\deg(N/D)} a_D,$$

  where the sum is over all **monic** divisors D of N.
- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

$$w(\mathbf{a}) = (w_N(\mathbf{a}))_{N \in \mathbb{F}_q[T]_+}.$$

- We define $W_q(A)$ to be the $\mathcal{F}$-module
  - that equals $A^{\mathbb{F}_q[T]_+}$ as a set, but
  - which is functorial in A (that is, we are really defining a functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$), and
  - whose $\mathcal{F}$-module structure is such that w is a natural (in A) homomorphism of $\mathcal{F}$-modules from $W_q(A)$ to $A^{\mathbb{F}_q[T]_+}$.

An $\mathcal{F}$-ghost-Witt integrality theorem, 1

- Again, there is a "ghost-Witt integrality theorem" that helps prove the existence of the W_q functors.

- **$\mathcal{F}$-ghost-Witt integrality theorem.**
  Let A be a (left) $\mathcal{F}$-module. For every $P \in \mathbb{F}_q[T]_+$, let $\varphi_P : A \to A$ be an endomorphism of the $\mathcal{F}$-module A.
  Assume that:
  - We have $\varphi_\pi(a) \equiv F^{\deg \pi} a \mod \pi A$ for every $a \in A$ and every monic irreducible $\pi \in \mathbb{F}_q[T]_+$.
  - We have $\varphi_1 = \mathrm{id}$, and we have $\varphi_N \circ \varphi_M = \varphi_{NM}$ for every $N, M \in \mathbb{F}_q[T]_+$. (Thus, $N \mapsto \varphi_N$ is an action of the multiplicative monoid $\mathbb{F}_q[T]_+$ on A by $\mathcal{F}$-module endomorphisms.)

- **$\mathcal{F}$-ghost-Witt integrality theorem.**
  Let A be a (left) $\mathcal{F}$-module. For every $P \in \mathbb{F}_q[T]_+$, let $\varphi_P : A \to A$ be an endomorphism of the $\mathcal{F}$-module A.
  Assume that:
  - We have $\varphi_\pi(a) \equiv F^{\deg \pi} a \mod \pi A$ for every $a \in A$ and every monic irreducible $\pi \in \mathbb{F}_q[T]_+$.
  - We have $\varphi_1 = \mathrm{id}$, and we have $\varphi_N \circ \varphi_M = \varphi_{NM}$ for every $N, M \in \mathbb{F}_q[T]_+$. (Thus, $N \mapsto \varphi_N$ is an action of the multiplicative monoid $\mathbb{F}_q[T]_+$ on A by $\mathcal{F}$-module endomorphisms.)
- Let $\mathbf{b} = (b_n)_{n \in \mathbb{N}_+} \in A^{\mathbb{F}_q[T]_+}$ be a family of elements of A. Then, the following assertions are equivalent: [continued on next page]

An $\mathcal{F}$-ghost-Witt integrality theorem, 3

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:
  - $\mathcal{C}$: Every $N \in \mathbb{F}_q[T]_+$ and every monic irreducible divisor π of N satisfy
  $$\varphi_\pi\left(b_{N/\pi}\right) \equiv b_N \bmod \pi^{v_\pi(N)} A.$$

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:
  $\mathcal{C}$: Every $N \in \mathbb{F}_q[T]_+$ and every monic irreducible divisor π of N satisfy
  $$\varphi_\pi\left(b_{N/\pi}\right) \equiv b_N \mod \pi^{v_\pi(N)} A.$$
  $\mathcal{D}_2$: There exists a family $\mathbf{x} = (x_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ of elements of A such that
  $$b_N = \sum_{D \mid N} DF^{\deg(N/D)} x_D = w_N(\mathbf{x}) \quad \text{for every } N \in \mathbb{F}_q[T]_+.$$
  In other words, $\mathbf{x}$ belongs to the image of the ghost map w.

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:

  $\mathcal{C}$: Every $N \in \mathbb{F}_q[T]_+$ and every monic irreducible divisor π of N satisfy

  $$\varphi_\pi\left(b_{N/\pi}\right) \equiv b_N \operatorname{mod} \pi^{v_\pi(N)} A.$$

  $\mathcal{D}_1$: There exists a family $\mathbf{x} = (x_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ of elements of A such that

  $$b_N = \sum_{D \mid N} D \frac{N}{D}[T+F]\, x_D \qquad \text{for every } N \in \mathbb{F}_q[T]_+.$$

  [This is mainly interesting due to the connection to Carlitz polynomials.]

