COHOMOLOGY AND CENTRAL SIMPLE ALGEBRAS

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Errata (Darij Grinberg)

§1

- On page 2, replace "it's homology" by "its homology". (This is in the definition of "homology", at the middle of the page.)
- On page 2, example 1.3 (2) works only if $a \neq 0$.
- On page 4, replace "it's" by "its" in Example 1.6.
- On page 5, in Example 1.7, "n-plane" means "affine n-plane". Maybe you should specify this.
- On page 5, in the definition of "simplicial complex", you require: "(1) If $s \in K$ then so is every face of K." What you mean is: "... so is every face of s."
- On page 7, you write "$Z_1(C) =$ is set of [...]" (in the middle of the page). The "is" is redundant here.
- On page 7, you write "$B_1(C)$ is set of $\zeta([24] - [14] + [12])$ with $\zeta, \eta \in \mathbb{Z}$" (again, in the middle of this page). But there is no η in $\zeta([24] - [14] + [12])$.
- On page 9, in the commutative diagram in Definition 1.10 the lower exact sequence should be $... \to D_{n+1} \to D_n \to D_{n-1} \to ...$, not $... \to C_{n+1} \to C_n \to C_{n-1} \to ...$.
- On page 11, in Corollary 1.15, the $H^{n+1}(C, N)$ term should be $H^{n+1}(C, L)$. Also, in the same Corollary, you write "you get an exact sequence [...]". This is, in my opinion, somewhat misleading; it would be better to say that "the sequence [...] is exact" (so it becomes clear that the sequence always exists, even if C_n were not projective, but its exactness requires the projectivity of C_n). But that's just my opinion.
- On page 12, in Definition 1.19, "gf is homotopic to Id_D and fg is homotopic to Id_C" should be the other way round (gf is homotopic to Id_C and fg is homotopic to Id_D).
- On page 14, the full stop after the exact sequence

  $$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

  makes no sense.
- On page 15, you write:

  $$= s_{n-1}\partial_n x + \partial_{n+1} s_n x + (x - s_{n-1}\partial_n x) = x.$$

  The $\partial_{n+1} s_n x$ term should be removed from this.

§2

- On page 17, in Proposition 2.3, replace "along θ" by "along ψ".
- On page 23, in the proof of Proposition 2.15, you write:

  $\operatorname{Ext}^n(M,N) \cong \operatorname{Ext}^{n-1}\left(M, @^i N\right) \cong \ldots \cong \operatorname{Ext}^1\left(M, @^{n-1}N\right) \qquad \text{(Dimension shifting)}.$

  The i here should be a 1 instead.
- On page 24, in Lemma 2.18, replace "Fix a projective resolution of N" by "Fix a projective resolution $P.$ of N" (so that the notation $P.$ is defined).
- On page 27, in the second line from the bottom of the page, you write: "There are maps θ, ϕ as in the Comparison Theorem." Maybe you mean the maps $\Omega^1\psi, \psi_0$ instead?
- On page 27, in the third line from the top of the page, replace α by α''.
- On page 27, in the third line from the top of the page, replace θ by ψ.
- On page 30, in Example 2.27 (1), how do you conclude that R is artinian if R has global dimension 0 ?
- On page 31, in the part (2) of the proof, you write: "there is an exact sequence of $R[x]$-modules [...]". I am totally nitpicking this time, you should make clear that the exact sequence if $0 \to R[x] \otimes_R M \xrightarrow{\alpha} R[x] \otimes_R M \xrightarrow{\beta} M \to 0$, not $0 \leftarrow R[x] \otimes_R M \xleftarrow{h_1} R[x] \otimes_R M \xleftarrow{h_0} M \leftarrow 0$.

