Cohn's theorem on the kernel of the Dynkin operator

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Abstract

Abstract. In his 1951 thesis, P. M. Cohn described the kernel of the Dynkin operator on the tensor algebra. This note gives a modern exposition of this result with its proof, both in the case of a free module and in the case of an arbitrary module over a $\mathbb{Q}$-algebra. (In the fully general case, a counterexample is given.) No novelty is claimed.

Manifest. This note has been written by GPT-5.5 based on an excerpt of Cohn's thesis and some of my own work; it has then been substantially edited by myself. I thank Christophe Reutenauer for sending me the relevant part of Cohn's thesis. – DG[1]

Let $\mathbf{k}$ be a commutative ring, and let V be a $\mathbf{k}$-module. The tensor algebra of V is

$$T(V) := \bigoplus_{n \in \mathbb{N}} V^{\otimes n},$$

and its positive part is its ideal

$$T(V)^+ := \bigoplus_{n \geq 1} V^{\otimes n}.$$

We write multiplication in $T(V)$ by juxtaposition, and we use the notation

$$[a, b] := ab - ba$$

for its commutator bracket.

The *Dynkin operator* is the $\mathbf{k}$-linear map

$$\begin{aligned} D : T(V)^+ &\longrightarrow T(V)^+, \\ v_1 v_2 \cdots v_n &\longmapsto [[\cdots [[v_1, v_2], v_3], \ldots], v_n] \qquad \text{for } v_1, v_2, \ldots, v_n \in V. \end{aligned}$$

This is recursively defined by

$$D(v) = v \qquad \text{for all } v \in V, \qquad \text{and} \tag{1}$$

$$D(av) = [D(a), v] \qquad \text{for all } a \in T(V)^+ \text{ and } v \in V. \tag{2}$$

Let Z be the right ideal of $T(V)$ generated by the elements $xD(x)$ with $x \in T(V)^+$. In other words,

$$Z := \sum_{x \in T(V)^+} xD(x)\, T(V). \tag{3}$$

The main result is the following theorem of P. M. Cohn. It appears, in a somewhat compressed form, as Theorem 10.1 of his Ph.D. thesis [Coh51, pp. 88–89].

Theorem 1 (Cohn's kernel theorem)**.** *We always have*

$$Z \subseteq \operatorname{Ker} D. \tag{4}$$

Moreover, equality holds in each of the following two cases:

1. *The* $\mathbf{k}$*-module* V *is free.*
2. *The ring* $\mathbf{k}$ *is a* $\mathbb{Q}$*-algebra.*

Here and below, $\operatorname{Ker} D$ means the kernel of the map D; thus, it is a submodule of $T(V)^+$.

The proof below follows Cohn's three auxiliary maps, but replaces the last induction in his proof by a short ideal argument. We shall also prove two additional facts. First, Cohn's proposed homogeneous spanning description of Z is valid without any hypothesis on $\mathbf{k}$ or V. Second, the equality $Z = \operatorname{Ker} D$ is false for arbitrary modules; the obstruction is already visible in degree 2.

1 Two identities for the Dynkin operator

We begin with an identity that Cohn attributes to Baker. We include a proof, since this identity drives the whole argument.

Lemma 2 (Baker's identity)**.** *Let* $u, v \in T(V)^+$*. Then,*

$$D(uD(v)) = [D(u), D(v)]. \tag{5}$$

Proof. By $\mathbf{k}$-linearity, it is enough to prove (5) when u and v are pure tensors. We induct on the degree of v.

If $v \in V$, then $D(v) = v$, and the recursion (2) yields $D(uv) = [D(u), v]$. In view of $D(v) = v$, we can rewrite this as $D(uD(v)) = [D(u), D(v)]$. Thus, (5) is proved in the base case $v \in V$.

Now, assume that $v = v'a$ for some pure tensor $v' \in T(V)^+$ and some $a \in V$. Then, v' has lower degree than v, so our induction hypothesis yields

$$D(uD(v')) = [D(u), D(v')] \qquad \text{and} \qquad D(uaD(v')) = [D(ua), D(v')].$$

But $v = v'a$, so that

$$\begin{aligned} D(v) &= D(v'a) = [D(v'), a] \qquad \text{(by (2))} \\ &= D(v')a - aD(v'). \end{aligned}$$

Hence,

$$\begin{aligned} D(uD(v)) &= D(u(D(v')a - aD(v'))) = \underbrace{D(uD(v')a)}_{\substack{=[D(uD(v')),a] \\ \text{(by (2))}}} - \underbrace{D(uaD(v'))}_{=[D(ua),D(v')]} \\ &= \left[\underbrace{D(uD(v'))}_{=[D(u),D(v')]}, a\right] - \left[\underbrace{D(ua)}_{\substack{=[D(u),a] \\ \text{(by (2))}}}, D(v')\right] \\ &= [[D(u), D(v')], a] - [[D(u), a], D(v')] = \left[D(u), \underbrace{[D(v'), a]}_{=D(v)}\right] \\ &\qquad \left(\begin{array}{c}\text{by the Jacobi identity for commutators in the} \\ \text{form } [[x,y],z] - [[x,z],y] = [x,[y,z]]\end{array}\right) \\ &= [D(u), D(v)]. \end{aligned}$$

This completes the induction. $\square$

Corollary 3. *The submodule* $\operatorname{Ker} D$ *is a right ideal of* $T(V)$*. Moreover,* $Z \subseteq \operatorname{Ker} D$*.*

Proof. Let $g \in \operatorname{Ker} D$, and let $a \in V$. Then, (2) yields

$$D(ga) = [D(g), a] = 0 \qquad (\text{since } D(g) = 0).$$

An induction on the degree of a pure tensor $w \in T(V)$ now shows that $D(gw) = 0$ for each pure tensor $w \in T(V)$. By linearity, this must therefore hold for every $w \in T(V)$. Hence, $\operatorname{Ker} D$ is a right ideal.

