Function-field symmetric functions: In search of an $\mathbb{F}_q[T]$-combinatorics

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slides:
http://www.cip.ifi.lmu.de/~grinberg/algebra/caac17.pdf
preprint (WIP, and currently a mess):
http://www.cip.ifi.lmu.de/~grinberg/algebra/schur-ore.pdf

- The connection between symmetric functions and (big) Witt vectors is due to Cartier around 1970 (vaguely; made explicit by Reutenauer in 1995), and can be used to the benefit of either.
- Modern references: e.g., Hazewinkel's *Witt vectors, part 1* (arXiv:0804.3888v1, see also errata), and works of James Borger (mainly arXiv:0801.1691v6, as well as arXiv:math/0407227v1 joint with Wieland).

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- Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. The *(big) Witt vector functor* is a functor W : **CRing** $\to$ **CRing**, sending any commutative ring A to a new commutative ring $W(A)$ with some extra structure.
- Note that $W(A)$ is a ring, not an A-algebra.

Definition of Witt vectors, 1: ghost maps

- Let A be a commutative ring.
  We abbreviate a family $(a_k)_{k \in \mathbb{N}_+} \in A^{\mathbb{N}_+}$ as $\mathbf{a}$. Similarly for other letters.
- For each $n \in \mathbb{N}_+$, define a map $w_n : A^{\mathbb{N}_+} \to A$ by

$$w_n(\mathbf{a}) = \sum_{d|n} d a_d^{n/d}.$$

  The map w_n is called the *n-th ghost projection*.
- **Examples:**
  - $w_1 = a_1$.
  - If p is a prime, then $w_p = a_1^p + p a_p$.
  - $w_6 = a_1^6 + 2a_2^3 + 3a_3^2 + 6a_6$.

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- Let $w : A^{\mathbb{N}_+} \to A^{\mathbb{N}_+}$ be the map given by

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  We call w the *ghost map*.
- This ghost map w is not linear and in general not injective or surjective. **However**, its image turns out to be a subring of $A^{\mathbb{N}_+}$. It is called the *ring of ghost-Witt vectors*.

Definition of Witt vectors, 2: addition

- For example, for any $\mathbf{a}, \mathbf{b} \in A^{\mathbb{N}_+}$, we have $w(\mathbf{a}) + w(\mathbf{b}) = w(\mathbf{c})$ for some $\mathbf{c} \in A^{\mathbb{N}_+}$. How to compute this $\mathbf{c}$?

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- Good news:
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- Hence, we can compute $\mathbf{c}$ back from $w(\mathbf{c})$ by recursion (coordinate by coordinate). Miraculously, the denominators vanish.

  Examples:
  - $w_1(\mathbf{c}) = w_1(\mathbf{a}) + w_1(\mathbf{b}) \iff c_1 = a_1 + b_1$.
  - $w_2(\mathbf{c}) = w_2(\mathbf{a}) + w_2(\mathbf{b}) \iff$ $c_1^2 + 2c_2 = (a_1^2 + 2a_2) + (b_1^2 + 2b_2) \overset{\text{naturality}}{\iff}$ $c_2 = a_2 + b_2 + \frac{1}{2}\left(a_1^2 + b_1^2 - (a_1 + b_1)^2\right)$, and the RHS is indeed a $\mathbb{Z}$-polynomial.

- Let's make a new ring out of this: We define $W(A)$ to be the ring that equals $A^{\mathbb{N}_+}$ as a set, but whose ring structure is such that $W : \mathbf{CRing} \to \mathbf{CRing}$ is a functor, and w is a natural (in A) ring homomorphism from $W(A)$ to $A^{\mathbb{N}_+}$.

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- This looks abstract and confusing, but the underlying idea is simple: Define addition on $W(A)$ so that
  $w_n(\mathbf{a} + \mathbf{b}) = w_n(\mathbf{a}) + w_n(\mathbf{b})$ for all n.
  Thus, $\mathbf{a} + \mathbf{b}$ is the $\mathbf{c}$ from last page.
- Functoriality is needed, because there might be several choices for a given A (if A is not torsionfree), but only one consistent choice for all rings A. Functoriality forces us to pick the consistent choice.

