Lecture Notes on Cherednik Algebras
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Errata and questions by Darij Grinberg

These are corrections and comments to the "Lecture Notes on Cherednik Algebras" by Pavel Etingof and Xiaoguang Ma. In their current form, they cover only the first ca. 10 pages of the notes.

Section 1

- **Page 4:** Replace "irrieducible" by "irreducible".

- **Page 5:** Replace "shperical" by "spherical".

Section 2

- **Page 6, Theorem 2.1:** I think the words "rational coefficients", "lower order terms" and "homogeneous" need some more explanations. Here is how I understand them; please correct me if I am getting something wrong:

  "rational coefficients" means "coefficients which are rational functions in the variables $x_1, x_2, ..., x_n$" (not "coefficients which are rational numbers" or "coefficients which are polynomials over $\mathbb{Q}$").

  "lower order terms" means the following: Let $\mathbf{D}$ be the $\mathbb{C}$-algebra of all partial differential operators in the variables $x_1, x_2, ..., x_n$ whose coefficients are rational functions in the variables $x_1, x_2, ..., x_n$. Define a $\mathbb{C}$-algebra filtration on $\mathbf{D}$ by requiring that all rational functions in $x_1, x_2, ..., x_n$ are in filtration degree 0, and all $\dfrac{\partial}{\partial x_j}$ are in filtration degree 1. Then,

  $$L_j = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^j + \text{lower order terms}$$

  means that

  $$L_j \equiv \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^j \mod ((j-1)\text{-th filtered part of } \mathbf{D}).$$

  And the *order* of a partial differential operator $E \in \mathbf{D}$ means the smallest $n \in \mathbb{N}$ such that E lies in the n-th filtered part of $\mathbf{D}$. Am I seeing this right?

  Note that this $\mathbb{C}$-algebra filtration on $\mathbf{D}$ can be also characterized differently: Let $\mathbf{D}_{\text{const}}$ denote the $\mathbb{C}$-algebra of all partial differential operators in the variables $x_1, x_2, ..., x_n$ whose coefficients are constant. Let the unadorned $\otimes$ sign denote $\otimes_{\mathbb{C}}$. Then, $\mathbf{D} = \mathbb{C}(x_1, x_2, ..., x_n) \otimes \mathbf{D}_{\text{const}}$ as vector spaces. Since the algebra $\mathbf{D}_{\text{const}}$ is canonically graded (by giving all $\dfrac{\partial}{\partial x_j}$ the degree 1) and the

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algebra $\mathbb{C}(x_1, x_2, ..., x_n)$ is trivially graded (by giving its every element the degree 0), the tensor product $\mathbb{C}(x_1, x_2, ..., x_n) \otimes \mathbf{D}_{\mathrm{const}}$ is also graded. Since $\mathbf{D} = \mathbb{C}(x_1, x_2, ..., x_n) \otimes \mathbf{D}_{\mathrm{const}}$ as vector spaces, this yields that the vector space $\mathbf{D}$ is also graded (albeit this is not a grading of the $\mathbb{C}$-**algebra** $\mathbf{D}$, since generally $\mathbf{D} \neq \mathbb{C}(x_1, x_2, ..., x_n) \otimes \mathbf{D}_{\mathrm{const}}$ as algebras), hence filtered. This filtration is easily seen to be the same filtration on $\mathbf{D}$ as defined above.)

Note that as vector spaces,

$(j$-th filtered part of $\mathbf{D}) \diagup ((j-1)$-th filtered part of $\mathbf{D})$
$\cong (j$-th graded part of $\mathbf{D})$

   (since the filtration of $\mathbf{D}$ comes from a vector space grading on $\mathbf{D}$)
$= (j$-th graded part of $\mathbb{C}(x_1, x_2, ..., x_n) \otimes \mathbf{D}_{\mathrm{const}})$
$= \mathbb{C}(x_1, x_2, ..., x_n) \otimes (j$-th graded part of $\mathbf{D}_{\mathrm{const}})$

   (since $\mathbb{C}(x_1, x_2, ..., x_n)$ is concentrated in degree 0) .

"homogeneous" means the following: Let $\mathbf{D}_{\mathrm{hom}}$ be the $\mathbb{C}$-subalgebra of the algebra $\mathbf{D}$ (defined above) generated by all homogeneous rational functions in $x_1, x_2, ..., x_n$ and the derivations $\dfrac{\partial}{\partial x_j}$. This is a graded algebra, where the degree of a homogeneous rational function is its usual degree, and the degree of a derivation $\dfrac{\partial}{\partial x_j}$ is -1. Then, when we say that a differential operator in $\mathbf{D}$ is *homogeneous of degree k* (for some integer k), we mean that this operator lies in $\mathbf{D}_{\mathrm{hom}}$ and has degree k.

- **Page 6, four lines above Definition 2.3:** You speak of an "inner product". Maybe point out that it is supposed to be bilinear, not sesquilinear (some people might be confused).

- **Page 6, two lines above Definition 2.3:** You say "equivalently, s is conjugate to $\mathrm{diag}(-1, 1, ..., 1)$". Conjugate where? in $\mathrm{GL}(\mathfrak{h})$ or in $\mathrm{O}(\mathfrak{h})$? In this case, both are true (as long as we suppose s to lie in $\mathrm{O}(\mathfrak{h})$), but it would be better if you would point this out more explicitly.

- **Page 6, Theorem 2.4, and many times after:** I think Theorem 2.4 is called the Chevalley-Shep**h**ard-Todd theorem, with two "h"'s in "Shephard" (cf. `http://en.wikipedia.org/wiki/Geoffrey_Colin_Shephard`).

- **Page 6, one line below Theorem 2.4:** Maybe add "if G is a complex reflection group" into the sentence that comes directly after Theorem 2.4.

- **Page 6, two lines below Theorem 2.4:** You write: "The numbers d_i are uniquely determined". You need to add here that you require $d_1 \leq d_2 \leq ... \leq d_{\dim \mathfrak{h}}$ (else, the "$L_1 = H$" part of Theorem 2.9 makes no sense).

- **Page 7, Example 2.5:** It is not clear what p_i are, and why you write $P_i(p_1, ..., p_n)$ (the p_i are definitely not polynomial variables, since they are algebraically dependent). Let me just record the answer (which you explained in an email): You

want $p_i = e_i - \dfrac{e_1 + e_2 + \cdots + e_n}{n} \in \mathfrak{h}$ (where $e_1, e_2, ..., e_n$ are the standard basis vectors of $\mathbb{C}^n$), and instead of $P_i\,(p_1, ..., p_n)$ you simply want to write P_i.

- **Page 7, between Definition 2.6 and Example 2.7:** You write: "Note that by Chevalley's theorem, a parabolic subgroup of a complex (respectively, real) reflection group is itself a complex (respectively, real) reflection group." What Chevalley's theorem do you mean? If you are applying Theorem 2.4, isn't it quite an overkill? (Or is there really no simpler proof?)

- **Page 7, between Example 2.7 and §2.4:** You write: "and we can define the open set $\mathfrak{h}_{\mathrm{reg}}^{*G'}$ of all $\lambda \in \mathfrak{h}^{G'}$ for which $G_\lambda = G'$". I think the "$\mathfrak{h}^{G'}$" should be "$\mathfrak{h}^{*G'}$" here.

- **Page 7, first line of §2.4:** Replace "Let $s \subset \mathrm{GL}\,(\mathfrak{h})$" by "Let $s \in \mathrm{GL}\,(\mathfrak{h})$".

- **Page 7, second line of §2.4:** You might want to point out that a "nontrivial eigenvalue" of a reflection means an eigenvalue $\neq 1$. (Normally, in linear algebra, I tend to mean $\neq 0$ by "nontrivial".)

- **Page 7, one line above Definition 2.8:** What do you mean by a "conjugation invariant function"? Invariant under conjugation by elements of W, or by conjugation by any element of $\mathrm{O}\,(\mathfrak{h})$ (or even $\mathrm{GL}\,(\mathfrak{h})$?) that happens to send an element of $\mathcal{S}$ to another element of $\mathcal{S}$?

- **Page 7, Definition 2.8:** This is hardly an error, but maybe it would improve the exposition if you would define what $\Delta_\mathfrak{h}$ means. (It's just that I don't like algebra texts relying on geometry preknowledge.) I assume we can define it by $\Delta_\mathfrak{h} = \sum_{i=1}^{r} \partial_{y_i}^2$ for any orthonormal basis $(y_1, y_2, ..., y_r)$ of $\mathfrak{h}$?

- **Page 7, one line above Theorem 2.9:** When you write "$P_1\,(\mathbf{p}) = \mathbf{p}^2$", it wouldn't hurt to point out that $\mathbf{p}$ is a variable vector in $\mathfrak{h}^*$ (not $\mathfrak{h}$), so "$P_1\,(\mathbf{p}) = \mathbf{p}^2$" describes P_1 as a polynomial function on $\mathfrak{h}^*$ (that is, an element of $\mathbb{C}\,[\mathfrak{h}^*] = S\mathfrak{h}$).

- **Page 8, two lines above Remark 2.10:** You write: "This theorem is obviously a generalization of Theorem 1 about $W = \mathfrak{S}_n$." Given that the representation $\mathbb{C}^n$ of $\mathfrak{S}_n$ is not irreducible, while lifting the L_j from the representation $\mathbb{C}^{n-1}$ of $\mathfrak{S}_n$ to $\mathbb{C}^n$ requires some work (as our emails showed), I don't think the word "obviously" is justified here. See below for a proposal how to improve this (by getting rid of the standing assumption that $\mathfrak{h}$ be irreducible).

- **Page 8, fourth line of §2.5:** You write: "We normalize them in such a way that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$." At this point, I had to think for a while about why this is possible (i. e., why we don't have $\langle \alpha_s, \alpha_s^\vee \rangle = 0$). This is quite easy to see by diagonalizing the matrix s, but maybe you should make this an explicit exercise. (Remark 2.13, too, could be an exercise.)

- **Page 8, fifth line of §2.5:** Again, you speak of a "function invariant with respect to conjugation", and it is not clear by what you allow to conjugate. (I will henceforth assume that you allow conjugation by G.)

- **Page 8, Definition 2.11:** Please say that $\mathbb{C}(\mathfrak{h})$ means the quotient field of $S(\mathfrak{h}^*)$. (I know that this follows from standard algebraic geometry notation, but I didn't expect that you are using algebraic geometry notation.) Also, please add "Let $a \in \mathfrak{h}$." at the beginning of this Definition.

- **Page 8, Proposition 2.14:** Beginning with part (i) of this proposition, you seem to systematically write $(\cdot, \cdot)$ for the bilinear form on $\mathfrak{h}^* \times \mathfrak{h}$ that you formerly denoted by $\langle \cdot, \cdot \rangle$. I don't like this notation very much, because $(\cdot, \cdot)$ already means two different bilinear forms (one on $\mathfrak{h} \times \mathfrak{h}$ and one on $\mathfrak{h}^* \times \mathfrak{h}^*$) in the case when $G \subseteq O(\mathfrak{h})$, but it's okay since one can always infer types. But you should point out the change in notation, or else it appears as if you suddenly switched to the case $G \subseteq O(\mathfrak{h})$!

- **Page 9, proof of Theorem 2.15:** I think that

$$-\sum_{s \in \mathcal{S}} c_s \left(a, \alpha_s\right) \left(x, \alpha_s^\vee\right) \left(b, \alpha_s\right) s D_{\alpha_s^\vee} \cdot \frac{1 - \lambda_s^{-1}}{2}$$

should be

$$-\sum_{s \in \mathcal{S}} c_s \left(a, \alpha_s\right) \left(x, \alpha_s^\vee\right) \left(b, \alpha_s\right) s D_{\alpha_s^\vee} \cdot \frac{1 - \lambda_s}{2}$$

(leaving aside the fact that you are still using the notation $(\cdot, \cdot)$ for what was formerly called $\langle \cdot, \cdot \rangle$). To make sure that I haven't done any mistakes, let me write up the details of this computation. (They are completely straightforward and I don't think you should explicit them in the paper, but I am doing them here so you can tell me where I am going wrong.)

It is clearly enough to prove that every $s \in \mathcal{S}$ satisfies

$$[s, D_b] = \langle b, \alpha_s \rangle \, s D_{\alpha_s^\vee} \cdot \frac{1 - \lambda_s}{2}. \tag{1}$$

First, we show that

$$\text{every } b \in \mathfrak{h} \text{ satisfies } b - s^{-1}b = \frac{1 - \lambda_s}{2} \langle b, \alpha_s \rangle \alpha_s^\vee. \tag{2}$$

(This is similar to Proposition 2.14 (i), but with $\mathfrak{h}$ instead of $\mathfrak{h}^*$.)