For each $x \in T(V)^+$, Lemma 2 gives

$$D(xD(x)) = [D(x), D(x)] = 0.$$

Thus, every generator $xD(x)$ of Z belongs to $\operatorname{Ker} D$. Since $\operatorname{Ker} D$ is a right ideal, we obtain $Z \subseteq \operatorname{Ker} D$. $\square$

We shall also need the usual Dynkin–Specht–Wever identity.

Lemma 4 (Dynkin–Specht–Wever identity)**.** *Let $n \geq 1$. Then,*

$$D^2 = nD \qquad \text{on } V^{\otimes n}. \tag{6}$$

Proof. We induct on n. The claim is clear for $n = 1$. Let $n > 1$, and let $u \in V^{\otimes(n-1)}$ and $a \in V$. Then, $D^2(u) = (n-1) D(u)$ by the induction hypothesis. But (2) yields $D(ua) = [D(u), a] = D(u)a - aD(u)$. Applying D to this equality, we obtain

$$\begin{aligned} D^2(ua) &= D(D(u)a - aD(u)) = \underbrace{D(D(u)a)}_{\substack{=[D(D(u)),a] \\ \text{(by (2))}}} - \underbrace{D(aD(u))}_{\substack{=[D(a),D(u)] \\ \text{(by Lemma 2)}}} \\ &= \left[\underbrace{D(D(u))}_{=D^2(u)=(n-1)D(u)}, a\right] - \left[\underbrace{D(a)}_{=a}, D(u)\right] = \underbrace{[(n-1)D(u), a]}_{=(n-1)[D(u),a]} - \underbrace{[a, D(u)]}_{=-[D(u),a]} \\ &= (n-1)[D(u), a] + [D(u), a] = n \underbrace{[D(u), a]}_{=D(ua)} = nD(ua). \end{aligned}$$

This proves that $D^2 = nD$ holds on each pure tensor $ua \in V^{\otimes n}$ with $u \in V^{\otimes(n-1)}$ and $a \in V$. By linearity, it follows that $D^2 = nD$ on all of $V^{\otimes n}$. Thus the induction is complete. ☐

Remark 5. *Let $n \geq 1$. On $V^{\otimes n}$, the map $\frac{1}{n}D$ is an idempotent whenever n is invertible in $\mathbf{k}$. Indeed, Lemma 4 gives*

$$\left(\frac{1}{n}D\right)^2 = \frac{1}{n}D \qquad \text{on } V^{\otimes n}.$$

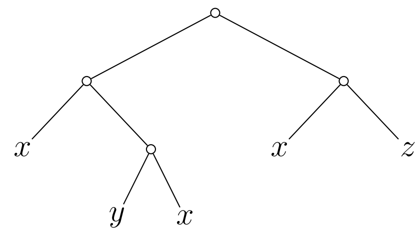
When $\mathbf{k}$ is a $\mathbb{Q}$-algebra, the endomorphism of $T(V)^+$ that is given by $\frac{1}{n}D$ on each $V^{\otimes n}$ is known as the Dynkin idempotent. *(Usually it is extended to $T(V)$ by letting it act as 0 on the 0-th degree component.)*

2 The free-module case

Throughout this section, assume that V is free as a $\mathbf{k}$-module. Fix a basis X of V. Let $\mathcal{M}$ be the free magmatic $\mathbf{k}$-algebra[2] on symbols $\overline{x}$ indexed by $x \in X$. Thus, $\mathcal{M}$ is the free $\mathbf{k}$-module on all nonassociative (fully parenthesized) words in the symbols $\overline{x}$. Combinatorialists will recognize the latter words as plane binary trees with their

[2] A *magmatic* $\mathbf{k}$*-algebra* means a $\mathbf{k}$-module A equipped with a $\mathbf{k}$-bilinear operation $A \times A \to A$ that is written as multiplication, but is not required to satisfy any further axioms; i.e., it is a nonunital nonassociative $\mathbf{k}$-algebra.

leaves colored by the elements of X. For instance, if x, y, z are three elements of X, then $(\overline{x}\,(\overline{y}\overline{x}))\,(\overline{x}\overline{z})$ is such a word, corresponding to the binary tree



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We use the following standard description of the free Lie algebra. This is the only structural result about free Lie algebras that will enter our proof.