§3

- On page 33, in Definition 3.2 replace "long exact sequence sequence" by "long exact sequence".
- On page 35, at the very beginning of this page, you write:

  $$= [g_0 \mid \ldots \mid g_n] - \sum\nolimits_{i=0}^{n} (-1)^n [g_0 \mid \ldots \mid \widehat{g_i} \mid \ldots \mid g_n].$$

  Two typos here: first, $(-1)^n$ should be $(-1)^i$; then, $[g_0 \mid \ldots \mid \widehat{g_i} \mid \ldots \mid g_n]$ should be $[1 \mid g_0 \mid \ldots \mid \widehat{g_i} \mid \ldots \mid g_n]$.
- On page 38, in Definition 3.11, you write: "A crossed homomorphism is principal if [...]". This should better be "A crossed homomorphism f is principal if [...]".
- On page 38, in the third line from the bottom of the page, you use the term "G-group". (You also use it some pages later.) I assume that it just a synonym for "multiplicative G-module"?

§4

- On page 43, in Definition 4.3, you define the notion of "A-B-bimodule". Maybe it makes sense to add "where A and B are R-algebras" somewhere here (although it's pretty much obvious).

- On page 43, in Definition 4.4, replace "S_n the the tensor product" by "S_n is the tensor product".

- On page 46, when you define the notion of "equivalent" for two algebra extensions, it is not directly clear that this is an equivalence relation. (This could be improved by putting this notion in relation with Definition 2.1, with the only difference being that this time $E \to E'$ is supposed to be an algebra morphism as well.)

- On page 46, you claim: "The split extensions form one equivalence class." Is there a quick proof for this? It is easy to see that every extension equivalent to $0 \to M \to A \oplus M \to A \to 0$ is split (where $A \oplus M$ has the obvious algebra structure), but it took me quite a while to show that every split extension is equivalent to $0 \to M \to A \oplus M \to A \to 0$. Am I missing something obvious?

- On page 46, in Theorem 4.11, you write: "(For comparison, $H^1(A, M)$ classifies the extensions $0 \to M \to E \to A \to 0$ of A-A-bimodules.)" To be completely precise, it doesn't classify these extensions, but it classifies their equivalence classes. If you think the word "equivalence class" is already implicit in "classify", then you could remove "equivalence classes" from the formulation "$H^2(A, M)$ classifies the equivalence classes of algebra extension".

- In the middle of page 46, the full stop at the end of "Its failure is given by the map." makes no sense.

- On page 46, in the third line from the bottom of the page, you write: $(a, x)(b, y) = (ab, ay + xb + f(a, b))$. This doesn't seem to work for me; I need

  $$(a, x)(b, y) = (ab, ay + xb - f(a, b)).$$

  One of us made a tiny calculation mistake.

- On page 47, in the proof of Proposition 4.14, replace "$A \otimes_R A$ is projective as a left A-module" by "$A \otimes_R A$ is projective as a right A-module", and replace "and all $\Omega^n A$ are projective left A-modules" by "and all $\Omega^n A$ are projective right A-modules". Of course, this is true both for left and for right A-modules, but what you need are the right ones, not the left (since P_i must be a right A-module in order for $P_i \otimes_A M$ to make sense).

- On page 47, on the last line of this page, replace $\operatorname{proj.dim} M$ by $\operatorname{proj.dim}_A M$.

§5

- On page 51, in the third line from the top of the page, replace "the separability idempotent" by "a separability idempotent e".

- On page 51, in the fifth line from the top of the page, replace $(a \otimes 1)\, e = (a \otimes 1)\, e$ by $(a \otimes 1)\, e = e\, (a \otimes 1)$.

- On page 51, in the sixth line from the top of the page, replace $z_0 e$ by $z_0 a$.

- On page 51, in the proof of Proposition 5.3, read "some irreducible polynomial f in $K[x]$" for "some irreducible polynomial in $K[x]$".

- On page 52, in the second line from the top of the page, replace $d = \lambda \in K$ by $d = \lambda \in \overline{K}$.