Proof of (2): WLOG, assume that $\mathfrak{h} = \mathbb{C}^n$, $s = \mathrm{diag}\left(\lambda_s, 1, 1, ..., 1\right)$, $\alpha_s = e_1^*$ and $\alpha_s^\vee = 2e_1$, where $(e_1, e_2, ..., e_n)$ is the standard basis of $\mathbb{C}^n$ and $(e_1^*, e_2^*, ..., e_n^*)$ is its dual basis. (This situation can always be achieved by an appropriate change of basis in $\mathfrak{h}$.) By linearity, it is enough to prove (2) in the cases when $b = e_i$ for $i \in \{1, 2, ..., n\}$. So consider this case. If $i > 1$, then both sides of (2) are 0, and thus (2) holds. Remains the case $i = 1$. In this case, $b = e_1 = \frac{1}{2}\alpha_s^\vee$, so that

$$b - s^{-1}b = \frac{1}{2}\left(\alpha_s^\vee - \underbrace{s^{-1}\alpha_s^\vee}_{=\lambda_s \alpha_s^\vee}\right) = \frac{1 - \lambda_s}{2}\alpha_s^\vee$$

and

$$\frac{1-\lambda_s}{2}\left\langle \underbrace{b}_{=e_1}, \underbrace{\alpha_s}_{=e_1^*}\right\rangle \alpha_s^\vee = \frac{1-\lambda_s}{2}\underbrace{\langle e_1, e_1^*\rangle}_{=1}\alpha_s^\vee = \frac{1-\lambda_s}{2}\alpha_s^\vee.$$

Thus, (2) holds in the case $i = 1$ as well, and thus (2) is proven.

Proof of (1): We have

$$[s, D_b] = sD_b - D_b s = s\left(D_b - \underbrace{s^{-1}D_b s}_{\substack{=D_{s^{-1}b}\\ \text{(by Proposition 2.14 (ii))}}}\right)$$

$$= s\underbrace{(D_b - D_{s^{-1}b})}_{\substack{=D_{b-s^{-1}b}=D_{\frac{1-\lambda_s}{2}\langle b,\alpha_s\rangle\alpha_s^\vee}\\ \text{(by (1))}}} = sD_{\frac{1-\lambda_s}{2}\langle b,\alpha_s\rangle\alpha_s^\vee}$$

$$= \langle b, \alpha_s\rangle\, sD_{\alpha_s^\vee}\cdot\frac{1-\lambda_s}{2}, \qquad \text{and (1) is proven.}$$

- **Page 9, proof of Theorem 2.15:** You write: "since this algebra acts faithfully on $\mathbb{C}(\mathfrak{h})$" (where "this algebra" is the semidirect product $\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$). I am wondering how you prove this. I have a proof, but it is rather messy: First, the claim that $\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ acts faithfully on $\mathbb{C}(\mathfrak{h})$ can be rewritten as follows: If $(D_g)_{g\in G}$ is a family of differential operators indexed by elements of G such that $\sum_{g\in G} gD_g$ is 0 as an endomorphism of $\mathbb{C}(\mathfrak{h})$, then each $g \in G$ satisfies $D_g = 0$. To prove this, we first notice that we can WLOG assume that every D_g has polynomial coefficients (because we can move denominators to the left, moving them past derivations by means of the quotient rule and moving them past the g's by using the formula

$$g\circ f = (gf)\,g \qquad \text{for any } g\in G \text{ and } f \in \mathbb{C}(\mathfrak{h})$$

). Now, let $v \in \mathfrak{h}$ be a point which is not fixed by any $g \in G\setminus\{\mathrm{id}\}$. Recall that $\sum_{g\in G} gD_g$ is 0 as an endomorphism of $\mathbb{C}(\mathfrak{h})$. In particular, $\sum_{g\in G} gD_g$ acts as 0 on $\mathbb{C}[\mathfrak{h}]$. Thus, for every $p \in \mathbb{C}[\mathfrak{h}]$, a certain $\mathbb{C}[\mathfrak{h}]$-linear combination of the partial derivatives of p (of various orders) taken at the points gv for varying $g \in G$ is identically 0 (and the coefficients of this combination don't depend on p). But since we can find a polynomial with any given set of finitely many prescribed values of partial derivatives at finitely many points[1], this yields that the $\mathbb{C}[\mathfrak{h}]$-linear combination must be trivial at v; in other words, D_g is identically 0 at v

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[1]This follows from the Chinese remainder theorem, applied to the ring $\mathbb{C}[\mathfrak{h}]$. In fact, by prescribing the values of finitely many partial derivatives of a polynomial $p \in \mathbb{C}[\mathfrak{h}]$ at some point $w \in \mathfrak{h}$, we put a condition on the residue class of p modulo a certain power of the maximal ideal $\mathfrak{m}_w \subseteq \mathfrak{h}$ (where $\mathfrak{m}_w$ is the ideal of all polynomials that vanish at w). Such a condition is always satisfiable. Thus, if we prescribe the values of finitely many partial derivatives of a polynomial $p \in \mathbb{C}[\mathfrak{h}]$ at finitely many points $w_1, w_2, ..., w_\ell \in \mathfrak{h}$, we put conditions on the residue classes of p modulo powers of $\mathfrak{m}_{w_1}, \mathfrak{m}_{w_2},$ ..., $\mathfrak{m}_{w_\ell}$. Each of these ℓ conditions alone is satisfiable; thus, the conjunction of these ℓ conditions

for every $g \in G$. Since this holds for every point $v \in \mathfrak{h}$ which is not fixed by any $g \in G \setminus \{\mathrm{id}\}$, and since the set of such points is Zariski-dense in $\mathfrak{h}$, this yields that D_g is identically 0 everywhere for every $g \in G$. This proves that each $g \in G$ satisfies $D_g = 0$, qed.

- **Page 9, §2.6, just before Proposition 2.16:** It would be nice to explain when an element of $\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ or an operator on the space of regular functions of $\mathfrak{h}_{\mathrm{reg}}$ is said to be W-*invariant*. (Short answer: When it commutes with every $g \in W$.)

- **Page 10, two lines above Corollary 2.17:** You write: "the algebra $(S\mathfrak{h})^W$ is free". By "free", you mean "free as a commutative algebra", not "free as an algebra". (I know, this is some nitpicking.)

- **Page 10, Corollary 2.17:** In my opinion, you should explain what $P_j (D_{y_1}, ..., D_{y_r})$ means, because P_j is just an element of $S\mathfrak{h}$, and not a polynomial. (The meaning of $P_j (D_{y_1}, ..., D_{y_r})$ is the following: Since $\{y_1, y_2, ..., y_r\}$ is a basis of $\mathfrak{h}$, we can identify the symmetric algebra $S\mathfrak{h}$ with the ring of polynomials in the r variables y_1, y_2, ..., y_r over $\mathbb{C}$. Thus, $P_j \in S\mathfrak{h}$ becomes a polynomial in the r variables y_1, y_2, ..., y_r. If we now substitute D_{y_1}, D_{y_2}, ..., D_{y_r} for these variables y_1, y_2, ..., y_r in P_j (this is allowed because the Dunkl operators D_a commute), we obtain an element of $\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$. This element is what you denote by $P_j (D_{y_1}, ..., D_{y_r})$.

- **Page 10, proof of Corollary 2.17:** Replace "L_j" by "$\overline{L}_j$" twice in this proof.

- **Page 10, proof of Corollary 2.17:** This proof would be more readable if you would explain why the $P_j (D_{y_1}, ..., D_{y_r})$ is W-invariant for all j. The *proof* is not too immediate:

First, it is easy to see that the map

$$\mathfrak{h} \to \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}),$$
$$a \mapsto D_a$$

is $\mathbb{C}$-linear[2]. Denote this map by T.

Since $\{y_1, y_2, ..., y_r\}$ is a basis of $\mathfrak{h}$, we can identify the symmetric algebra $S\mathfrak{h}$ with the ring of polynomials in the r variables y_1, y_2, ..., y_r over $\mathbb{C}$. Thus, every $P \in S\mathfrak{h}$ becomes a polynomial in the r variables y_1, y_2, ..., y_r. As a consequence, for every $P \in S\mathfrak{h}$, we will denote by $P(D_{y_1}, D_{y_2}, ..., D_{y_r})$ the result of substituting D_{y_1}, D_{y_2}, ..., D_{y_r} for these variables y_1, y_2, ..., y_r in P. When

is also satisfiable (because the Chinese remainder theorem says that $(\mathbb{C}[\mathfrak{h}]) \big/ (\mathfrak{m}_{w_1}^{\alpha_1} \mathfrak{m}_{w_2}^{\alpha_2} ... \mathfrak{m}_{w_\ell}^{\alpha_\ell}) = \prod_{i=1}^{\ell} (\mathbb{C}[\mathfrak{h}]) \big/ \mathfrak{m}_{w_i}^{\alpha_i}$, so that every ℓ-tuple in $\prod_{i=1}^{\ell} (\mathbb{C}[\mathfrak{h}]) \big/ \mathfrak{m}_{w_i}^{\alpha_i}$ has a common representative in $\mathbb{C}[\mathfrak{h}]$), i. e., we can find a polynomial with our given set of prescribed values.

[2]This is because the map

$$\mathfrak{h} \to \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}),$$
$$a \mapsto \partial_a$$

is $\mathbb{C}$-linear, and because α_s is $\mathbb{C}$-linear for every $s \in W$.

$P \in \mathfrak{h}$, then $P\left(D_{y_1}, D_{y_2}, ..., D_{y_r}\right)$ is a $\mathbb{C}$-linear combination of D_{y_1}, D_{y_2}, ..., D_{y_r} (here, we regard $\mathfrak{h}$ as a subspace of $S\mathfrak{h}$, so $P \in \mathfrak{h}$ yields $P \in S\mathfrak{h}$).

It is easy to see that

$$\text{every } a \in \mathfrak{h} \text{ satisfies } D_a = a\left(D_{y_1}, D_{y_2}, ..., D_{y_r}\right) \tag{3}$$

(where $a\left(D_{y_1}, D_{y_2}, ..., D_{y_r}\right)$ is to be understood as just explained, with a being regarded as an element of $S\mathfrak{h}$). (In fact, the equation (3) is $\mathbb{C}$-linear in a (because of the $\mathbb{C}$-linearity of T), and thus in order to prove it for all $a \in \mathfrak{h}$, it is enough to prove it in the case when $a \in \{y_1, y_2, ..., y_r\}$ (since $\{y_1, y_2, ..., y_r\}$ is a basis of $\mathfrak{h}$), but in this case it is trivial.)

Now, for any $j \in \{1, 2, ..., \dim \mathfrak{h}\}$ and any $g \in W$, we have

$gP_j\left(D_{y_1}, ..., D_{y_r}\right)g^{-1}$
$= P_j\left(gD_{y_1}g^{-1}, ..., gD_{y_r}g^{-1}\right)$ (since conjugation by g is an algebra automorphism)
$= P_j\left(D_{gy_1}, ..., D_{gy_r}\right)$ $\left(\text{since } gD_{y_i}g^{-1} = D_{gy_i} \text{ for every } i \text{ due to Proposition 2.14 (ii)}\right)$
$= P_j\left((gy_1)\left(D_{y_1}, D_{y_2}, ..., D_{y_r}\right), ..., (gy_r)\left(D_{y_1}, D_{y_2}, ..., D_{y_r}\right)\right)$
        (since (3) yields that $D_{gy_i} = (gy_i)\left(D_{y_1}, D_{y_2}, ..., D_{y_r}\right)$ for every i)
$= \underbrace{\left(P_j\left(gy_1, ..., gy_r\right)\right)}_{\substack{=gP_j=P_j \\ (\text{since } P_j \in (S\mathfrak{h})^W)}}\left(D_{y_1}, ..., D_{y_r}\right)$

$$\left(\begin{array}{c}\text{here (as explained above) } \left(P_j\left(gy_1, ..., gy_r\right)\right)\left(D_{y_1}, ..., D_{y_r}\right) \text{ means} \\ \text{"the polynomial } P_j\left(gy_1, ..., gy_r\right) \text{ with } D_{y_1}, D_{y_2}, ..., D_{y_r} \text{ substituted for } y_1, y_2, ..., y_r\text{"}\end{array}\right)$$

$= P_j\left(D_{y_1}, ..., D_{y_r}\right),$

so that $P_j\left(D_{y_1}, ..., D_{y_r}\right)$ is W-invariant, qed.

The main idea of this proof, I think, was the $\mathbb{C}$-linearity of T. While trivial, it is (in my opinion) unexpected for such a complicated map.

- **Pages 7-10:** Here are various suggested changes to make the proofs clearer (these changes should be made at the same time, as they depend one on another):

  – In Theorem 2.9, add the claim that the L_j are W-invariant. (Otherwise, Theorem 2.1 doesn't directly follow from Theorem 2.9, because Theorem 2.1 claims the $\mathfrak{S}_n$-invariance of the L_j.)