Lemma 6 (The defining relations of a free Lie algebra)**.** *Define a* **k***-linear map*

$$\lambda : \mathcal{M} \longrightarrow T(V)^+$$

recursively by

$$\lambda(\overline{x}) := x \qquad \text{for each } x \in X, \qquad \text{and}$$
$$\lambda(pq) := [\lambda(p), \lambda(q)] \qquad \text{for all nonassociative words } p, q.$$

(Thus, λ transforms products in $\mathcal{M}$ into commutators in $T(V)^+$. For instance, $\lambda(\overline{x}\,(\overline{y}\overline{z})) = [x, [y, z]]$ for any $x, y, z \in X$.) Then, $\operatorname{Ker}\lambda$ *is the two-sided ideal of the nonassociative algebra $\mathcal{M}$ generated by the elements*

$$p^2 \qquad \text{and} \qquad (pq)\,r + (qr)\,p + (rp)\,q \tag{7}$$

with $p, q, r \in \mathcal{M}$.

Proof. The quotient of $\mathcal{M}$ by the ideal generated by (7) is the free Lie **k**-algebra on X (by abstract nonsense: quotienting out (7) imposes precisely the axioms of a Lie algebra on the quotient). Therefore, the standard embedding theorem for a free Lie algebra (see, e.g., [Reu93, Theorem 0.5], [Bou89, §II.3.1, Theorem 1 (a)], [Ser06, Chapter IV, Theorem 4.2 part 2], [Lau13, Hauptsatz 4.10 (4)] or [GR26, Remark 6.1.42]) identifies this quotient with the Lie subalgebra of $T(V)$ generated by V. Thus, the projection of $\mathcal{M}$ onto this quotient can be identified with the map $\lambda : \mathcal{M} \to T(V)^+$. Consequently, the ideal generated by (7) (being the kernel of this projection) is precisely $\operatorname{Ker}\lambda$. See, for example, [Coh51, §§9–10] or [Reu93, Chapter 0]. □

We now introduce Cohn's other two maps. Define a **k**-linear map

$$\delta : T(V)^+ \longrightarrow \mathcal{M},$$
$$x_1 x_2 \cdots x_n \longmapsto (\cdots((\overline{x_1}\overline{x_2})\,\overline{x_3}) \cdots \overline{x_n}) \qquad \text{for } x_1, x_2, \ldots, x_n \in X.$$

Thus, δ puts a basis word into completely left-associated form. (Of course, this is again shorthand for a recursive definition: $\delta(x) = \overline{x}$ for any single letter $x \in X$, and $\delta(ax) = \delta(a)\overline{x}$ for any $a \in T(V)^+$ and $x \in X$.)

Define another $\mathbf{k}$-linear map

$$\begin{aligned} \gamma : \mathcal{M} &\longrightarrow T(V)^+ \qquad \text{by} \\ \gamma(\overline{x}) &:= x \qquad \text{for all } x \in X, \qquad \text{and} \\ \gamma(pq) &:= \gamma(p)\lambda(q) \qquad \text{for all } p, q \in \mathcal{M} \end{aligned}$$

(where the product in $\gamma(p)\lambda(q)$ is the associative product in $T(V)$). (Strictly speaking, the equality $\gamma(pq) = \gamma(p)\lambda(q)$ is imposed only on nonassociative words p and q, since each nonassociative word of length > 1 can be uniquely factored as pq for two nonassociative words p and q. But then, by linearity, the same equality holds for all $p, q \in \mathcal{M}$.) For instance, for any $x, y, z \in X$, we have

$$\begin{aligned} \gamma(\overline{x}\overline{y}) &= \underbrace{\gamma(\overline{x})}_{=x}\underbrace{\lambda(\overline{y})}_{=y} = xy; \\ \gamma((\overline{x}\overline{y})\overline{z}) &= \underbrace{\gamma(\overline{x}\overline{y})}_{=xy}\underbrace{\lambda(\overline{z})}_{=z} = xyz; \\ \gamma(\overline{x}(\overline{y}\overline{z})) &= \underbrace{\gamma(\overline{x})}_{=x}\underbrace{\lambda(\overline{y}\overline{z})}_{=[y,z]} = x[y,z]. \end{aligned}$$

Lemma 7. *The maps δ, λ, and γ satisfy*

$$\gamma \circ \delta = \mathrm{id}_{T(V)^+}, \tag{8}$$

$$\lambda \circ \delta = D, \tag{9}$$

$$D \circ \gamma = \lambda. \tag{10}$$

Proof. Equation (8) follows immediately by induction on the length of a basis word. Equation (9) is simply the definition of D.