- On page 52, in Theorem 5.7, replace

  $$\alpha_g (\ell z) = g (\ell)\, z \text{ for } z \in Z,\ \ell \in L$$

  by

  $$\alpha_g (\ell z) = g (\ell)\, \alpha_g (z) \text{ for } z \in Z,\ \ell \in L.$$

- On page 53, the first comma in "Let $v_{1,...,}v_n$ be a basis of L over K." should be one level higher (it should be on the same level with the v's, not with the indices).

- On page 53, at the end of the proof of Theorem 5.7, the formula

  $$\alpha_{g_j} = \sum\nolimits_i b_{ij} x_j$$

  is wrong; it should be

  $$\alpha_{g_i} = \sum\nolimits_j b_{ij} x_j.$$

- In the line directly after this formula, you write $z = \alpha_{g_1}(z) = \sum_i b_{1j} x_j$. The summation index should be j, not i.

- On page 54, in the long computation (which proves $\rho_\psi (gg') = \rho_\psi (g)\, (g \rho_\psi (g'))$), you write:

  $$= \rho_\psi (g)\, (1 \otimes g)\, \rho_Y (g')\, \left(1 \otimes g^{-1}\right).$$

  Replace ρ_Y by ρ_ψ here.

- In the bottommost line of page 54, you forgot a bracket in $H^1(G, \mathrm{Aut}\left(X^L\right)$.

- On page 56, in the Proof of Corollary 5.11, you write: "if x has the indicated form that $N(x) = 1$". It would be better to replace the "that" by "then" here.

- On page 56, in the Proof of Proposition 5.12, you write: "you can make K^n into a different module by making $a \in \mathrm{M}_n (K)$ act on $v \in K^n$ as $\theta (a)\, (v)$." This formulation is somewhat fishy. I propose "[...] by making $a \in \mathrm{M}_n (K)$ send $v \in K^n$ to $\theta (a)\, (v)$." or "[...] by making $a \in \mathrm{M}_n (K)$ act on K^n as $\theta (a)$.".

- On page 56, in Corollary 5.13, replace $H^1 (G, \mathrm{PGL}_n (K))$ by $H^1 (G, \mathrm{PGL}_n (L))$.

§6

- On page 58, on the second line of Example 6.4, you write L^* instead of $L^\times$. This mistake is repeated a few times.

- On page 58, in Example 6.4, "As a set" should be "As a K-module" (because you don't specify the additive group structure anywhere else). Maybe it should also be said that the canonical embedding of K into that K-algebra $L *_f G$ is *not* given by $1 \mapsto \underbrace{1}_{\in L} \cdot \underbrace{e}_{\in G}$, unless f is a *normalized* 2-cocycle.

- On page 59, two lines above Lemma 6.6, replace $\operatorname{End}_{A^e}(K)$ by $\operatorname{End}_{A^e}(A)$.

- On page 59, in Definition 6.9, "division algebras" could better be replaced by "division algebras (constructed in Lemma 6.2)".

- On page 60, Theorem 6.10 would become clearer if you write "A^L is central simple as an L-algebra" instead of just "A^L is central simple".

- On page 60, one line above Definition 6.14, you have a typo: $M_{n_1}(D_2)$ should be $M_{n_2}(D_2)$.

- On page 61, in Theorem 6.15, maybe it is better to clarify that the division algebras are supposed to be f.d. over K_s, not over K.

- On page 61, in the proof of Theorem 6.15, you write: $\delta_x : K_s \to D$. But this would be the zero map, since K_s is in the center of D. You want $\delta_x : D \to D$ instead.

- On page 62, the proof of Theorem 6.20 talks about "central simple K-algebras of dimension n^2 split by L". You haven't defined what "split by L" means for a central simple K-algebra. (In fact, you call a central simple K-algebra A *split by* L if $A^L \cong \mathrm{M}_n(L)$ as L-algebras for some $n \in \mathbb{N}$.)