  – In Corollary 2.17, add the claim that the $\overline{L}_j$ are W-invariant. (Otherwise, Theorem 2.9 with the added claim that the L_j are W-invariant doesn't directly follow from Theorem 2.9.)

  – On page 9, you write:

  "For any element $B \in \mathbb{C}W \ltimes \mathcal{D}\left(\mathfrak{h}_{\text{reg}}\right)$, define $m(B)$ to be the differential operator $\mathbb{C}\left(\mathfrak{h}\right)^W \to \mathbb{C}\left(\mathfrak{h}\right)$, defined by B. That is, if $B = \sum_{g \in W} B_g g$, $B_g \in \mathcal{D}\left(\mathfrak{h}_{\text{reg}}\right)$, then $m(B) = \sum_{g \in W} B_g$."

  This is slightly confusing, since you later (in the proof of Corollary 2.17) want $m(B)$ to be defined on the whole $\mathbb{C}\left(\mathfrak{h}\right)$ rather than just on $\mathbb{C}\left(\mathfrak{h}\right)^W$. In my opinion, you should replace the text I've just quoted by the following:

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"For any element $B \in \mathbb{C}W \ltimes \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$, define a differential operator $m\left(B\right) \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ by $m\left(B\right) = \sum\limits_{g \in W} B_g$, where B is being written in the form $B = \sum\limits_{g \in W} B_g g$ with $B_g \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$. The differential operator B defined this way satisfies the following properties:

(i) If $f \in \mathbb{C}\left(\mathfrak{h}\right)^W$, then $m\left(B\right) f = Bf$.

(ii) If $B \in \mathbb{C}W \ltimes \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ is W-invariant, then $m\left(B\right)$ is W-invariant as well[3].

(iii) Any $s \in W$ and any $B \in \mathbb{C}W \ltimes \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ satisfy $m\left(B\right) = m\left(Bs\right)$. (This is used, e. g., in the proof that $m\left(D_y^2\right) = m\left(D_y \partial_y\right)$ in the proof of Proposition

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[3]*Proof.* Let $B \in \mathbb{C}W \ltimes \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ be W-invariant. Write B in the form $B = \sum\limits_{g \in W} B_g g$ with $B_g \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$. Then, $m\left(B\right) = \sum\limits_{g \in W} B_g$. Let $h \in W$. Since B is W-invariant, we have $hB = Bh$, so that

$$\sum_{g \in W} hB_g g = h \underbrace{\sum_{g \in W} B_g g}_{=B} = hB = \underbrace{B}_{=\sum_{g \in W} B_g g} h = \sum_{g \in W} B_g gh = \sum_{g \in W} B_{gh^{-1}} g \underbrace{h^{-1}h}_{=\mathrm{id}}$$

$$\left(\begin{array}{c} \text{here, we substituted } gh^{-1} \text{ for } g \text{ in the sum} \\ \text{(since the map } W \to W, \ g \mapsto gh^{-1} \text{ is a bijection)} \end{array} \right)$$

$$= \sum_{g \in W} B_{gh^{-1}} g.$$

Compared to

$$\sum_{g \in W} hB_g \underbrace{g}_{=h^{-1}hg} = \sum_{g \in W} hB_g h^{-1}hg = \sum_{g \in W} hB_g h^{-1}hg = \sum_{g \in W} hB_{h^{-1}g}h^{-1} \underbrace{hh^{-1}}_{=\mathrm{id}}g$$

$$\left(\begin{array}{c} \text{here, we substituted } h^{-1}g \text{ for } g \text{ in the sum} \\ \text{(since the map } W \to W, \ g \mapsto h^{-1}g \text{ is a bijection)} \end{array} \right)$$

$$= \sum_{g \in W} hB_{h^{-1}g}h^{-1}g,$$

this yields $\sum\limits_{g \in W} B_{gh^{-1}} g = \sum\limits_{g \in W} hB_{h^{-1}g}h^{-1}g$. Notice that every $g \in W$ satisfies $B_{gh^{-1}} \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ and $hB_{h^{-1}g}h^{-1} \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$.

But any element of $\mathbb{C}W \ltimes \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ can be **uniquely** written in the form $\sum\limits_{g \in W} C_g g$ with $C_g \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$. Hence, if we have $\sum\limits_{g \in W} C_g g = \sum\limits_{g \in W} D_g g$ for some choice of $C_g \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$ and $D_g \in \mathcal{D}\left(\mathfrak{h}_{\mathrm{reg}}\right)$, then every $g \in W$ satisfies $C_g = D_g$. Applied to $C_g = B_{gh^{-1}}$ and $C_g = hB_{h^{-1}g}h^{-1}$, this yields that

$$\text{every } g \in W \text{ satisfies } B_{gh^{-1}} = hB_{h^{-1}g}h^{-1}$$

(because $\sum\limits_{g \in W} B_{gh^{-1}} g = \sum\limits_{g \in W} hB_{h^{-1}g}h^{-1}g$). Hence,

$$\sum_{g \in W} B_{gh^{-1}} = \sum_{g \in W} hB_{h^{-1}g}h^{-1} = \sum_{g \in W} hB_g h^{-1}$$

$$\left(\begin{array}{c} \text{here, we substituted } g \text{ for } h^{-1}g \text{ in the sum} \\ \text{(since the map } W \to W, \ g \mapsto h^{-1}g \text{ is a bijection)} \end{array} \right)$$

$$= h \underbrace{\left(\sum_{g \in W} B_g \right)}_{=m(B)} h^{-1} = hm\left(B\right) h^{-1}.$$

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2.16.)

(iv) Any $A \in \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ and any W-invariant $B \in \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ satisfy $m(AB) = m(A)\, m(B)$."

- **Pages 7-10:** In much of Section 2, you work with a standing assumption requiring that $\mathfrak{h}$ be an irreducible W-module. This makes deducing Theorem 2.1 from Theorem 2.9 unnecessarily hard. There is a very easy way to get rid of the standing assumption:

  – On page 7, replace "Let us assume that $\mathfrak{h}$ is an irreducible representation of W (i. e. W is an irreducible finite Coxeter group, and $\mathfrak{h}$ is its reflection representation.) In this case, we can take $P_1(\mathbf{p}) = \mathbf{p}^2$" by "Note that if $\mathfrak{h}$ is an irreducible representation of W (i. e. W is an irreducible finite Coxeter group, and $\mathfrak{h}$ is its reflection representation), then we can take $P_1(\mathbf{p}) = \mathbf{p}^2$. If $W = \mathfrak{S}_n$ and $\mathfrak{h} = \mathbb{C}^n$ (with the standard permutation representation of $\mathfrak{S}_n$), then we can take $P_2(\mathbf{p}) = \mathbf{p}^2$".

  – In Theorem 2.9, replace "$L_1 = H$" by "if $\ell \in \{1, 2, ..., \dim \mathfrak{h}\}$ is such that $P_\ell(\mathbf{p}) = \mathbf{p}^2$, then $L_\ell = H$".

  – One line above Corollary 2.17, replace "$P_1 = \mathbf{p}^2$" by "P_1".

  – In Corollary 2.17, replace "$\overline{L}_1 = \overline{H}$" by "if $\ell \in \{1, 2, ..., \dim \mathfrak{h}\}$ is such that $P_\ell(\mathbf{p}) = \mathbf{p}^2$, then $\overline{L}_\ell = \overline{H}$".

- **Page 10, proof of Proposition 2.18:** The expression "$\sum_{i=1}^{r} \partial_{y_i} (\log \delta_c)\, \partial_{y_i}$" is ambiguous: Does $\partial_{y_i}(\log \delta_c)$ mean the product $\partial_{y_i} \cdot (\log \delta_c)$ in $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ or the y_i-derivative of $\log \delta_c$? (I know it means the latter.)

- **Page 10, proof of Proposition 2.18:** As I don't like this proof (it uses a strange function δ_c, which is in general not algebraic and doesn't have a very obvious interpretation as a power series), let me reformulate it in a more algebraic way. First, I will show some lemmas:

Lemma 2.18a. Let A and D be $\mathbb{C}$-algebras. Let $\mathcal{G}$ be a subset of A which generates A as a $\mathbb{C}$-algebra. Let $f : A \to D$ be a $\mathbb{C}$-linear map. Assume that

$$f(ab) = f(a)\, f(b) \qquad \text{for every } a \in \mathcal{G} \text{ and } b \in A.$$

Also, assume that $f(1) = 1$. Then, f is a $\mathbb{C}$-algebra homomorphism.

Proof of Lemma 2.18a. Let $\mathcal{H}$ be the subset

$$\{x \in A \mid f(xb) = f(x)\, f(b) \text{ for every } b \in A\}.$$

Compared to

$$\sum_{g \in W} B_{gh^{-1}} = \sum_{g \in W} B_g \qquad \left(\begin{array}{c} \text{here, we substituted } g \text{ for } gh^{-1} \text{ in the sum} \\ \text{(since the map } W \to W,\ g \mapsto gh^{-1} \text{ is a bijection)} \end{array} \right)$$
$$= m(B),$$

this yields $m(B) = hm(B)\, h^{-1}$, so that $m(B)\, h = hm(B)$.

Since this holds for every $h \in W$, this yields that $m(B)$ is W-invariant, qed.

Every $a \in \mathcal{G}$ satisfies $a \in \mathcal{H}$ (because every $a \in \mathcal{G}$ satisfies $f(ab) = f(a)f(b)$ for every $b \in A$, and thus $a \in \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\} = \mathcal{H}$). In other words, $\mathcal{G} \subseteq \mathcal{H}$.

Also, any $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$, $a \in \mathcal{H}$ and $a' \in \mathcal{H}$ satisfy $\lambda a + \mu a' \in \mathcal{H}$ [4]. Combined with the trivial fact that $0 \in \mathcal{H}$ (this quickly follows from $f(0) = 0$), this yields that $\mathcal{H}$ is a $\mathbb{C}$-vector subspace of A.

Also, $1 \in \mathcal{H}$ (since $f(\underbrace{1b}_{=b}) = f(b) = \underbrace{1}_{=f(1)} f(b) = f(1)f(b)$ for every $b \in A$, so that $1 \in \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\} = \mathcal{H}$). Besides, any $a \in \mathcal{H}$ and $a' \in \mathcal{H}$ satisfy $aa' \in \mathcal{H}$ [5]. Combining this with the fact that $1 \in \mathcal{H}$ and that $\mathcal{H}$ is a $\mathbb{C}$-vector subspace of A, we conclude that $\mathcal{H}$ is a $\mathbb{C}$-subalgebra of A. Combined with $\mathcal{G} \subseteq \mathcal{H}$, this yields that $\mathcal{H}$ is a $\mathbb{C}$-subalgebra of A containing $\mathcal{G}$ as a subset. But since every $\mathbb{C}$-subalgebra of A containing $\mathcal{G}$ as a subset must contain A as a subset[6], this yields that $\mathcal{H}$ contains A as a subset. In other words, $A \subseteq \mathcal{H}$. Thus, every $a \in A$ satisfies $a \in A \subseteq \mathcal{H} = \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\}$, so that $f(ab) = f(a)f(b)$ for

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[4] *Proof.* Let $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$, $a \in \mathcal{H}$ and $a' \in \mathcal{H}$. Since $a \in \mathcal{H} = \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\}$, we have $f(ab) = f(a)f(b)$ for every $b \in A$. Since $a' \in \mathcal{H} = \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\}$, we have $f(a'b) = f(a')f(b)$ for every $b \in A$. Now,

$$f(\underbrace{(\lambda a + \mu a')b}_{=\lambda ab + \mu a'b}) = f(\lambda ab + \mu a'b) = \lambda \underbrace{f(ab)}_{=f(a)f(b)} + \mu \underbrace{f(a'b)}_{=f(a')f(b)} \qquad (\text{since } f \text{ is } \mathbb{C}\text{-linear})$$

$$= \lambda f(a)f(b) + \mu f(a')f(b) = \underbrace{(\lambda f(a) + \mu f(a'))}_{\substack{=f(\lambda a + \mu a') \\ (\text{since } f \text{ is } \mathbb{C}\text{-linear})}} f(b) = f(\lambda a + \mu a')f(b)$$

for every $b \in A$. In other words, $\lambda a + \mu a' \in \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\} = \mathcal{H}$, qed.