It remains to prove (10). By linearity, it suffices to prove that $D(\gamma(w)) = \lambda(w)$ for any nonassociative word w in $\mathcal{M}$. We induct on the length (or the structure) of w. The claim is clear if w is a generator $\overline{x}$. Let p and q be nonassociative words, and assume that the claim holds for $w = p$ and $w = q$. We must prove that it also holds for $w = pq$. The induction hypothesis yields $D(\gamma(p)) = \lambda(p)$ and $D(\gamma(q)) = \lambda(q)$. Now,

$$\begin{aligned} D(\gamma(pq)) &= D(\gamma(p)\lambda(q)) \qquad \text{(by the recursive definition of } \gamma\text{)} \\ &= D(\gamma(p)D(\gamma(q))) \qquad (\text{since } \lambda(q) = D(\gamma(q))) \\ &= [D(\gamma(p)), D(\gamma(q))] \qquad \text{(by Lemma 2)} \\ &= [\lambda(p), \lambda(q)] \qquad (\text{since } D(\gamma(p)) = \lambda(p) \text{ and } D(\gamma(q)) = \lambda(q)) \\ &= \lambda(pq) \qquad \text{(by the recursive definition of } \lambda\text{)}. \end{aligned}$$

This completes the induction, and thus proves (10). $\square$

For $a, b \in T(V)^+$, set

$$P(a,b) := aD(b) + bD(a). \tag{11}$$

Linearity of D yields

$$\begin{aligned}(a+b)D(a+b) &= (a+b)(D(a)+D(b)) = aD(a) + \underbrace{aD(b)+bD(a)}_{=P(a,b)} + bD(b)\\ &= aD(a) + P(a,b) + bD(b),\end{aligned}$$

so that

$$P(a,b) = \underbrace{(a+b)D(a+b)}_{\in Z} - \underbrace{aD(a)}_{\in Z} - \underbrace{bD(b)}_{\in Z} \in Z. \tag{12}$$

This simple computation (which did not use the freeness of V) will be used several times in what follows.

Lemma 8. *We have*

$$\gamma(\operatorname{Ker}\lambda) \subseteq Z. \tag{13}$$

Proof. Let

$$\mathcal{J} := \{p \in \mathcal{M} \mid \gamma(p) \in Z\}. \tag{14}$$

Clearly, $\mathcal{J}$ is a $\mathbf{k}$-submodule of $\mathcal{M}$ (since γ is $\mathbf{k}$-linear, and Z is a $\mathbf{k}$-module). We shall now show that $\mathcal{J}$ is a two-sided ideal of $\mathcal{M}$.

Let $p, q \in \mathcal{M}$.

If $p \in \mathcal{J}$, then $\gamma(p) \in Z$ and

$$\gamma(pq) = \gamma(p)\lambda(q) \in Z,$$

since Z is a right ideal. Thus, $pq \in \mathcal{J}$.

If $q \in \mathcal{J}$, then $\gamma(q) \in Z \subseteq \operatorname{Ker} D$ by Corollary 3. Hence, $D(\gamma(q)) = 0$, so that (10) gives $\lambda(q) = D(\gamma(q)) = 0$, and therefore

$$\gamma(pq) = \gamma(p)\lambda(q) = 0 \in Z.$$

Thus, $pq \in \mathcal{J}$ again. So we have proved that $pq \in \mathcal{J}$ holds both if $p \in \mathcal{J}$ and if $q \in \mathcal{J}$. This proves that $\mathcal{J}$ is a two-sided ideal.

We next show that $\mathcal{J}$ contains the generators in (7). For $p \in \mathcal{M}$, the recursive definition of γ and the equation (10) give

$$\gamma\left(p^2\right) = \gamma(p)\lambda(p) = \gamma(p)D(\gamma(p)) \in Z.$$

Hence, $p^2 \in \mathcal{J}$.

Now, let $p, q, r \in \mathcal{M}$, and abbreviate

$$a := \gamma(p), \qquad b := \gamma(q), \qquad c := \gamma(r).$$

Then, $D(a) = D(\gamma(p)) = \lambda(p)$ by (10), and similarly $D(b) = \lambda(q)$ and $D(c) = \lambda(r)$. Using the recursive definition of γ and the equation (10), we now have

$$\gamma((pq)r) = \underbrace{\gamma(pq)}_{=\gamma(p)\lambda(q)} \lambda(r) = \underbrace{\gamma(p)}_{=a} \underbrace{\lambda(q)}_{=D(b)} \underbrace{\lambda(r)}_{=D(c)} = aD(b)D(c)$$

and similarly

$$\gamma((qr)p) = bD(c)D(a) \qquad \text{and}$$
$$\gamma((rp)q) = cD(a)D(b).$$

Adding these three equalities together, we obtain

$$\gamma((pq)r + (qr)p + (rp)q)$$
$$= aD(b)D(c) + bD(c)D(a) + cD(a)D(b). \tag{15}$$

On the other hand, Lemma 2 gives

$$D(cD(a)) = [D(c), D(a)]. \tag{16}$$

Consequently, (15) becomes

$$\begin{aligned}
&\gamma((pq)r + (qr)p + (rp)q) \\
&= \underbrace{aD(b)}_{\substack{=P(a,b)-bD(a) \\ (\text{since } P(a,b)=aD(b)+bD(a))}} D(c) + bD(c)D(a) + \underbrace{cD(a)D(b)}_{\substack{=P(b,cD(a))-bD(cD(a)) \\ (\text{since } P(b,cD(a))=bD(cD(a))+cD(a)D(b))}} \\
&= (P(a,b) - bD(a))D(c) + bD(c)D(a) + P(b,cD(a)) - bD(cD(a)) \\
&= P(a,b)D(c) + \underbrace{bD(c)D(a) - bD(a)D(c)}_{=b[D(c),D(a)]} + P(b,cD(a)) - b\underbrace{D(cD(a))}_{\substack{=[D(c),D(a)] \\ (\text{by (16)})}} \\
&= P(a,b)D(c) + b[D(c),D(a)] + P(b,cD(a)) - b[D(c),D(a)] \\
&= \underbrace{P(a,b)D(c)}_{\substack{\in Z \\ (\text{since (12) yields } P(a,b)\in Z, \\ \text{and since } Z \text{ is a right ideal})}} + \underbrace{P(b,cD(a))}_{\substack{\in Z \\ (\text{by (12)})}} \\
&\in Z.
\end{aligned}$$

Thus, $(pq)r + (qr)p + (rp)q$ belongs to $\mathcal{J}$.