- **Ghost-Witt integrality theorem, continued.**
  The following are equivalent:
  - $\mathcal{C}$: Every $N \in \mathbb{F}_q[T]_+$ and every monic irreducible divisor π of N satisfy
  $$\varphi_\pi \left(b_{N/\pi} \right) \equiv b_N \bmod \pi^{v_\pi(N)} A.$$
  - $\mathcal{D}_2$: There exists a family $\mathbf{x} = (x_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ of elements of A such that
  $$b_N = \sum_{D \mid N} D F^{\deg(N/D)} x_D = w_N(\mathbf{x}) \text{ for every } N \in \mathbb{F}_q[T]_+.$$
  In other words, $\mathbf{x}$ belongs to the image of the ghost map w.
  - $\mathcal{E}$: There exists a family $\mathbf{y} = (y_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ of elements of A such that
  $$b_N = \sum_{D \mid N} D \varphi_{N/D}(y_D) \text{ for every } N \in \mathbb{F}_q[T]_+.$$

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$**:** Every $N \in \mathbb{F}_q[T]_+$ satisfies

  $$\sum_{D|N} \mu(D) \varphi_D(b_{N/D}) \in NA.$$

  Here, μ is an $\mathbb{F}_q[T]$-version of the Möbius function, defined as the usual one (i.e., squarefree $\mapsto$ number of distinct irreducible factors; non-squarefree $\mapsto$ 0).

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$: Every $N \in \mathbb{F}_q[T]_+$ satisfies

$$\sum_{D|N} \mu(D) \varphi_D(b_{N/D}) \in NA.$$

  Here, μ is an $\mathbb{F}_q[T]$-version of the Möbius function, defined as the usual one (i.e., squarefree $\mapsto$ number of distinct irreducible factors; non-squarefree $\mapsto$ 0).

  $\mathcal{G}$: Every $N \in \mathbb{F}_q[T]_+$ satisfies

$$\sum_{D|N} \phi(D) \varphi_D(b_{N/D}) \in NA,$$

  where ϕ is one of two reasonable $\mathbb{F}_q[T]$-versions of the Euler totient function.

- **Ghost-Witt integrality theorem, continued.**

$\mathcal{F}$: Every $N \in \mathbb{F}_q[T]_+$ satisfies

$$\sum_{D \mid N} \mu(D) \varphi_D\left(b_{N/D}\right) \in NA.$$

Here, μ is an $\mathbb{F}_q[T]$-version of the Möbius function, defined as the usual one (i.e., squarefree $\mapsto$ number of distinct irreducible factors; non-squarefree $\mapsto$ 0).

$\mathcal{G}$: Every $N \in \mathbb{F}_q[T]_+$ satisfies

$$\sum_{D \mid N} \phi(D) \varphi_D\left(b_{N/D}\right) \in NA,$$

where ϕ is one of two reasonable $\mathbb{F}_q[T]$-versions of the Euler totient function.

$\mathcal{J}$: ???

- **Ghost-Witt integrality theorem, continued.**

  $\mathcal{F}$: Every $N \in \mathbb{F}_q[T]_+$ satisfies

  $$\sum_{D \mid N} \mu(D) \varphi_D\left(b_{N/D}\right) \in NA.$$

  Here, μ is an $\mathbb{F}_q[T]$-version of the Möbius function, defined as the usual one (i.e., squarefree $\mapsto$ number of distinct irreducible factors; non-squarefree $\mapsto$ 0).

  $\mathcal{G}$: Every $N \in \mathbb{F}_q[T]_+$ satisfies

  $$\sum_{D \mid N} \phi(D) \varphi_D\left(b_{N/D}\right) \in NA,$$

  where ϕ is one of two reasonable $\mathbb{F}_q[T]$-versions of the Euler totient function.

  $\mathcal{J}$: ???

- To state $\mathcal{J}$, we need an $\mathbb{F}_q[T]$-analogue of the symmetric functions.

- Now, back to the question: We have found two functors

$$W_q : \mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{CRing}_{\mathbb{F}_q[T]} \qquad \text{and}$$
$$W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}.$$

What are their representing objects? Call them $\Lambda'_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$.

- Now, back to the question: We have found two functors

$$W_q : \mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{CRing}_{\mathbb{F}_q[T]} \qquad \text{and}$$
$$W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}.$$

  What are their representing objects? Call them $\Lambda'_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$.

- Both objects (they are distinct) have good claims on the name "$\mathbb{F}_q[T]$-symmetric functions".

Tinfoil, 1: What is $\Lambda_{\mathcal{F}}$?