[5] *Proof.* Let $a \in \mathcal{H}$ and $a' \in \mathcal{H}$. Since $a \in \mathcal{H} = \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\}$, we have

$$f(ab) = f(a)f(b) \qquad \text{for every } b \in A \qquad (4)$$

for every $b \in A$. Since $a' \in \mathcal{H} = \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\}$, we have $f(a'b) = f(a')f(b)$ for every $b \in A$. Now, every $b \in A$ satisfies

$$f(aa'b) = f(a)\underbrace{f(a'b)}_{=f(a')f(b)} \qquad (\text{by (4), applied to } a'b \text{ instead of } b)$$

$$= f(a)f(a')f(b)$$

and

$$\underbrace{f(aa')}_{\substack{=f(a)f(a') \\ (\text{by (4), applied to } a' \\ \text{instead of } b)}} f(b) = f(a)f(a')f(b).$$

Hence, $f(aa'b) = f(a)f(a')f(b) = f(aa')f(b)$ for every $b \in A$. Hence, $aa' \in \{x \in A \mid f(xb) = f(x)f(b) \text{ for every } b \in A\} = \mathcal{H}$, qed.

[6] *Proof.* We know that $\mathcal{G}$ generates A as a $\mathbb{C}$-algebra. In other words, A is the smallest $\mathbb{C}$-subalgebra of A containing $\mathcal{G}$ as a subset. Hence, every $\mathbb{C}$-subalgebra of A containing $\mathcal{G}$ as a subset must contain A as a subset, qed.

every $b \in A$.

We thus have proven that every $a \in A$ and $b \in A$ satisfy $f(ab) = f(a) f(b)$. Combined with $f(1) = 1$ and with the $\mathbb{C}$-linearity of the map f, this yields that f is a $\mathbb{C}$-algebra homomorphism. Lemma 2.18a is proven.

Lemma 2.18b. Let V be a finite-dimensional $\mathbb{C}$-vector space, and let U be a Zariski-dense open subset of V. Let $\mathbb{C}[U]$ and $\mathbb{C}[V]$ be the coordinate rings of U and V, respectively (so that $\mathbb{C}[V] = S(V^*)$, and $\mathbb{C}[U]$ is a localization of $\mathbb{C}[V]$). Let $\tau : V \to \mathbb{C}[U]$ be a $\mathbb{C}$-linear map. Assume that

$$[\partial_a + \tau(a), \partial_b + \tau(b)] = 0 \qquad \text{for any } a \in V \text{ and } b \in V. \tag{5}$$

Then, there exists a unique $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(U) \to \mathcal{D}(U)$ which satisfies the following two conditions:

Condition 1: We have $\varsigma(f) = f$ for every $f \in \mathbb{C}[U]$ (where $\mathbb{C}[U]$ is canonically embedded into $\mathcal{D}(U)$).

Condition 2: We have $\varsigma(\partial_a) = \partial_a + \tau(a)$ for every $a \in V$.

Proof of Lemma 2.18b. Since the $\mathbb{C}$-algebra $\mathcal{D}(U)$ is generated by the elements of $\mathbb{C}[U]$ and the elements ∂_a for $a \in V$, it is clear that there exists **at most one** $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(U) \to \mathcal{D}(U)$ satisfying Conditions 1 and 2. Hence, in order to prove that there exists **exactly one** such homomorphism, we need only check that there exists **at least one** such homomorphism. Let us do this now.

Let $\mathcal{D}_{\text{const}}(V)$ be the $\mathbb{C}$-algebra of differential operators on V with constant coefficients. Recall that $\mathcal{D}(U) = \mathbb{C}[U] \otimes \mathcal{D}_{\text{const}}(V)$ as a vector space (where $\otimes$ means $\otimes_{\mathbb{C}}$). In particular, for any $f \in \mathbb{C}[U]$ and any $D \in \mathcal{D}_{\text{const}}(V)$, the operator $fD \in \mathcal{D}(U)$ is the tensor product $f \otimes D \in \mathbb{C}[U] \otimes \mathcal{D}_{\text{const}}(V)$.

Moreover, we can define a map $\partial : V \to \mathcal{D}_{\text{const}}(V)$ by

$$\partial(v) = \partial_v \qquad \text{for every } v \in V.$$

Then, ∂ is a $\mathbb{C}$-linear injection, and the image $\partial(V)$ is the space of all degree-1 differential operators on V with constant coefficients. Denote by $\partial^{-1} : \partial(V) \to V$ the inverse of ∂ on $\partial(V)$.

Let D' be the $\mathbb{C}$-subalgebra of $\mathcal{D}(U)$ generated by $\{\partial_v + \tau(v) \mid v \in V\}$. Then, the algebra D' is commutative (because (5) shows that its generators commute). Define a k-linear map $\xi : V \to D'$ by

$$\xi(v) = \partial_v + \tau(v) \qquad \text{for every } v \in V.$$

Then, $\xi \circ \partial^{-1}$ is a $\mathbb{C}$-linear map $\partial(V) \to D'$. By the universal property of the symmetric algebra, the $\mathbb{C}$-linear map $\xi \circ \partial^{-1} : \partial(V) \to D'$ can be extended to a $\mathbb{C}$-algebra homomorphism $\Xi : S(\partial(V)) \to D'$ such that

$$\Xi(z) = (\xi \circ \partial^{-1})(z) \qquad \text{for every } z \in \partial(V) \tag{6}$$

(because D' is commutative). Consider this Ξ.

Since $\partial(V)$ is the space of all degree-1 differential operators on V with constant coefficients, we have $\mathcal{D}_{\text{const}}(V) \cong S(\partial(V))$. Hence, we can regard $\Xi : S(\partial(V)) \to D'$ as a $\mathbb{C}$-algebra homomorphism $\mathcal{D}_{\text{const}}(V) \to D'$. Since Ξ is a $\mathbb{C}$-algebra homomorphism, we have $\Xi(1) = 1$.

Now, define a $\mathbb{C}$-linear map $\varsigma : \mathbb{C}[U] \otimes \mathcal{D}_{\text{const}}(V) \to \mathcal{D}(U)$ by

$$\varsigma(f \otimes D) = f\Xi(D) \qquad \text{for every } f \in \mathbb{C}[U] \text{ and } D \in \mathcal{D}_{\text{const}}(V).$$

Since $\mathbb{C}[U] \otimes \mathcal{D}_{\text{const}}(V) = \mathcal{D}(U)$, this map ς is a $\mathbb{C}$-linear map $\mathcal{D}(U) \to \mathcal{D}(U)$. We claim that ς is a $\mathbb{C}$-algebra homomorphism satisfying Conditions 1 and 2.

In fact, every $f \in \mathbb{C}[U]$ satisfies

$$\varsigma\left(\underbrace{f}_{=f \otimes 1}\right) = \varsigma(f \otimes 1) = f\underbrace{\Xi(1)}_{=1} \qquad \text{(by the definition of } \varsigma)$$
$$= f.$$

Thus, ς satisfies Condition 1. Applied to $f = 1$, Condition 1 yields $\varsigma(1) = 1$.

Every $a \in V$ satisfies

$$\varsigma\left(\underbrace{\partial_a}_{=1 \otimes \partial_a}\right) = \zeta(1 \otimes \partial_a) = 1\Xi(\partial_a) \qquad \text{(by the definition of } \varsigma)$$
$$= \Xi(\partial_a) = \left(\xi \circ \partial^{-1}\right)(\partial_a) \qquad \text{(by (6), applied to } z = \partial_a)$$
$$= \xi\left(\underbrace{\partial^{-1}(\partial_a)}_{\substack{=a \\ (\text{since } \partial_a = \partial(a))}}\right) = \xi(a) = \partial_a + \tau(a) \qquad \text{(by the definition of } \xi).$$

Thus, ς satisfies Condition 2.

We now will prove that ς is a $\mathbb{C}$-algebra homomorphism. For this, define a subset $\mathcal{G}$ of $\mathcal{D}(U)$ by $\mathcal{G} = \mathbb{C}[U] \cup \partial(V)$. Then, $\mathcal{G}$ generates $\mathcal{D}(U)$ as a $\mathbb{C}$-algebra. Hence, Lemma 2.18a (applied to $A = \mathcal{D}(U)$, $D = \mathcal{D}(U)$ and $f = \varsigma$), in order to prove that ς is a $\mathbb{C}$-algebra homomorphism, it will be enough to prove that

$$\varsigma(ab) = \varsigma(a)\varsigma(b) \qquad \text{for every } a \in \mathcal{G} \text{ and } b \in \mathcal{D}(U). \tag{7}$$

So let us prove this now:

Proof of (7): Let $a \in \mathcal{G}$ and $b \in \mathcal{D}(U)$. Since the equality (7) is $\mathbb{C}$-linear in b, we can WLOG assume that b has the form gE for some $g \in \mathbb{C}[U]$ and $E \in \mathcal{D}(U)$ (because every element of $\mathcal{D}(U)$ is a $\mathbb{C}$-linear combination of elements of this form). Assume this. Thus, $b = gE = g \otimes E$. Hence,

$$\varsigma(b) = \varsigma(g \otimes E) = g\Xi(E) \qquad \text{(by the definition of } \varsigma). \tag{8}$$

Since $a \in \mathcal{G} = \mathbb{C}[U] \cup \partial(V)$, we have $a \in \mathbb{C}[U]$ or $a \in \partial(V)$. Thus, we must be in one of the following cases:

Case 1: We have $a \in \mathbb{C}[U]$.

Case 2: We have $a \in \partial(V)$.

Let us consider Case 1 first. In this case, $a \in \mathbb{C}[U]$. Hence, $\varsigma(a) = a$ (by Condition 1, applied to $f = a$), and

$$\varsigma\left(a \underbrace{b}_{=gE}\right) = \varsigma\left(\underbrace{agE}_{=ag\otimes E}\right) = \varsigma(ag \otimes E) = \underbrace{a}_{=\varsigma(a)}\underbrace{g\Xi(E)}_{\substack{=\varsigma(b) \\ \text{(by (8))}}} \qquad \text{(by the definition of }\varsigma\text{)}$$

$$= \varsigma(a)\varsigma(b).$$

Hence, (7) is proven in Case 1.

Let us now consider Case 2. In this case, $a \in \partial(V)$. Thus, there exists some $v \in V$ such that $a = \partial_v$. Consider this v. Let $\partial_v g$ denote the product of the elements ∂_v and g in the $\mathbb{C}$-algebra $\mathcal{D}(V)$, whereas $\partial_v(g)$ denotes the image of g under the differential operator ∂_v. Then,

$$\partial_v g = g\partial_v + \partial_v(g),$$

so that

$$\underbrace{a}_{=\partial_v}\underbrace{b}_{=gE} = \underbrace{\partial_v g}_{=g\partial_v + \partial_v(g)} E = g\partial_v E + \partial_v(g)E = g\otimes\partial_v E + \partial_v(g)\otimes E,$$

and thus

$$\varsigma(ab) = \varsigma(g\otimes\partial_v E + \partial_v(g)\otimes E) = \underbrace{\varsigma(g\otimes\partial_v E)}_{\substack{=g\Xi(\partial_v E) \\ \text{(by the definition of }\varsigma\text{)}}} + \underbrace{\varsigma(\partial_v(g)\otimes E)}_{\substack{=\partial_v(g)\Xi(E) \\ \text{(by the definition of }\varsigma\text{)}}}$$

$$= g\underbrace{\Xi(\partial_v E)}_{\substack{=\Xi(\partial_v)\Xi(E) \\ \text{(since }\Xi\text{ is a }\mathbb{C}\text{-algebra} \\ \text{homomorphism)}}} + \partial_v(g)\Xi(E) = g\underbrace{\Xi(\partial_v)}_{\substack{=(\xi\circ\partial^{-1})(\partial_v) \\ \text{(by (6), applied} \\ \text{to } z=\partial_v)}}\Xi(E) + \partial_v(g)\Xi(E)$$

$$= g\underbrace{(\xi\circ\partial^{-1})(\partial_v)}_{=\xi(\partial^{-1}(\partial_v))}\Xi(E) + \partial_v(g)\Xi(E) = g\xi\left(\underbrace{\partial^{-1}(\partial_v)}_{\substack{=v \\ \text{(since }\partial_v=\partial(v))}}\right)\Xi(E) + \partial_v(g)\Xi(E)$$

$$= g\underbrace{\xi(v)}_{\substack{=\partial_v+\tau(v) \\ \text{(by the definition of }\xi\text{)}}}\Xi(E) + \partial_v(g)\Xi(E) = g(\partial_v + \tau(v))\Xi(E) + \partial_v(g)\Xi(E)$$

$$= \left(\underbrace{g(\partial_v + \tau(v))}_{=g\partial_v+g\tau(v)} + \partial_v(g)\right)\Xi(E) = (g\partial_v + g\tau(v) + \partial_v(g))\Xi(E).$$

On the other hand, Condition 2 (applied to v instead of a) yields $\varsigma(\partial_v) = \partial_v +$

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$\tau(v)$, so that

$$\varsigma\left(\underbrace{a}_{=\partial_v}\right)\varsigma\left(\underbrace{b}_{=gE=g\otimes E}\right) = \underbrace{\varsigma\left(\partial_v\right)}_{=\partial_v+\tau(v)}\underbrace{\varsigma\left(g\otimes E\right)}_{=g\Xi(E)}$$
$$\text{(by the definition of }\varsigma\text{)}$$
$$= \left(\partial_v+\tau(v)\right)g\Xi(E) = \underbrace{\partial_v g}_{=g\partial_v+\partial_v(g)}\Xi(E) + \underbrace{\tau(v)\,g}_{=g\tau(v)}\Xi(E)$$
$$= g\partial_v\Xi(E) + \partial_v(g)\Xi(E) + g\tau(v)\Xi(E)$$
$$= \left(g\partial_v + \partial_v(g) + g\tau(v)\right)\Xi(E) = \left(g\partial_v + g\tau(v) + \partial_v(g)\right)\Xi(E)$$
$$= \varsigma(ab).$$

Hence, (7) is proven in Case 2.