We have shown that the two-sided ideal $\mathcal{J}$ contains all the elements p^2 and $(pq)r + (qr)p + (rp)q$ from (7). Since these elements generate the two-sided ideal $\operatorname{Ker}\lambda$ (by Lemma 6), we thus conclude that $\operatorname{Ker}\lambda \subseteq \mathcal{J}$. But this is precisely (13). $\square$

Proof of Theorem 1 when V is free. Let $f \in \operatorname{Ker} D$. By (9), we then have

$$\lambda(\delta(f)) = D(f) = 0.$$

Thus, $\delta(f) \in \operatorname{Ker}\lambda$, so that $\gamma(\delta(f)) \in \gamma(\operatorname{Ker}\lambda) \subseteq Z$ by Lemma 8. Now, (8) yields

$$f = \gamma(\delta(f)) \in Z.$$

Since we have shown this for each $f \in \operatorname{Ker} D$, we thus obtain $\operatorname{Ker} D \subseteq Z$. The reverse inclusion is Corollary 3. $\square$

The same argument immediately yields a slight extension.

Corollary 9. *If the* $\mathbf{k}$*-module* V *is projective, then* $Z = \operatorname{Ker} D$.

Proof. We denote the Dynkin operator D and the right ideal Z by D_V and Z_V, respectively, in order to contrast them for different choices of V. Note that the Dynkin operator is natural in V. That is, if M and N are two $\mathbf{k}$-modules, and if $f : M \to N$ is a $\mathbf{k}$-linear map, then the canonically induced $\mathbf{k}$-algebra morphism $T(f) : T(M) \to T(N)$ commutes with the respective Dynkin operators D, in the sense that the diagram

$$\begin{tikzcd} T(M)^+ \arrow[r, "D_M"] \arrow[d, "T(f)"'] & T(M)^+ \arrow[d, "T(f)"] \\ T(N)^+ \arrow[r, "D_N"'] & T(N)^+ \end{tikzcd} \tag{17}$$

is commutative. Similarly, the right ideal Z_V is natural in V, meaning that any $\mathbf{k}$-linear map $f : M \to N$ satisfies

$$T(f)(Z_M) \subseteq Z_N \tag{18}$$

(since $T(f)$ sends the generators $xD_M(x)$ of Z_M to generators $T(f)(x) \cdot D_N(T(f)(x))$ of Z_N).

Now, assume that V is projective. Thus, V is a direct addend of a free $\mathbf{k}$-module F. Let $i : V \to F$ and $r : F \to V$ be the corresponding injection and projection, so that $r \circ i = \mathrm{id}_V$. The induced tensor algebra maps $T(i) : T(V) \to T(F)$ and $T(r) : T(F) \to T(V)$ commute with the Dynkin operators D_V and D_F and send the corresponding right ideals Z_V and Z_F into one another, as we saw in the preceding paragraph.

Hence, $T(r)(Z_F) \subseteq Z_V$ and $T(i)(\operatorname{Ker} D_V) \subseteq \operatorname{Ker} D_F$. But F is a free $\mathbf{k}$-module, so we have $\operatorname{Ker} D_F = Z_F$ by Theorem 1. Hence, $T(i)(\operatorname{Ker} D_V) \subseteq \operatorname{Ker} D_F = Z_F$. Applying $T(r)$ to this relation, we obtain

$$(T(r) \circ T(i))(\operatorname{Ker} D_V) \subseteq T(r)(Z_F) \subseteq Z_V.$$

Since $T(r) \circ T(i) = \mathrm{id}_{T(V)}$ (because $r \circ i = \mathrm{id}_V$), this rewrites as $\operatorname{Ker} D_V \subseteq Z_V$. The reverse inclusion again follows from Corollary 3. $\square$

3 The case of a $\mathbb{Q}$-algebra

Now we shall no longer assume that V is free. We work towards the proof of Theorem 1 in the case when $\mathbf{k}$ is a $\mathbb{Q}$-algebra. A weaker statement holds in full generality:

Theorem 10 (torsion version of Cohn's kernel theorem)**.** *Let $n \geq 1$. Then,*

$$n\left(V^{\otimes n} \cap \operatorname{Ker} D\right) \subseteq V^{\otimes n} \cap Z \subseteq V^{\otimes n} \cap \operatorname{Ker} D.$$

First proof of Theorem 10. Corollary 3 yields $Z \subseteq \operatorname{Ker} D$, from which $V^{\otimes n} \cap Z \subseteq V^{\otimes n} \cap \operatorname{Ker} D$ immediately follows. It remains to show that $n\left(V^{\otimes n} \cap \operatorname{Ker} D\right) \subseteq V^{\otimes n} \cap Z$.