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- Functoriality is needed, because there might be several choices for a given A (if A is not torsionfree), but only one consistent choice for all rings A. Functoriality forces us to pick the consistent choice.
- If $\mathbf{a} \in W(A)$, then the a_n are called the *Witt coordinates* of $\mathbf{a}$, while the $w_n(\mathbf{a})$ are called the *ghost coordinates* of $\mathbf{a}$.

- The ring $W(A)$ is called the *ring of (big) Witt vectors over* A.
- The functor $\mathbf{CRing} \to \mathbf{CRing}$, $A \mapsto W(A)$ is called the *(big) Witt vector functor*.

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- For any given prime p, there is a canonical quotient $W_p(A)$ of $W(A)$ called the *ring of p-typical Witt vectors of A*. Number theorists usually care about the latter ring. For example, $W_p(\mathbb{F}_p) = \mathbb{Z}_p$ (the p-adics). We have nothing to say about it here.
- $W(A)$ comes with more structure: Frobenius and Verschiebung endomorphisms, a comonad comultiplication map $W(A) \to W(W(A))$, etc.

- There are some equivalent ways to define $W(A)$. Let me show two.
- One is the Grothendieck construction using power series (see, again, Hazewinkel, or Rabinoff's arXiv:1409.7445):
- Let $\Lambda(A)$ be the topological ring defined as follows:
  - As topological spaces, $\Lambda(A) = 1 + tA[[t]] = \{\text{power series with constant term } 1\}$.
  - Addition $\widehat{+}$ in $\Lambda(A)$ is multiplication of power series.
  - Multiplication $\widehat{\cdot}$ in $\Lambda(A)$ is given by

$$(1 - at)\widehat{\cdot}(1 - bt) = 1 - abt$$

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- Canonical ring isomorphism

$$W(A) \to \Lambda(A), \qquad \mathbf{a} \mapsto \prod_{n=1}^{\infty} (1 - a_n t^n) .$$

- Here is another: Let Λ be the Hopf algebra of symmetric functions over $\mathbb{Z}$. (No direct relation to $\Lambda(A)$; just traditional notations clashing.)
- Define ring $\operatorname{Alg}(\Lambda, A)$ as follows:
  - As set, $\operatorname{Alg}(\Lambda, A) = \{\text{algebra homomorphisms } \Lambda \to A\}$.
  - Addition = convolution.
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- The elements of $\operatorname{Alg}(\Lambda, A)$ are known as *characters* of Λ (as in Aguiar-Bergeron-Sottile) or *virtual alphabets* (to the Lascoux school) or as *specializations of symmetric functions* (as in Stanley's EC2).

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  - As set, $\mathrm{Alg}(\Lambda, A) = \{\text{algebra homomorphisms } \Lambda \to A\}$.
  - Addition $=$ convolution.
  - Multiplication $=$ convolution using the *second comultiplication* on Λ ($=$ Kronecker comultiplication $=$ Hall dual of Kronecker multiplication).
- There is a unique family $(w_n)_{n\in\mathbb{N}_+}$ of symmetric functions satisfying $p_n = \sum_{d|n} d w_d^{n/d}$ for all $n \in \mathbb{N}_+$. (Equivalently, it is determined by $h_n = \sum_{\lambda \vdash n} w_\lambda$, where $w_\lambda = w_{\lambda_1} w_{\lambda_2} \cdots$.) These are called the *Witt coordinates*.
- We have a ring isomorphism

$$\mathrm{Alg}(\Lambda, A) \to W(A), \qquad f \mapsto (f(w_n))_{n\in\mathbb{N}_+}.$$

Avatars of Witt vectors, 2: Characters of Λ, cont'd

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- We have a ring isomorphism

$$\operatorname{Alg}(\Lambda, A) \to W(A), \qquad f \mapsto (f(w_n))_{n\in\mathbb{N}_+}.$$

- We also have a ring homomorphism (isomorphism when A is a $\mathbb{Q}$-algebra)

$$\operatorname{Alg}(\Lambda, A) \to A^{\mathbb{N}_+}, \qquad f \mapsto (f(p_n))_{n\in\mathbb{N}_+}.$$