So we have proven (7) in each of the Cases 1 and 2. Since Cases 1 and 2 are the only possible cases, this yields that (7) always holds.

Thus, Lemma 2.18a (applied to $A = \mathcal{D}(U)$, $D = \mathcal{D}(U)$ and $f = \varsigma$) yields that ς is a $\mathbb{C}$-algebra homomorphism. Hence, ς is a $\mathbb{C}$-algebra homomorphism satisfying Conditions 1 and 2. We thus have verified the existence of a $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(U) \to \mathcal{D}(U)$ satisfying Conditions 1 and 2. This completes the proof of Lemma 2.18b.

Corollary 2.18c. Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space. Let $\mathcal{S}$ be a finite set. For every $s \in \mathcal{S}$, let c_s be an element of $\mathbb{C}$ and let α_s be an element of $\mathfrak{h}^*$. Let $\mathfrak{h}_{\mathrm{reg}}$ be a Zariski-dense open subset of $\mathfrak{h}$ such that every $a \in \mathfrak{h}_{\mathrm{reg}}$ and every $s \in \mathcal{S}$ satisfy $\alpha_s(a) \neq 0$. Then, there exists a unique $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \to \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ which satisfies the following two conditions:

Condition 1: We have $\varsigma(f) = f$ for every $f \in \mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]$ (where $\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]$ is canonically embedded into $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$).

Condition 2: We have $\varsigma(\partial_a) = \partial_a + \sum_{s \in \mathcal{S}} \dfrac{c_s \alpha_s(a)}{\alpha_s}$ for every $a \in \mathfrak{h}$.

Proof of Corollary 2.18c. Let $V = \mathfrak{h}$ and $U = \mathfrak{h}_{\mathrm{reg}}$. Define a $\mathbb{C}$-linear map $\tau : \mathfrak{h} \to \mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]$ by

$$\tau(a) = \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s} \qquad \text{for every } a \in \mathfrak{h}.$$

Then, obviously, Conditions 1 and 2 of Corollary 2.18c are equivalent to Conditions 1 and 2 of Lemma 2.18b, respectively.

Every $a \in \mathfrak{h}$ and $b \in \mathfrak{h}$ satisfy

$$[\partial_a + \tau(a), \partial_b + \tau(b)]$$

$$= \left[\partial_a + \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s}, \partial_b + \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(b)}{\alpha_s}\right]$$

$$\left(\text{since } \tau(a) = \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s} \text{ and } \tau(b) = \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(b)}{\alpha_s}\right)$$

$$= \underbrace{[\partial_a, \partial_b]}_{=0} + \left[\partial_a, \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(b)}{\alpha_s}\right] + \underbrace{\left[\sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s}, \partial_b\right]}_{=-\left[\partial_b, \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s}\right]} + \underbrace{\left[\sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s}, \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(b)}{\alpha_s}\right]}_{=0}$$

$$= \underbrace{\left[\partial_a, \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(b)}{\alpha_s}\right]}_{=\partial_a\left(\sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(b)}{\alpha_s}\right) = \sum_{s \in \mathcal{S}} c_s \alpha_s(b) \cdot \partial_a\left(\frac{1}{\alpha_s}\right)} - \underbrace{\left[\partial_b, \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s}\right]}_{=\partial_b\left(\sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s}\right) = \sum_{s \in \mathcal{S}} c_s \alpha_s(a) \cdot \partial_b\left(\frac{1}{\alpha_s}\right)}$$

$$= \sum_{s \in \mathcal{S}} c_s \alpha_s(b) \cdot \underbrace{\partial_a\left(\frac{1}{\alpha_s}\right)}_{\substack{=-\frac{\partial_a(\alpha_s)}{\alpha_s^2}=-\frac{\alpha_s(a)}{\alpha_s^2} \\ (\text{since } \partial_a(\alpha_s)=\alpha_s(a) \text{ (because } \alpha_s \\ \text{is linear))}}} - \sum_{s \in \mathcal{S}} c_s \alpha_s(a) \cdot \underbrace{\partial_b\left(\frac{1}{\alpha_s}\right)}_{\substack{=-\frac{\partial_b(\alpha_s)}{\alpha_s^2}=\cdot\frac{\alpha_s(b)}{\alpha_s^2} \\ (\text{since } \partial_b(\alpha_s)=\alpha_s(b) \text{ (because } \alpha_s \\ \text{is linear))}}}$$

$$= \sum_{s \in \mathcal{S}} c_s \alpha_s(b) \cdot \underbrace{\left(-\frac{\alpha_s(a)}{\alpha_s^2}\right)}_{=\frac{-c_s\alpha_s(a)\alpha_s(b)}{\alpha_s^2}} - \sum_{s \in \mathcal{S}} c_s \alpha_s(a) \cdot \underbrace{\left(-\frac{\alpha_s(b)}{\alpha_s^2}\right)}_{=\frac{-c_s\alpha_s(a)\alpha_s(b)}{\alpha_s^2}}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a) \alpha_s(b)}{\alpha_s^2} - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a) \alpha_s(b)}{\alpha_s^2} = 0.$$

Hence, Lemma 2.18b yields that there exists a unique $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(U) \to \mathcal{D}(U)$ which satisfies the Conditions 1 and 2 of Lemma 2.18b. In other words, there exists a unique $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(U) \to \mathcal{D}(U)$ which satisfies the Conditions 1 and 2 of Corollary 2.18c (because we know that Conditions 1 and 2 of Corollary 2.18c are equivalent to Conditions 1 and 2 of Lemma 2.18b, respectively). Corollary 2.18c is thus proven.

Definition. Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space with a nondegenerate bilinear inner product $(\cdot, \cdot)$. Let $W \subseteq O(\mathfrak{h})$ be a real reflection group, and $\mathcal{S} \subseteq W$ the set of reflections. Let $c : \mathcal{S} \to \mathbb{C}$ be a function invariant under conjugation (by elements of W). For every $s \in \mathcal{S}$, we will write c_s for $c(s)$. For every $s \in \mathcal{S}$, let $\alpha_s \in \mathfrak{h}^*$ be the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of s (acting on $\mathfrak{h}^*$) with eigenvalue -1, and let $\alpha_s^\vee \in \mathfrak{h}$ be the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of s (acting on $\mathfrak{h}$) with eigenvalue -1. Define

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H as in Definition 2.8, and define $\overline{H}$ as in Proposition 2.16. Let $\mathfrak{h}_{\mathrm{reg}}$ be the subset $\{x \in \mathfrak{h} \mid W_x = \{\mathrm{id}\}\}$ of $\mathfrak{h}$. According to Corollary 2.18c, there exists a unique $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \to \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ which satisfies the Conditions 1 and 2 of Corollary 2.18c[7]. This homomorphism ς will be denoted by ς_c. Due to Condition 1, it satisfies

$$\varsigma_c(f) = f \qquad \text{for every } f \in \mathbb{C}[\mathfrak{h}_{\mathrm{reg}}] \tag{9}$$

(where $\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]$ is canonically embedded into $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$). Due to Condition 2, it satisfies

$$\varsigma_c(\partial_a) = \partial_a + \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(a)}{\alpha_s} \qquad \text{for every } a \in \mathfrak{h}. \tag{10}$$

Remark. In terms of your Proposition 2.18, this homomorphism ς_c is the conjugation by δ_c (that is, it is given by $D \mapsto \delta_c^{-1} \circ D \circ \delta_c$). However, our definition of ς_c was purely algebraic, while your δ_c is a transcendental function (in general).

Now, our elementary version of Proposition 2.18 rewrites as follows:

Proposition 2.18d. We have $\varsigma_c(\overline{H}) = H$.

Before we prove this, another lemma:

Lemma 2.18e. Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space with a nondegenerate bilinear inner product $(\cdot, \cdot)$. Let $W \subseteq \mathrm{O}(\mathfrak{h})$ be a real reflection group, and $\mathcal{S} \subseteq W$ the set of reflections. Let $c : \mathcal{S} \to \mathbb{C}$ be a function invariant under conjugation (by elements of W). For every $s \in \mathcal{S}$, we will write c_s for $c(s)$. For every $s \in \mathcal{S}$, let $\alpha_s \in \mathfrak{h}^*$ be the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of s (acting on $\mathfrak{h}^*$) with eigenvalue -1. Let $\mathfrak{h}_{\mathrm{reg}}$ be the subset $\{x \in \mathfrak{h} \mid G_x = \{\mathrm{id}\}\}$ of $\mathfrak{h}$. Then:

(a) Every $t \in \mathcal{S}$ satisfies

$$t\left(\prod_{s \in \mathcal{S}} \alpha_s\right) = -\prod_{s \in \mathcal{S}} \alpha_s$$

(where $t\left(\prod_{s \in \mathcal{S}} \alpha_s\right)$ denotes the action of $t \in W$ on $\prod_{s \in \mathcal{S}} \alpha_s \in S(\mathfrak{h}^*)$).

(b) We have

$$\sum_{\substack{s \in \mathcal{S}; \ u \in \mathcal{S}; \\ s \neq u}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u} = 0.$$

(c) We have

$$\sum_{s \in \mathcal{S}; \ u \in \mathcal{S}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u} = \sum_{s \in \mathcal{S}} \frac{c_s^2 (\alpha_s, \alpha_s)}{\alpha_s^2}.$$

Proof of Lemma 2.18e. Let us notice that

$$\text{if two } t \in \mathcal{S} \text{ and } s \in \mathcal{S} \text{ satisfy } \mathrm{Ker}(\alpha_t) \subseteq \mathrm{Ker}(\alpha_s), \text{ then } t = s \tag{11}$$

---

[7]In fact, every $a \in \mathfrak{h}_{\mathrm{reg}}$ and every $s \in \mathcal{S}$ satisfy $\alpha_s(a) \neq 0$ (because otherwise, a would be fixed under s, contradicting $W_a = \{\mathrm{id}\}$).

[8]. As a consequence, the polynomials $\alpha_s \in \mathbb{C}[\mathfrak{h}]$ for $s \in \mathcal{S}$ are pairwise coprime.[9]

Also, for every $t \in \mathcal{S}$ and every $s \in \mathcal{S}$, we have $t\alpha_s \in \mathbb{C}^\times \alpha_{tst^{-1}}$ [10]. In other words, for every $t \in \mathcal{S}$ and every $s \in \mathcal{S}$, there exists a $\mu_{t,s} \in \mathbb{C}^\times$ such that

$$t\alpha_s = \mu_{t,s}\alpha_{tst^{-1}}. \tag{12}$$

Consider these $\mu_{t,s}$.

(a) Let $t \in \mathcal{S}$. Then,

$$t\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right) = \prod_{s\in\mathcal{S}\setminus\{t\}}\underbrace{(t\alpha_s)}_{=\mu_{t,s}\alpha_{tst^{-1}}} = \prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s}\underbrace{\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_{tst^{-1}}}_{\substack{=\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s \\ \text{(because the map} \\ \mathcal{S}\setminus\{t\}\to\mathcal{S}\setminus\{t\},\ s\mapsto tst^{-1} \\ \text{is a bijection)}}}$$

$$= \prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s}\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s.$$

---

[8] *Proof of (11):* Let $t \in \mathcal{S}$ and $s \in \mathcal{S}$ satisfy $\mathrm{Ker}(\alpha_t) \subseteq \mathrm{Ker}(\alpha_s)$. Then, $\mathrm{Ker}(\alpha_t) = \mathrm{Ker}(\alpha_s)$ (since $\mathrm{Ker}(\alpha_t)$ and $\mathrm{Ker}(\alpha_s)$ are hyperplanes in $\mathfrak{h}$, and thus have the same dimension).

But s is the reflection in the hyperplane $\mathrm{Ker}(\alpha_s)$ (because s is a reflection, and $\alpha_s \in \mathfrak{h}^*$ is the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of s (acting on $\mathfrak{h}^*$) with eigenvalue -1). Similarly, t is the reflection in the hyperplane $\mathrm{Ker}(\alpha_t)$. Thus,

$$s = \left(\text{the reflection in the hyperplane } \underbrace{\mathrm{Ker}(\alpha_s)}_{=\mathrm{Ker}(\alpha_t)}\right) = (\text{the reflection in the hyperplane } \mathrm{Ker}(\alpha_t)) = t,$$

qed.