Let

$$z \in V^{\otimes n} \cap \operatorname{Ker} D.$$

We must prove that $nz \in V^{\otimes n} \cap Z$.

Choose a free $\mathbf{k}$-module F and a surjective $\mathbf{k}$-linear map $\pi : F \to V$. The induced map

$$\pi_n := \pi^{\otimes n} : F^{\otimes n} \longrightarrow V^{\otimes n}$$

is surjective. Choose $f \in F^{\otimes n}$ satisfying $\pi_n(f) = z$.

The Dynkin operator is natural in the module (see the proof of Corollary 9), so we have

$$\pi_n\left(D_F(f)\right) = D_V\left(\pi_n(f)\right) = D_V(z) = 0 \tag{19}$$

(since $z \in \operatorname{Ker} D = \operatorname{Ker} D_V$). Set

$$f_0 := nf - D_F(f). \tag{20}$$

Thus,

$$D_F(f_0) = nD_F(f) - D_F^2(f) = 0 \qquad \text{by Lemma 4.}$$

Thus, $f_0 \in \operatorname{Ker} D_F = Z_F$ (by the already-proved free-module case of Theorem 1, since F is a free $\mathbf{k}$-module).

But (20) gives $\pi_n(f_0) = n\underbrace{\pi_n(f)}_{=z} - \underbrace{\pi_n(D_F(f))}_{\substack{=0 \\ \text{(by (19))}}} = nz$. Thus, $nz = \pi_n(f_0)$. In terms of the tensor algebra map $T(\pi) : T(F) \to T(V)$, we can rewrite this as $nz = T(\pi)(f_0)$ (since $\pi_n = T(\pi)\mid_{F^{\otimes n}}$).

But as we saw in the above proof of Corollary 9, the tensor algebra map $T(\pi) : T(F) \to T(V)$ satisfies $T(\pi)(Z_F) \subseteq Z_V$. We conclude that $nz = T(\pi)\underbrace{(f_0)}_{\in Z_F} \in T(\pi)(Z_F) \subseteq Z_V = Z$. Hence, $nz \in V^{\otimes n} \cap Z$ (since $nz \in V^{\otimes n}$).

Since we have chosen z to be an arbitrary element of $V^{\otimes n} \cap \operatorname{Ker} D$, we thus have proved that $n\left(V^{\otimes n} \cap \operatorname{Ker} D\right) \subseteq V^{\otimes n} \cap Z$. This completes our proof. $\square$

Second proof of Theorem 10. We give a proof based on cyclic rotation, avoiding the use of the free-module case of Theorem 1 entirely.

As in the first proof, it suffices to show that

$$n\left(V^{\otimes n}\cap\operatorname{Ker}D\right)\subseteq V^{\otimes n}\cap Z. \tag{21}$$

The case $n=1$ is clear, since D is the identity on $V^{\otimes 1}=V$. Thus, fix $n\geq 2$.

Let

$$c_n:V^{\otimes n}\longrightarrow V^{\otimes n}$$

be the cyclic rotation defined by

$$c_n\left(v_1v_2\cdots v_n\right):=v_nv_1v_2\cdots v_{n-1} \tag{22}$$

for $v_1,v_2,\ldots,v_n\in V$. Also, define the $\mathbf{k}$-linear map

$$\rho_n:V^{\otimes n}\longrightarrow V^{\otimes n}$$

by

$$\rho_n\left(uv\right):=D\left(u\right)v \tag{23}$$

for $u\in V^{\otimes(n-1)}$ and $v\in V$. Thus, ρ_n is the map $D\mid_{V^{\otimes(n-1)}}\otimes\operatorname{id}_V$ under the canonical identification $V^{\otimes n}=V^{\otimes(n-1)}\otimes V$. Since $\left(D\mid_{V^{\otimes(n-1)}}\right)^2=(n-1)\left(D\mid_{V^{\otimes(n-1)}}\right)$ (by Lemma 4, applied to $n-1$ instead of n), we thus obtain

$$\rho_n^2=(n-1)\,\rho_n. \tag{24}$$

However, using the recursion (2), we can see that

$$D=\left(\operatorname{id}-c_n\right)\circ\rho_n\qquad\text{on }V^{\otimes n}. \tag{25}$$

Indeed, any $u\in V^{\otimes(n-1)}$ and $v\in V$ satisfy

$$\begin{aligned}\left(\left(\operatorname{id}-c_n\right)\circ\rho_n\right)\left(uv\right)&=\left(\operatorname{id}-c_n\right)\underbrace{\left(\rho_n\left(uv\right)\right)}_{=D(u)v}=\left(\operatorname{id}-c_n\right)\left(D\left(u\right)v\right)\\&=D\left(u\right)v-\underbrace{c_n\left(D\left(u\right)v\right)}_{=vD(u)}=D\left(u\right)v-vD\left(u\right)=\left[D\left(u\right),v\right]\\&=D\left(uv\right)\qquad\left(\text{by (2)}\right),\end{aligned}$$

so that (25) follows by linearity.