- Now, back to the question: We have found two functors

$$W_q : \mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{CRing}_{\mathbb{F}_q[T]} \qquad \text{and}$$
$$W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}.$$

  What are their representing objects? Call them $\Lambda'_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$.
- Both objects (they are distinct) have good claims on the name "$\mathbb{F}_q[T]$-symmetric functions".
- I shall focus on $\Lambda_{\mathcal{F}}$, since it is smaller.

Tinfoil, 2: "Re"constructing $\Lambda_{\mathcal{F}}$

- Proceed in the same way as when we reconstructed Λ from the functor W, but now reconstruct the representing object $\Lambda_{\mathcal{F}}$ of the functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$:
  - The functor $\text{Forget} \circ W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Set}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-module (by Yoneda).
  - The functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-$\mathcal{F}$-bimodule.
  - There is an additional comonad structure on W_q, which determines a "plethysm" on $\Lambda_{\mathcal{F}}$, but I know nothing about it.

Tinfoil, 2: “Re”constructing $\Lambda_{\mathcal{F}}$

- Proceed in the same way as when we reconstructed Λ from the functor W, but now reconstruct the representing object $\Lambda_{\mathcal{F}}$ of the functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$:
  - The functor $\text{Forget} \circ W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Set}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-module (by Yoneda).
  - The functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-$\mathcal{F}$-bimodule.
  - There is an additional comonad structure on W_q, which determines a “plethysm” on $\Lambda_{\mathcal{F}}$, but I know nothing about it.
- So what is this $\Lambda_{\mathcal{F}}$?

Tinfoil, 2: "Re" constructing $\Lambda_{\mathcal{F}}$

- Proceed in the same way as when we reconstructed Λ from the functor W, but now reconstruct the representing object $\Lambda_{\mathcal{F}}$ of the functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$:
    - The functor $\text{Forget} \circ W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Set}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-module (by Yoneda).
    - The functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-$\mathcal{F}$-bimodule.
    - There is an additional comonad structure on W_q, which determines a "plethysm" on $\Lambda_{\mathcal{F}}$, but I know nothing about it.
- So what is this $\Lambda_{\mathcal{F}}$?
- I don't really know.

- At least, we can compute in $\Lambda_{\mathcal{F}}$ (in theory):
- The left $\mathcal{F}$-module $\Lambda_{\mathcal{F}}$ has a basis $(w_N)_{N \in \mathbb{F}_q[T]_+}$, similarly to the generating set $(w_n)_{n \in \mathbb{N}_+}$ of the commutative ring Λ.
- The left $\mathcal{F}$-module $\Lambda_{\mathcal{F}}$ has an "almost-basis" $(p_N)_{N \in \mathbb{F}_q[T]_+}$, similarly to the "almost-generating set" $(p_n)_{n \in \mathbb{N}_+}$ of the commutative ring Λ.

  Here, "almost-basis" means "basis after localizing so that elements of $\mathbb{F}_q[T]_+$ become invertible". (Noncommutative localization, but a harmless case thereof.)

- At least, we can compute in $\Lambda_{\mathcal{F}}$ (in theory):
- The left $\mathcal{F}$-module $\Lambda_{\mathcal{F}}$ has a basis $(w_N)_{N \in \mathbb{F}_q[T]_+}$, similarly to the generating set $(w_n)_{n \in \mathbb{N}_+}$ of the commutative ring Λ.
- The left $\mathcal{F}$-module $\Lambda_{\mathcal{F}}$ has an "almost-basis" $(p_N)_{N \in \mathbb{F}_q[T]_+}$, similarly to the "almost-generating set" $(p_n)_{n \in \mathbb{N}_+}$ of the commutative ring Λ.

  Here, "almost-basis" means "basis after localizing so that elements of $\mathbb{F}_q[T]_+$ become invertible". (Noncommutative localization, but a harmless case thereof.)
- The right $\mathcal{F}$-module structure is easily expressed on the p_N's (just as the second comultiplication of Λ is easily expressed on the p_n's):

$$p_N f = f p_N \qquad \text{for all } f \in \mathcal{F} \text{ and } N \in \mathbb{F}_q[T]_+.$$

- You can thus express qf for each $f \in \mathcal{F}$ and $q \in \Lambda_{\mathcal{F}}$ by recursion (all fractions will turn out polynomial at the end), but nothing really explicit.

- Here are some of these expressions:

$$w_\pi T = T w_\pi - \frac{T^{q^{\deg \pi}} - T}{\pi} F^{\deg \pi} w_1;$$
$$w_\pi F = \pi^{q-1} F w_\pi$$

for any irreducible $\pi \in \mathbb{F}_q[T]_+$.