[9] *Proof.* Assume the contrary. Then, there exist two distinct elements $t \in \mathcal{S}$ and $s \in \mathcal{S}$ such that the polynomials α_s and α_t have a nontrivial common divisor. Consider these t and s. The polynomials α_s and α_t have a nontrivial common divisor, but are both linear. Therefore, α_s and α_t must be proportional to each other, i. e., there exists a $\lambda \in \mathbb{C}^\times$ such that $\alpha_s = \lambda\alpha_t$. Therefore, $\mathrm{Ker}(\alpha_t) = \mathrm{Ker}(\alpha_s)$, so that $t = s$ (by (11)), contradicting the assumption that t and s be distinct. This contradiction proves that our assumption was wrong, qed.

[10] *Proof.* Let $t \in \mathcal{S}$ and $s \in \mathcal{S}$. Then, α_s is a nonzero eigenvector of s (acting on $\mathfrak{h}^*$) with eigenvalue -1. Thus, $s\alpha_s = -1\alpha_s = -\alpha_s$, so that $(tst^{-1})(t\alpha_s) = t\underbrace{s\alpha_s}_{=-\alpha_s} = -t\alpha_s$. In other words, $t\alpha_s$ is an eigenvector of tst^{-1} (acting on $\mathfrak{h}^*$) with eigenvalue -1. Also, $t\alpha_s \neq 0$ (since $\alpha_s \neq 0$). Hence, $t\alpha_s$ is a nonzero eigenvector of tst^{-1} (acting on $\mathfrak{h}^*$) with eigenvalue -1. Thus, $t\alpha_s \in \mathbb{C}^\times\alpha_{tst^{-1}}$ (because $\alpha_{tst^{-1}}$ is the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of tst^{-1} (acting on $\mathfrak{h}^*$) with eigenvalue -1), qed.

Hence,

$$t^2 \left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right) = t \cdot \underbrace{t\left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right)}_{= \prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s}$$

$$= \prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} \cdot \underbrace{t\left(\prod_{s\in\mathcal{S}} \alpha_s \right)}_{= \prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s} = \left(\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} \right)^2 \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s.$$

Compared with $t^2 \left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right) = \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s$ (because t is a reflection and thus

satisfies $t^2 = \mathrm{id}$), this yields $\left(\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} \right)^2 \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s = \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s$. Since $\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s$

is nonzero, this yields $\left(\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} \right)^2 = 1$. Hence, $\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} = 1$ or $\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} = -1$.

Let us first assume that $\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s} = -1$. In this case,

$$t\left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right) = \underbrace{\prod_{s\in\mathcal{S}\setminus\{t\}} \mu_{t,s}}_{=-1} \prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s = -\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s. \tag{13}$$

Now, $\mathrm{Ker}\,(\alpha_t) \not\subseteq \bigcup_{s\in\mathcal{S};\ s\neq t} \mathrm{Ker}\,(\alpha_s)$ [11]. Hence, there exists a $p \in \mathrm{Ker}\,(\alpha_t)$ such that $p \notin \bigcup_{s\in\mathcal{S};\ s\neq t} \mathrm{Ker}\,(\alpha_s)$. Pick such a p.

Now, t is the reflection in the hyperplane $\mathrm{Ker}\,(\alpha_t)$ (because t is a reflection, and $\alpha_t \in \mathfrak{h}^*$ is the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of t (acting on $\mathfrak{h}^*$) with eigenvalue -1). Thus, $\mathrm{Ker}\,(\alpha_t) = \{\text{set of fixed points of } t \text{ in } \mathfrak{h}\}$.

Since $p \in \mathrm{Ker}\,(\alpha_t) = \{\text{set of fixed points of } t \text{ in } \mathfrak{h}\}$, the point p is fixed under t, so that $tp = p$ and thus $t^{-1}p = p$. Thus,

$$\left(t\left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right) \right)(p) = \left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right) \underbrace{(t^{-1}p)}_{=p} = \left(\prod_{s\in\mathcal{S}\setminus\{t\}} \alpha_s \right)(p).$$

---

[11] *Proof.* Assume the contrary. Then, $\mathrm{Ker}\,(\alpha_t) \subseteq \bigcup_{s\in\mathcal{S};\ s\neq t} \mathrm{Ker}\,(\alpha_s)$. Since $\mathrm{Ker}\,(\alpha_t)$ and $\mathrm{Ker}\,(\alpha_s)$ are vector subspaces of $\mathfrak{h}$, this yields that there exists some $s \in \mathcal{S}$ such that $s \neq t$ and $\mathrm{Ker}\,(\alpha_t) \subseteq \mathrm{Ker}\,(\alpha_s)$ (because there is a well-known linear-algebraic fact that if a vector subspace U of a finite-dimensional $\mathbb{C}$-vector space V is a subset of the union $\bigcup_{i\in I} W_i$ of finitely many subspaces W_i of V, then there exists some $i \in I$ such that $U \subseteq W_i$). Consider this s. Then, $\mathrm{Ker}\,(\alpha_t) \subseteq \mathrm{Ker}\,(\alpha_s)$, so that $t = s$ (by (11)), contradicting $s \neq t$. This contradiction shows that our assumption was wrong, qed.

Compared to

$$\underbrace{\left(t\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right)\right)(p)}_{\substack{=-\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\\ \text{(by (13))}}}=-\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right)(p),$$

this yields $\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right)(p) = -\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right)(p)$. Thus, $\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right)(p) = 0$.
In other words, $\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s(p) = 0$. Hence, there exists some $s \in \mathcal{S}\setminus\{t\}$ such that
$p \in \mathrm{Ker}\,(\alpha_s)$. In other words, $p \in \bigcup_{s\in\mathcal{S}\setminus\{t\}}\mathrm{Ker}\,(\alpha_s) = \bigcup_{s\in\mathcal{S};\ s\neq t}\mathrm{Ker}\,(\alpha_s)$, contradict-
ing $p \notin \bigcup_{s\in\mathcal{S};\ s\neq t}\mathrm{Ker}\,(\alpha_s)$.

This contradiction shows that our assumption (the assumption that $\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s} = -1$) was wrong. So we don't have $\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s} = -1$. Since we know that we have
$\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s} = 1$ or $\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s} = -1$, this yields that $\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s} = 1$.

But we know that α_t is an eigenvector of t (acting on $\mathfrak{h}^*$) with eigenvalue -1.
Thus, $t\alpha_t = -1\alpha_t = -\alpha_t$.

Now, $\prod_{s\in\mathcal{S}}\alpha_s = \alpha_t \cdot \prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s$, so that

$$t\left(\prod_{s\in\mathcal{S}}\alpha_s\right) = t\left(\alpha_t \cdot \prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right) = \underbrace{t\alpha_t}_{=-\alpha_t} \cdot \underbrace{t\left(\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s\right)}_{=\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s}\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s}$$

$$= -\underbrace{\prod_{s\in\mathcal{S}\setminus\{t\}}\mu_{t,s}}_{=1} \cdot \alpha_t \underbrace{\prod_{s\in\mathcal{S}\setminus\{t\}}\alpha_s}_{=\prod_{s\in\mathcal{S}}\alpha_s} = -\prod_{s\in\mathcal{S}}\alpha_s.$$

This proves Lemma 2.18e **(a)**.

(b) Let P be the function

$$\sum_{\substack{s\in\mathcal{S};\ u\in\mathcal{S};\\ s\neq u}}\frac{c_s c_u\,(\alpha_s,\alpha_u)}{\alpha_s\alpha_u} \in \mathbb{C}\,[\mathfrak{h}_{\mathrm{reg}}].$$

Then, every $t \in \mathcal{S}$ satisfies

$$tP = t \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\alpha_s, \alpha_u\right)}{\alpha_s \alpha_u} = \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\alpha_s, \alpha_u\right)}{\left(t\alpha_s\right)\left(t\alpha_u\right)} = \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(t\alpha_s, t\alpha_u\right)}{\left(t\alpha_s\right)\left(t\alpha_u\right)}$$

$$\left(\text{since } t \in \mathcal{S} \subseteq W \subseteq \mathrm{O}\left(\mathfrak{h}\right) \text{ and thus } \left(\alpha_s, \alpha_u\right) = \left(t\alpha_s, t\alpha_u\right)\right)$$

$$= \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\mu_{t,s} \alpha_{tst^{-1}}, \mu_{t,u} \alpha_{tut^{-1}}\right)}{\mu_{t,s} \alpha_{tst^{-1}} \cdot \mu_{t,u} \alpha_{tut^{-1}}}$$

$$\left(\text{since (12) yields } t\alpha_s = \mu_{t,s} \alpha_{tst^{-1}} \text{ and } t\alpha_u = \mu_{t,u} \alpha_{tut^{-1}}\right)$$

$$= \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \mu_{t,s} \mu_{t,u} \left(\alpha_{tst^{-1}}, \alpha_{tut^{-1}}\right)}{\mu_{t,s} \alpha_{tst^{-1}} \cdot \mu_{t,u} \alpha_{tut^{-1}}} = \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\alpha_{tst^{-1}}, \alpha_{tut^{-1}}\right)}{\alpha_{tst^{-1}} \alpha_{tut^{-1}}}$$

$$= \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_{tst^{-1}} c_{tut^{-1}} \left(\alpha_{tst^{-1}}, \alpha_{tut^{-1}}\right)}{\alpha_{tst^{-1}} \alpha_{tut^{-1}}} \qquad \left(\begin{array}{c} \text{since the function } c \text{ is invariant under} \\ \text{conjugation, and thus } c_s = c_{tst^{-1}} \text{ and } c_u = c_{tut^{-1}} \end{array}\right)$$

$$= \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\alpha_s, \alpha_u\right)}{\alpha_s \alpha_u}$$

$$\left(\begin{array}{c} \text{here, we substituted } (s, u) \text{ for } (tst^{-1}, tut^{-1}) \text{ in the sum, because the map} \\ \{(s, u) \in \mathcal{S} \times \mathcal{S} \mid s \neq u\} \to \{(s, u) \in \mathcal{S} \times \mathcal{S} \mid s \neq u\}, \ (s, u) \mapsto (tst^{-1}, tut^{-1}) \\ \text{is a bijection} \end{array}\right)$$

$$= P.$$

Moreover, since $P = \displaystyle\sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\alpha_s, \alpha_u\right)}{\alpha_s \alpha_u}$ and $\displaystyle\prod_{s \in \mathcal{S}} \alpha_s = \prod_{q \in \mathcal{S}} \alpha_q$, we have

$$P \cdot \prod_{s \in \mathcal{S}} \alpha_s = \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} \frac{c_s c_u \left(\alpha_s, \alpha_u\right)}{\alpha_s \alpha_u} \cdot \prod_{q \in \mathcal{S}} \alpha_q = \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} c_s c_u \left(\alpha_s, \alpha_u\right) \cdot \underbrace{\frac{\displaystyle\prod_{q \in \mathcal{S}} \alpha_q}{\alpha_s \alpha_u}}_{= \displaystyle\prod_{q \in \mathcal{S} \setminus \{s,u\}} \alpha_q}$$

$$= \sum_{\substack{s \in \mathcal{S};\ u \in \mathcal{S};\\ s \neq u}} c_s c_u \left(\alpha_s, \alpha_u\right) \cdot \prod_{q \in \mathcal{S} \setminus \{s,u\}} \alpha_q.$$

This yields immediately that $P \cdot \displaystyle\prod_{s \in \mathcal{S}} \alpha_s \in \mathbb{C}\left[\mathfrak{h}\right]$ and $\deg\left(P \cdot \displaystyle\prod_{s \in \mathcal{S}} \alpha_s\right) \leq |\mathcal{S}| - 2$.