Now, let

$$z\in V^{\otimes n}\cap\operatorname{Ker}D.$$

We must prove that $nz\in V^{\otimes n}\cap Z$. Let us set

$$q:=\rho_n\left(z\right). \tag{26}$$

Since $z \in \operatorname{Ker} D$, we have

$$\begin{aligned} 0 = D(z) &= (\operatorname{id} - c_n) \underbrace{(\rho_n(z))}_{=q} \qquad \text{(by (25))} \\ &= (\operatorname{id} - c_n)(q) = q - c_n(q), \end{aligned}$$

and therefore

$$c_n(q) = q. \tag{27}$$

Write z as a finite sum

$$z = \sum_i u_i v_i$$

with $u_i \in V^{\otimes(n-1)}$ and $v_i \in V$. Applying the map ρ_n to this equality (and recalling that $\rho_n(z) = q$), we find

$$q = \sum_i \underbrace{\rho_n(u_i v_i)}_{\substack{=D(u_i)v_i \\ \text{(by (23))}}} = \sum_i D(u_i) v_i. \tag{28}$$

Applying the map c_n to this equality, we find

$$c_n(q) = \sum_i \underbrace{c_n(D(u_i) v_i)}_{=v_i D(u_i)} = \sum_i v_i D(u_i).$$

Compared with (27), this becomes

$$q = \sum_i v_i D(u_i). \tag{29}$$

For any $a, b \in T(V)^+$, we have $aD(b) + bD(a) = P(a,b) \in Z$ (by (12)). Applying this with $a = u_i$ and $b = v_i$, and then summing over i, we obtain

$$\sum_i (u_i D(v_i) + v_i D(u_i)) \in Z.$$

Since

$$\begin{aligned} &\sum_i (u_i D(v_i) + v_i D(u_i)) \\ &= \sum_i u_i \underbrace{D(v_i)}_{\substack{=v_i \\ \text{(since } v_i \in V)}} + \underbrace{\sum_i v_i D(u_i)}_{\substack{=q \\ \text{(by (29))}}} \\ &= \underbrace{\sum_i u_i v_i}_{=z} + q = z + q, \end{aligned}$$

this rewrites as

$$z+q\in Z. \tag{30}$$

Since $q=\rho_n(z)$, we can rewrite this as

$$z+\rho_n(z)\in Z. \tag{31}$$

We next apply the same observation to q. First, (24) shows that ρ_n acts on the image of ρ_n as multiplication by $n-1$. Since $q=\rho_n(z)$ belongs to the image of ρ_n, we thus obtain

$$\rho_n(q)=(n-1)\,q. \tag{32}$$

Hence, (25) yields

$$\begin{aligned} D(q) &= (\mathrm{id}-c_n)\underbrace{(\rho_n(q))}_{=(n-1)q} = (n-1)\left((\mathrm{id}-c_n)(q)\right) \\ &= (n-1)\underbrace{(q-c_n(q))}_{\substack{=0 \\ \text{(by (27))}}} = 0. \end{aligned}$$

Thus, $q\in\operatorname{Ker}D$, so that $q\in V^{\otimes n}\cap\operatorname{Ker}D$.

The argument that led from $z\in V^{\otimes n}\cap\operatorname{Ker}D$ to (31) can therefore be applied with q in place of z. It gives

$$q+\rho_n(q)\in Z.$$

Using (32), we can rewrite this as $q+(n-1)\,q\in Z$. In other words,

$$nq\in Z. \tag{33}$$

On the other hand, (30) gives

$$n(z+q)\in Z.$$

Subtracting (33), we obtain $nz\in Z$. Hence, $nz\in V^{\otimes n}\cap Z$ (since $nz\in V^{\otimes n}$ is obvious). Since we have proved this for every $z\in V^{\otimes n}\cap\operatorname{Ker}D$, we thus have shown that $n\left(V^{\otimes n}\cap\operatorname{Ker}D\right)\subseteq V^{\otimes n}\cap Z$. This completes the second proof. $\square$

Proof of Theorem 1 when $\mathbf{k}$ *is a* $\mathbb{Q}$*-algebra.* Assume that $\mathbf{k}$ is a $\mathbb{Q}$-algebra. We must show that $\operatorname{Ker}D\subseteq Z$ (since $Z\subseteq\operatorname{Ker}D$ was already shown in Corollary 3). Since the map D is graded (i.e., preserves degree), it suffices to prove the claim in each positive degree, i.e., to show that

$$V^{\otimes n}\cap\operatorname{Ker}D\subseteq Z \qquad \text{for each } n\geq 1.$$

But this follows from Theorem 10: Since n is invertible in $\mathbf{k}$, we have

$$\begin{aligned} V^{\otimes n}\cap\operatorname{Ker}D &= n\left(V^{\otimes n}\cap\operatorname{Ker}D\right)\subseteq V^{\otimes n}\cap Z \qquad \text{(by Theorem 10)} \\ &\subseteq Z. \end{aligned}$$

$\square$

4 A homogeneous spanning set for Z

The following proposition is a nicer version of Cohn's [Coh51, Theorem 10.1]. It does not require either of the hypotheses in Theorem 1.