Bonus oddity (back in $\mathbb{Z}$): a "ghost-Burnside theorem"

- This is not $\mathbb{F}_q[T]$-related, but I find it curious.
- Remember how the ghost-Witt equivalence theorem generalizes the divisibility $n \mid \sum_{d|n} \mu(d) q^{n/d}$ for $n \in \mathbb{N}_+$ and $q \in \mathbb{Z}$:

Bonus oddity (back in $\mathbb{Z}$): a "ghost-Burnside theorem"

- This is not $\mathbb{F}_q[T]$-related, but I find it curious.
- Remember how the ghost-Witt equivalence theorem generalizes the divisibility $n \mid \sum_{d|n} \mu(d) q^{n/d}$ for $n \in \mathbb{N}_+$ and $q \in \mathbb{Z}$:
- **Ghost-Witt:** The following (among others) are equivalent:

  $\mathcal{C}$**:** Every $n \in \mathbb{N}_+$ and every prime divisor p of n satisfy

  $$\varphi_p\left(b_{n/p}\right) \equiv b_n \bmod p^{v_p(n)} A$$

  (where $v_p(n)$ is the multiplicity of p in the factorization of n).

  $\mathcal{G}$**:** Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d|n} \phi(d)\, \varphi_d\left(b_{n/d}\right) \in nA.$$

[Remember that you can pick $\varphi_n = \mathrm{id}$ when $A = \mathbb{Z}$.]

Bonus oddity (back in $\mathbb{Z}$): a "ghost-Burnside theorem"

- This is not $\mathbb{F}_q[T]$-related, but I find it curious.
- The following strange equivalence also generalizes the divisibility $n \mid \sum_{d \mid n} \mu(d) q^{n/d}$ for $n \in \mathbb{N}_+$ and $q \in \mathbb{Z}$:
- **Ghost-Burnside:** The following are equivalent:

  $\mathcal{R}$**:** Every $n \in \mathbb{N}_+$, every $d \mid n$ and every prime divisor p of d satisfy

  $$\phi(d)\, b_d^{n/d} \equiv \phi(d)\, b_{d/p}^{n/(d/p)} \mod p^{v_p(n)} A$$

  (where $v_p(n)$ is the multiplicity of p in the factorization of n).

  $\mathcal{S}$**:** Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d \mid n} \phi(d)\, b_d^{n/d} \in nA.$$

Bonus oddity (back in $\mathbb{Z}$): a "ghost-Burnside theorem"

- This is not $\mathbb{F}_q[T]$-related, but I find it curious.
- The following strange equivalence also generalizes the divisibility $n \mid \sum_{d|n} \mu(d) q^{n/d}$ for $n \in \mathbb{N}_+$ and $q \in \mathbb{Z}$:
- **Ghost-Burnside:** The following are equivalent:

  $\mathcal{R}$: When A is "nice" (viz., $px \in p^k A \Longrightarrow x \in p^{k-1} A$, and the quotient ring A/pA is reduced), this simplifies to: Every $d \in \mathbb{N}_+$ and every prime divisor p of d satisfy

  $$b_{d/p}^p \equiv b_d \mod pA.$$

  $\mathcal{S}$: Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d|n} \phi(d)\, b_d^{n/d} \in nA.$$

Bonus oddity (back in $\mathbb{Z}$): a "ghost-Burnside theorem"

- This is not $\mathbb{F}_q[T]$-related, but I find it curious.
- The following strange equivalence also generalizes the divisibility $n \mid \sum_{d \mid n} \mu(d) q^{n/d}$ for $n \in \mathbb{N}_+$ and $q \in \mathbb{Z}$:
- **Ghost-Burnside:** The following are equivalent:

  $\mathcal{R}$: Every $n \in \mathbb{N}_+$, every $d \mid n$ and every prime divisor p of d satisfy

  $$\phi(d)\, b_d^{n/d} \equiv \phi(d)\, b_{d/p}^{n/(d/p)} \mod p^{v_p(n)} A$$

  (where $v_p(n)$ is the multiplicity of p in the factorization of n).

  $\mathcal{S}$: Every $n \in \mathbb{N}_+$ satisfies

  $$\sum_{d \mid n} \phi(d)\, b_d^{n/d} \in nA.$$

- This leads to a notion of "ghost-Burnside vectors", which also form a subring of $A^{\mathbb{N}_+}$. Not sure yet what they are good for...

Thanks to James Borger for some inspiring discussions. Thanks to Christophe Reutenauer for historiographical comments.
And thank you!