Also, every $t \in \mathcal{S}$ satisfies

$$t\left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s\right) = \underbrace{tP}_{=P} \cdot \underbrace{t\left(\prod_{s \in \mathcal{S}} \alpha_s\right)}_{\substack{= - \prod_{s \in \mathcal{S}} \alpha_s \\ \text{(by Lemma 2.18e (a))}}} = -P \cdot \prod_{s \in \mathcal{S}} \alpha_s. \qquad (14)$$

Thus, for every $t \in \mathcal{S}$, we have $\alpha_t \mid P \cdot \prod_{s \in \mathcal{S}} \alpha_s$ in $\mathbb{C}[\mathfrak{h}]$ [12]. Since the polynomials $\alpha_s \in \mathbb{C}[\mathfrak{h}]$ for $s \in \mathcal{S}$ are pairwise coprime, this yields that $\prod_{t \in \mathcal{S}} \alpha_t \mid P \cdot \prod_{s \in \mathcal{S}} \alpha_s$ in $\mathbb{C}[\mathfrak{h}]$ (because $\mathbb{C}[\mathfrak{h}]$ is a unique factorization domain). Since $\deg \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right) \leq |\mathcal{S}| - 2 < |\mathcal{S}| = \deg \left(\prod_{t \in \mathcal{S}} \alpha_t \right)$, this leads to $P \cdot \prod_{s \in \mathcal{S}} \alpha_s = 0$ (because if a polynomial is divisible by a polynomial of greater degree, then the former polynomial must be 0). Hence, $P = 0$ (since $\mathbb{C}[\mathfrak{h}_{\text{reg}}]$ is an integral domain, and $\prod_{s \in \mathcal{S}} \alpha_s \neq 0$). Since

$$P = \sum_{\substack{s \in \mathcal{S}; \ u \in \mathcal{S}; \\ s \neq u}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u},$$ this rewrites as $\sum_{\substack{s \in \mathcal{S}; \ u \in \mathcal{S}; \\ s \neq u}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u} = 0.$ Lemma 2.18e **(b)** is proven.

(c) We have

$$\sum_{s \in \mathcal{S}; \ u \in \mathcal{S}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u} = \underbrace{\sum_{\substack{s \in \mathcal{S}; \ u \in \mathcal{S}; \\ s \neq u}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u}}_{\substack{=0 \\ \text{(by Lemma 2.18e } \mathbf{(b)}\text{)}}} + \underbrace{\sum_{\substack{s \in \mathcal{S}; \ u \in \mathcal{S}; \\ s = u}} \frac{c_s c_u (\alpha_s, \alpha_u)}{\alpha_s \alpha_u}}_{= \sum_{s \in \mathcal{S}} \frac{c_s c_s (\alpha_s, \alpha_s)}{\alpha_s \alpha_s}}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s c_s (\alpha_s, \alpha_s)}{\alpha_s \alpha_s} = \sum_{s \in \mathcal{S}} \frac{c_s^2 (\alpha_s, \alpha_s)}{\alpha_s^2},$$

and thus Lemma 2.18e **(c)** is proven.

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[12] *Proof.* Let $t \in \mathcal{S}$. We know that t is the reflection in the hyperplane $\operatorname{Ker}(\alpha_t)$ (because t is a reflection, and $\alpha_t \in \mathfrak{h}^*$ is the unique (up to scaling by an element of $\mathbb{C}^\times$) nonzero eigenvector of t (acting on $\mathfrak{h}^*$) with eigenvalue -1). Thus, $\operatorname{Ker}(\alpha_t) = \{$set of fixed points of t in $\mathfrak{h}\}$.

Now, let $x \in \operatorname{Ker}(\alpha_t)$. Then, $tx = x$ (because $x \in \operatorname{Ker}(\alpha_t) = \{$set of fixed points of t in $\mathfrak{h}\}$) and thus $t^{-1}x = x$, so that

$$\left(t \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right) \right)(x) = \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right) \left(\underbrace{t^{-1}x}_{=x} \right) = \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right)(x).$$

Compared to

$$\underbrace{\left(t \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right) \right)(x)}_{\substack{= -P \cdot \prod_{s \in \mathcal{S}} \alpha_s \\ \text{(by (14))}}} = - \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right)(x),$$

this yields $\left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right)(x) = - \left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right)(x)$, so that $\left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right)(x) = 0$.

Now forget that we fixed x. We thus have proven that every $x \in \operatorname{Ker}(\alpha_t)$ satisfies $\left(P \cdot \prod_{s \in \mathcal{S}} \alpha_s \right)(x) = 0$. In other words, the polynomial $P \cdot \prod_{s \in \mathcal{S}} \alpha_s$ vanishes on the kernel of the linear function α_t. Thus, $\alpha_t \mid P \cdot \prod_{s \in \mathcal{S}} \alpha_s$ in $\mathbb{C}[\mathfrak{h}]$ (because a polynomial which vanishes on the kernel of a linear function must be divisible by that function), qed.

Proof of Proposition 2.18d. Let $\{y_1, y_2, ..., y_r\}$ be an orthonormal basis of $\mathfrak{h}$. Then, by the definition of the Laplace operator, $\Delta_{\mathfrak{h}} = \sum\limits_{i=1}^{r} \partial_{y_i}^2$.

For every $s \in \mathcal{S}$, we have

$$\sum_{i=1}^{r} \alpha_s(y_i) y_i = \frac{1}{2} (\alpha_s, \alpha_s) \alpha_s^{\vee}. \tag{15}$$

[13] Thus, for every $s \in \mathcal{S}$, we have

$$\sum_{i=1}^{r} \alpha_s(y_i) \partial_{y_i} = \partial_{\sum\limits_{i=1}^{r} \alpha_s(y_i) y_i} = \partial_{\frac{1}{2}(\alpha_s, \alpha_s)\alpha_s^{\vee}} \qquad \text{(by (15))}$$

$$= \frac{1}{2} (\alpha_s, \alpha_s) \partial_{\alpha_s^{\vee}}. \tag{16}$$

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[13] *Proof of (15):* Let $s \in \mathcal{S}$. Then, the bilinear form $(\cdot, \cdot)$ is W-invariant (because $W \subseteq \mathrm{O}(\mathfrak{h})$).

Since the bilinear form $(\cdot, \cdot)$ is nondegenerate, it induces an isomorphism $J : \mathfrak{h}^* \to \mathfrak{h}$. This isomorphism J is W-linear (since $(\cdot, \cdot)$ is W-invariant). Also, it satisfies

$$J(\varphi) = \sum_{i=1}^{r} \varphi(y_i) y_i \qquad \text{for every } \varphi \in \mathfrak{h}^*$$

(since $\{y_1, y_2, ..., y_r\}$ is an orthonormal basis of $\mathfrak{h}$). Applied to $\varphi = \alpha_s$, this yields

$$J(\alpha_s) = \sum_{i=1}^{r} \alpha_s(y_i) y_i.$$

But since α_s is an eigenvector of s (acting on $\mathfrak{h}^*$) with eigenvalue -1, we have $s\alpha_s = -1\alpha_s = -\alpha_s$. Thus, $J(s\alpha_s) = J(-\alpha_s) = -J(\alpha_s)$. Compared with $J(s\alpha_s) = sJ(\alpha_s)$ (since J is W-linear), this yields $sJ(\alpha_s) = -J(\alpha_s) = -1J(\alpha_s)$. In other words, $J(\alpha_s)$ is an eigenvector of s (acting on $\mathfrak{h}$) with eigenvalue -1. This yields that $J(\alpha_s) \in \mathbb{C}\alpha_s^{\vee}$ (because $\alpha_s^{\vee} \in \mathfrak{h}$ is the unique (up to scaling by an element of $\mathbb{C}^{\times}$) nonzero eigenvector of s (acting on $\mathfrak{h}$) with eigenvalue -1). In other words, there exists a $\lambda \in \mathbb{C}$ such that $J(\alpha_s) = \lambda\alpha_s^{\vee}$. We now will prove that $\lambda = \frac{1}{2}(\alpha_s, \alpha_s)$.

In fact, from $J(\alpha_s) = \sum\limits_{i=1}^{r} \alpha_s(y_i) y_i$, we deduce that

$$\langle \alpha_s, J(\alpha_s) \rangle = \left\langle \alpha_s, \sum_{i=1}^{r} \alpha_s(y_i) y_i \right\rangle = \sum_{i=1}^{r} \alpha_s(y_i) \underbrace{\langle \alpha_s, y_i \rangle}_{= \alpha_s(y_i)} = \sum_{i=1}^{r} (\alpha_s(y_i))^2 = (\alpha_s, \alpha_s)$$

(since $\{y_1, y_2, ..., y_r\}$ is an orthonormal basis of $\mathfrak{h}$). Compared with

$$\left\langle \alpha_s, \underbrace{J(\alpha_s)}_{= \lambda\alpha_s^{\vee}} \right\rangle = \lambda \underbrace{\langle \alpha_s, \alpha_s^{\vee} \rangle}_{=2} = 2\lambda,$$

this yields $2\lambda = (\alpha_s, \alpha_s)$, so that $\lambda = \frac{1}{2}(\alpha_s, \alpha_s)$. Now,

$$\sum_{i=1}^{r} \alpha_s(y_i) y_i = J(\alpha_s) = \underbrace{\lambda}_{= \frac{1}{2}(\alpha_s, \alpha_s)} \alpha_s^{\vee} = \frac{1}{2}(\alpha_s, \alpha_s) \alpha_s^{\vee}.$$

This proves (15).

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Also, for every $s \in \mathcal{S}$ and $g \in \mathfrak{h}^*$, we have

$$(\alpha_s, \alpha_s) \, g \, (\alpha_s^\vee) = 2 \, (\alpha_s, g) . \tag{17}$$

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[14] *Proof.* Let $s \in \mathcal{S}$ and $g \in \mathfrak{h}^*$. Then, (15) yields $\dfrac{1}{2} \, (\alpha_s, \alpha_s) \, \alpha_s^\vee = \sum\limits_{i=1}^{r} \alpha_s \, (y_i) \, y_i$, so that $(\alpha_s, \alpha_s) \, \alpha_s^\vee = 2 \sum\limits_{i=1}^{r} \alpha_s \, (y_i) \, y_i$, and thus

$$g \left((\alpha_s, \alpha_s) \, \alpha_s^\vee \right) = g \left(2 \sum_{i=1}^{r} \alpha_s \, (y_i) \, y_i \right) = 2 \underbrace{\sum_{i=1}^{r} \alpha_s \, (y_i) \, g \, (y_i)}_{\substack{= (\alpha_s, g) \\ \text{(since } \{y_1, y_2, \dots, y_r\} \text{ is an} \\ \text{orthonormal basis of } \mathfrak{h})} } = 2 \, (\alpha_s, g) .$$

Since $g \left((\alpha_s, \alpha_s) \, \alpha_s^\vee \right) = (\alpha_s, \alpha_s) \, g \, (\alpha_s^\vee)$, this rewrites as $(\alpha_s, \alpha_s) \, g \, (\alpha_s^\vee) = 2 \, (\alpha_s, g)$. This proves (17).

On the other hand, from $\Delta_{\mathfrak{h}} = \sum\limits_{i=1}^{r} \partial_{y_i}^2$, we obtain

$$\varsigma_c\left(\Delta_{\mathfrak{h}}\right) = \varsigma_c\left(\sum_{i=1}^{r} \partial_{y_i}^2\right) = \sum_{i=1}^{r}\left(\underbrace{\varsigma_c\left(\partial_{y_i}\right)}_{\substack{=\partial_{y_i}+\sum\limits_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\\ \text{(by (10), applied to } a=y_i\text{)}}}\right)^2$$

(since ς_c is a $\mathbb{C}$-algebra homomorphism)

$$= \sum_{i=1}^{r} \underbrace{\left(\partial_{y_i} + \sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\right)^2}_{=\partial_{y_i}^2 + \partial_{y_i}\sum\limits_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s} + \sum\limits_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\partial_{y_i} + \left(\sum\limits_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\right)^2}$$

$$= \sum_{i=1}^{r}\left(\partial_{y_i}^2 + \partial_{y_i}\sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s} + \sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\partial_{y_i} + \left(\sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\right)^2\right)$$

$$= \underbrace{\sum_{i=1}^{r}\partial_{y_i}^2}_{=\Delta_{\mathfrak{h}}} + \underbrace{\sum_{i=1}^{r}\partial_{y_i}\sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}}_{=\sum\limits_{s\in\mathcal{S}}c_s\sum\limits_{i=1}^{r}\alpha_s(y_i)\partial_{y_i}\frac{1}{\alpha_s}}$$

$$+ \underbrace{\sum_{i=1}^{r}\sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\partial_{y_i}}_{=\sum\limits_{s\in\mathcal{S}}c_s\sum\limits_{i=1}^{r}\alpha_s(y_i)\frac{1}{\alpha_s}\partial_{y_i}} + \sum_{i=1}^{r}\underbrace{\left(\sum_{s\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\right)^2}_{=\sum\limits_{s\in\mathcal{S};\ u\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\cdot\frac{c_u\alpha_u\left(y_i\right)}{\alpha_u}}$$

$$= \Delta_{\mathfrak{h}} + \sum_{s\in\mathcal{S}}c_s\sum_{i=1}^{r}\alpha_s\left(y_i\right)\underbrace{\partial_{y_i}\frac{1}{\alpha_s}}_{=\frac{1}{\alpha_s}\partial_{y_i}+\partial_{y_i}\left(\frac{1}{\alpha_s}\right)}$$