Proposition 11. *For every commutative ring* $\mathbf{k}$ *and every* $\mathbf{k}$*-module* V*, the* $\mathbf{k}$*-module* Z *is spanned by elements of the following two forms:*

- *elements of the form*
$$xD(x), \tag{34}$$
where $x \in T(V)^+$ *is homogeneous, and*
- *elements of the form*
$$(yD(z) + zD(y))w, \tag{35}$$
where $y, z \in T(V)^+$ *and* $w \in T(V)$ *are homogeneous.*

Proof. Every element of the form (34) belongs to Z by definition. Every element of the form (35) belongs to Z by (12) (indeed, $yD(z) + zD(y) = P(y,z) \in Z$ by (12), and therefore $(yD(z) + zD(y))w \in Z$ since Z is a right ideal). Thus, the span of these elements is contained in Z. It remains to prove the reverse inclusion.

By its definition, Z is spanned by the elements

$$xD(x)w \qquad \text{with } x \in T(V)^+ \text{ and } w \in T(V). \tag{36}$$

It thus suffices to show that every such element lies in the span of the elements of the forms (34) and (35).

Since (36) depends linearly on w, we WLOG assume that w is homogeneous. We cannot assume the same for x, but we can still break x into its homogeneous components:

Write x as $x = \sum_{n\geq 1} x_n$, where each $x_n \in V^{\otimes n}$. Then,

$$\begin{aligned} xD(x) &= \left(\sum_{n\geq 1} x_n\right) D\left(\sum_{n\geq 1} x_n\right) = \left(\sum_{n\geq 1} x_n\right) \sum_{n\geq 1} D(x_n) \\ &= \sum_{n\geq 1} x_n D(x_n) + \sum_{1\leq m<n} (x_m D(x_n) + x_n D(x_m)). \end{aligned}$$

Upon multiplication by w, this becomes

$$xD(x)w = \sum_{n\geq 1} x_n D(x_n)w + \sum_{1\leq m<n} (x_m D(x_n) + x_n D(x_m))w. \tag{37}$$

Every addend in the second sum on the right-hand side already has the form (35). It remains to show that the addends $x_n D(x_n)w$ of the first sum also lie in the span of the elements of the forms (34) and (35).

The degree-0 part of w merely gives a scalar multiple of an element of the form (34). Hence, it suffices to handle the positive-degree part of w. This part is a sum of elements of the form vu with $v \in V$ and $u \in T(V)$. Thus, it is enough to show that each of these elements $x_n D(x_n) vu$ has the form (35). Since (2) yields

$$D(x_n v) = [D(x_n), v] = D(x_n) v - vD(x_n),$$

we have

$$D(x_n) v = D(x_n v) + vD(x_n)$$

and therefore

$$x_n \underbrace{D(x_n) v}_{=D(x_n v)+vD(x_n)} u = \underbrace{(x_n D(x_n v) + x_n v D(x_n)) u}_{\text{an element of the form (35)}}.$$

This is precisely what we needed to show; thus, (37) expresses $xD(x) w$ as a linear combination of elements of the forms (34) and (35). This proves that the elements of the latter forms span Z. $\square$

As we said, Proposition 11 is a variant of [Coh51, Theorem 10.1]. In the latter theorem, Cohn uses $yD(y) w$ instead of $(yD(z) + zD(y)) w$ in (35), and does not impose any homogeneity conditions on x, y, z, w. It is easy to see that his generating set spans ours (thanks to the polarization identity $yD(z)+zD(y) = (y+z) D(y+z) - yD(y) - zD(z)$), and thus [Coh51, Theorem 10.1] follows from our Proposition 11.

5 The unrestricted statement is false

The question whether $Z = \operatorname{Ker} D$ holds for every module V (with no further restrictions) has a negative answer. The obstruction is already present in degree 2.

Let

$$\begin{aligned} \tau : V \otimes V &\longrightarrow V \otimes V, \\ v \otimes w &\longmapsto w \otimes v \end{aligned}$$

be the switch map. On $V^{\otimes 2}$, the Dynkin operator D acts as $\operatorname{id} - \tau$. Thus,

$$\operatorname{Ker}(D|_{V^{\otimes 2}}) = (V \otimes V)^{\langle \tau \rangle}, \tag{38}$$

the module of symmetric tensors. On the other hand, the degree-2 part of Z is

$$Z \cap V^{\otimes 2} = \operatorname{span}_{\mathbf{k}} \{v \otimes v \mid v \in V\} \tag{39}$$

(this is particularly easy to see from Proposition 11). The right-hand side of (39) is the image of the canonical map

$$\Gamma^2_{\mathbf{k}}(V) \longrightarrow \operatorname{TS}^2_{\mathbf{k}}(V) \tag{40}$$

from the second divided-powers module of V to the symmetric tensors. This map is an isomorphism when V is flat, and it is an isomorphism when 2 is invertible in $\mathbf{k}$; however, it need not be surjective for an arbitrary module. Explicit counterexamples are given in [Lun08, Example 4.6]. For any such counterexample, a symmetric tensor outside the image of (40) belongs to $\operatorname{Ker} D$ but not to Z. Hence, $Z = \operatorname{Ker} D$ fails in general.

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