$$+ \sum_{s\in\mathcal{S}}c_s\sum_{i=1}^{r}\alpha_s\left(y_i\right)\frac{1}{\alpha_s}\partial_{y_i} + \underbrace{\sum_{i=1}^{r}\sum_{s\in\mathcal{S};\ u\in\mathcal{S}}\frac{c_s\alpha_s\left(y_i\right)}{\alpha_s}\cdot\frac{c_u\alpha_u\left(y_i\right)}{\alpha_u}}_{=\sum\limits_{s\in\mathcal{S};\ u\in\mathcal{S}}\frac{c_sc_u}{\alpha_s\alpha_u}\sum\limits_{i=1}^{r}\alpha_s(y_i)\alpha_u(y_i)}$$

$$= \Delta_{\mathfrak{h}} + \underbrace{\sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \left(\frac{1}{\alpha_s}\partial_{y_i} + \partial_{y_i}\left(\frac{1}{\alpha_s}\right)\right)}_{= \sum\limits_{s \in \mathcal{S}} c_s \sum\limits_{i=1}^{r} \alpha_s(y_i)\frac{1}{\alpha_s}\partial_{y_i} + \sum\limits_{s \in \mathcal{S}} c_s \sum\limits_{i=1}^{r} \alpha_s(y_i)\partial_{y_i}\left(\frac{1}{\alpha_s}\right)}$$

$$+ \sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \frac{1}{\alpha_s}\partial_{y_i} + \sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s c_u}{\alpha_s \alpha_u} \underbrace{\sum_{i=1}^{r} \alpha_s \left(y_i\right) \alpha_u \left(y_i\right)}_{\substack{=(\alpha_s, \alpha_u) \\ \text{(since } \{y_1, y_2, \ldots, y_r\} \text{ is an orthonormal} \\ \text{basis of } \mathfrak{h})}}$$

$$= \Delta_{\mathfrak{h}} + \sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \frac{1}{\alpha_s}\partial_{y_i} + \sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \partial_{y_i}\left(\frac{1}{\alpha_s}\right)$$

$$+ \sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \frac{1}{\alpha_s}\partial_{y_i} + \sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s c_u}{\alpha_s \alpha_u} \left(\alpha_s, \alpha_u\right)$$

$$= \Delta_{\mathfrak{h}} + 2\sum_{s \in \mathcal{S}} c_s \underbrace{\sum_{i=1}^{r} \alpha_s \left(y_i\right) \frac{1}{\alpha_s}\partial_{y_i}}_{= \frac{1}{\alpha_s} \sum\limits_{i=1}^{r} \alpha_s(y_i)\partial_{y_i}} + \sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \underbrace{\partial_{y_i}\left(\frac{1}{\alpha_s}\right)}_{\substack{= -\frac{\partial_{y_i}\left(\alpha_s\right)}{\alpha_s^2} = -\frac{\alpha_s \left(y_i\right)}{\alpha_s^2} \\ \text{(since } \partial_{y_i}(\alpha_s) = \alpha_s(y_i) \text{ (because } \alpha_s \\ \text{is linear))}}}$$

$$+ \underbrace{\sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s c_u}{\alpha_s \alpha_u} \left(\alpha_s, \alpha_u\right)}_{\substack{= \sum\limits_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s c_u \left(\alpha_s, \alpha_u\right)}{\alpha_s \alpha_u} = \sum\limits_{s \in \mathcal{S}} \frac{c_s^2 \left(\alpha_s, \alpha_s\right)}{\alpha_s^2} \\ \text{(by Lemma 2.18e (c))}}}$$

$$= \Delta_{\mathfrak{h}} + 2\sum_{s \in \mathcal{S}} c_s \frac{1}{\alpha_s} \underbrace{\sum_{i=1}^{r} \alpha_s \left(y_i\right) \partial_{y_i}}_{\substack{= \frac{1}{2}(\alpha_s, \alpha_s)\partial_{\alpha_s^\vee} \\ \text{(by (16))}}} + \underbrace{\sum_{s \in \mathcal{S}} c_s \sum_{i=1}^{r} \alpha_s \left(y_i\right) \left(-\frac{\alpha_s \left(y_i\right)}{\alpha_s^2}\right)}_{= -\sum\limits_{s \in \mathcal{S}} \frac{c_s}{\alpha_s^2} \sum\limits_{i=1}^{r} (\alpha_s(y_i))^2} + \sum_{s \in \mathcal{S}} \frac{c_s^2 \left(\alpha_s, \alpha_s\right)}{\alpha_s^2}$$

$$= \Delta_{\mathfrak{h}} + 2 \underbrace{\sum_{s \in \mathcal{S}} c_s \frac{1}{\alpha_s} \cdot \frac{1}{2} (\alpha_s, \alpha_s) \partial_{\alpha_s^{\vee}}}_{= \sum_{s \in \mathcal{S}} \frac{c_s (\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^{\vee}}} - \sum_{s \in \mathcal{S}} \frac{c_s}{\alpha_s^2} \underbrace{\sum_{i=1}^{r} (\alpha_s (y_i))^2}_{\substack{= (\alpha_s, \alpha_s) \\ \text{(since } \{y_1, y_2, \ldots, y_r\} \text{ is an} \\ \text{orthonormal basis of } \mathfrak{h})}} + \sum_{s \in \mathcal{S}} \frac{c_s^2 (\alpha_s, \alpha_s)}{\alpha_s^2}$$

$$= \Delta_{\mathfrak{h}} + \sum_{s \in \mathcal{S}} \frac{c_s (\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^{\vee}} - \sum_{s \in \mathcal{S}} \underbrace{\frac{c_s}{\alpha_s^2} (\alpha_s, \alpha_s)}_{= \frac{c_s (\alpha_s, \alpha_s)}{\alpha_s^2}} + \sum_{s \in \mathcal{S}} \frac{c_s^2 (\alpha_s, \alpha_s)}{\alpha_s^2}$$

$$= \Delta_{\mathfrak{h}} + \sum_{s \in \mathcal{S}} \frac{c_s (\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^{\vee}} - \sum_{s \in \mathcal{S}} \frac{c_s (\alpha_s, \alpha_s)}{\alpha_s^2} + \sum_{s \in \mathcal{S}} \frac{c_s^2 (\alpha_s, \alpha_s)}{\alpha_s^2}. \tag{18}$$

On the other hand, every $t \in \mathcal{S}$ satisfies

$$\varsigma_c (\partial_{\alpha_t^{\vee}}) = \partial_{a_t^{\vee}} + \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s (\alpha_t^{\vee})}{\alpha_s} \qquad \text{(by (10), applied to } a = \alpha_t^{\vee})$$

$$= \partial_{a_t^{\vee}} + \sum_{u \in \mathcal{S}} \frac{c_u \alpha_u (\alpha_t^{\vee})}{\alpha_u} \qquad \text{(here, we renamed } s \text{ as } u \text{ in the sum)}.$$

$$\tag{19}$$

Now,

$$\varsigma_c \left(\sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{\alpha_s^\vee} \right)$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \underbrace{\varsigma_c \left(\partial_{\alpha_s^\vee} \right)}_{\substack{= \partial_{a_s^\vee} + \sum_{u \in \mathcal{S}} \frac{c_u \alpha_u \left(\alpha_s^\vee \right)}{\alpha_u} \\ \text{(by (19), applied to } t=s\text{)}}} = \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \left(\partial_{a_s^\vee} + \sum_{u \in \mathcal{S}} \frac{c_u \alpha_u \left(\alpha_s^\vee \right)}{\alpha_u} \right)$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{a_s^\vee} + \underbrace{\sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \sum_{u \in \mathcal{S}} \frac{c_u \alpha_u \left(\alpha_s^\vee \right)}{\alpha_u}}_{= \sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \cdot \frac{c_u \alpha_u \left(\alpha_s^\vee \right)}{\alpha_u}}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{a_s^\vee} + \sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \underbrace{\frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \cdot \frac{c_u \alpha_u \left(\alpha_s^\vee \right)}{\alpha_u}}_{= \frac{c_s c_u}{\alpha_s \alpha_u} (\alpha_s, \alpha_s) \alpha_u (\alpha_s^\vee)}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{a_s^\vee} + \sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s c_u}{\alpha_s \alpha_u} \underbrace{\left(\alpha_s, \alpha_s \right) \alpha_u \left(\alpha_s^\vee \right)}_{\substack{= 2(\alpha_s, \alpha_u) \\ \text{(by (17), applied to } g=\alpha_u\text{)}}}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{a_s^\vee} + 2 \sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \underbrace{\frac{c_s c_u}{\alpha_s \alpha_u} \left(\alpha_s, \alpha_u \right)}_{= \frac{c_s c_u \left(\alpha_s, \alpha_u \right)}{\alpha_s \alpha_u}}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{a_s^\vee} + 2 \underbrace{\sum_{s \in \mathcal{S};\ u \in \mathcal{S}} \frac{c_s c_u \left(\alpha_s, \alpha_u \right)}{\alpha_s \alpha_u}}_{\substack{= \sum_{s \in \mathcal{S}} \frac{c_s^2 \left(\alpha_s, \alpha_s \right)}{\alpha_s^2} \\ \text{(by Lemma 2.18e (c))}}}$$

$$= \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s \right)}{\alpha_s} \partial_{a_s^\vee} + 2 \sum_{s \in \mathcal{S}} \frac{c_s^2 \left(\alpha_s, \alpha_s \right)}{\alpha_s^2}. \tag{20}$$

Now, $\overline{H} = \Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s \left(\alpha_s, \alpha_s\right)}{\alpha_s} \partial_{\alpha_s^\vee}$, so that

$$\varsigma_c\left(\overline{H}\right) = \varsigma_c\left(\Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s} \partial_{\alpha_s^\vee}\right) = \varsigma_c\left(\Delta_{\mathfrak{h}}\right) - \varsigma_c\left(\sum_{s \in \mathcal{S}} \frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s} \partial_{\alpha_s^\vee}\right)$$

$$= \left(\Delta_{\mathfrak{h}} + \sum_{s \in \mathcal{S}} \frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s} \partial_{\alpha_s^\vee} - \sum_{s \in \mathcal{S}} \frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s^2} + \sum_{s \in \mathcal{S}} \frac{c_s^2\left(\alpha_s, \alpha_s\right)}{\alpha_s^2}\right)$$

$$- \left(\sum_{s \in \mathcal{S}} \frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s} \partial_{a_s^\vee} + 2\sum_{s \in \mathcal{S}} \frac{c_s^2\left(\alpha_s, \alpha_s\right)}{\alpha_s^2}\right)$$

(by (18) and (20))

$$= \Delta_{\mathfrak{h}} - \underbrace{\left(\sum_{s \in \mathcal{S}} \frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s^2} + \sum_{s \in \mathcal{S}} \frac{c_s^2\left(\alpha_s, \alpha_s\right)}{\alpha_s^2}\right)}_{= \sum_{s \in \mathcal{S}}\left(\frac{c_s\left(\alpha_s, \alpha_s\right)}{\alpha_s^2} + \frac{c_s^2\left(\alpha_s, \alpha_s\right)}{\alpha_s^2}\right) = \sum_{s \in \mathcal{S}} \frac{c_s\left(c_s + 1\right)\left(\alpha_s, \alpha_s\right)}{\alpha_s^2}}$$

$$= \Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s\left(c_s + 1\right)\left(\alpha_s, \alpha_s\right)}{\alpha_s^2} = H.$$

This proves Proposition 2.18d.

Proof of Theorem 2.9. The $\mathbb{C}$-algebra homomorphism $\varsigma_c : \mathcal{D}\left(\mathfrak{h}_{\text{reg}}\right) \to \mathcal{D}\left(\mathfrak{h}_{\text{reg}}\right)$ preserves the degree of homogeneous differential operators and their symbols and commutes with the action of W by conjugation (these facts all are easy to prove), and maps $\overline{H}$ to H (by Proposition 2.18d). Hence, applying ς_c to Corollary 2.17, we obtain Theorem 2.9 (at least, if we add to Corollary 2.17 the claim that the $\overline{L}_j$ are W-invariant, as I suggested above), with the L_j being given by $L_j = \varsigma_c\left(\overline{L}_j\right)$.

Theorem 2.9 is thus proven, and with it Theorem 2.1.

- **Page 10, Remark 2.20:** It would be better to replace "L_i" by "$\overline{L}_i$" here.

... [To be continued?]

- **Page 12, Example 2.25:** In this example, you regard $\mathfrak{h}$ as being embedded into $\mathbb{C}^n$ as the subspace consisting of the vectors whose coordinates sum to zero. (This is the same embedding as in Example 2.5.) The p_i are the same as in Example 2.5. The x_i (for each $i \in \{1, 2, \ldots, n\}$) is the linear map sending each element of $\mathfrak{h}$ to its i-th coordinate. This all is worth pointing out explicitly, since it is far from obvious.