These form commutative diagram

$$\begin{array}{ccc} \operatorname{Alg}(\Lambda, A) & \xrightarrow{\cong} & W(A) \\ & \searrow & \downarrow w \\ & & A^{\mathbb{N}_+} \end{array}$$

- This also works in reverse: We can reconstruct Λ from the functor W, as its representing object. Namely:
  - The functor $\mathrm{Forget} \circ W : \mathbf{CRing} \to \mathbf{Set}$ determines Λ as a ring (by Yoneda).
  - The functor $\mathrm{Forget} \circ W : \mathbf{CRing} \to \mathbf{Ab}$ (additive group of $W(A)$) determines Λ as a Hopf algebra.
  - The functor $W : \mathbf{CRing} \to \mathbf{CRing}$ determines Λ as a Hopf algebra equipped with a second comultiplication.
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  - The functor $W : \mathbf{CRing} \to \mathbf{CRing}$ determines Λ as a Hopf algebra equipped with a second comultiplication.
  - The comonad structure on W additionally determines plethysm on Λ.
- Thus, if symmetric functions hadn't been around, Witt vectors would have let us rediscover them.

$\mathbb{Z}$ and $\mathbb{F}_q[T]$: a tale of two rings

Now to something completely different...

- Fix a prime power q.
- There is a famous analogy between the elements of $\mathbb{Z}$ and the elements of $\mathbb{F}_q[T]$. (This is related to q-enumeration, the lore of the field with 1 element, etc.)
  All that matters to us is that
  - **positive** integers in $\mathbb{Z}$ correspond to **monic** polynomials in $\mathbb{F}_q[T]$;
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- Let $\mathbb{F}_q[T]_+$ be the set of all **monic** polynomials in $\mathbb{F}_q[T]$.
- Let's define an analogue of (big) Witt vectors for $\mathbb{F}_q[T]$ instead of $\mathbb{Z}$.

- Let A be a commutative $\mathbb{F}_q[T]$-algebra.
  We abbreviate a family $(a_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ as $\mathbf{a}$.
- For each $N \in \mathbb{F}_q[T]_+$, define a map $w_N : A^{\mathbb{F}_q[T]_+} \to A$ by

$$w_N(\mathbf{a}) = \sum_{D \mid N} D a_D^{q^{\deg(N/D)}},$$

  where the sum is over all **monic** divisors D of N.
- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

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Definition of $\mathbb{F}_q[T]$-Witt vectors, 1: ghost maps

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- This “ghost map” w is $\mathbb{F}_q$-linear, but not $\mathbb{F}_q[T]$-linear.

Definition of $\mathbb{F}_q[T]$-Witt vectors, 2: $W_q(A)$

- Let's make a new $\mathbb{F}_q[T]$-algebra out of this: We define $W_q(A)$ to be the $\mathbb{F}_q[T]$-algebra
  - that equals $A^{\mathbb{F}_q[T]_+}$ as a set, but
  - which is functorial in A (that is, we are really defining a functor $W_q : \mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{CRing}_{\mathbb{F}_q[T]}$, where $\mathbf{CRing}_R$ is the category of commutative R-algebras), and
  - whose $\mathbb{F}_q[T]$-algebra structure is such that w is a natural (in A) homomorphism of $\mathbb{F}_q[T]$-algebras from $W_q(A)$ to $A^{\mathbb{F}_q[T]_+}$.

Definition of $\mathbb{F}_q[T]$-Witt vectors, 2: $W_q(A)$, cont'd

- **Example:** The addition in $W_q(A)$ is the same as in $A^{\mathbb{F}_q[T]_+}$ (since w is $\mathbb{F}_q$-linear, and so $W_q(A) = A^{\mathbb{F}_q[T]_+}$ as $\mathbb{F}_q$-modules), so this would be boring.

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  Start with $c_1 = Ta_1$, which is easy to check.
  $w_\pi(\mathbf{c}) = Tw_\pi(\mathbf{a})$
  $\iff c_1^{q^{\deg \pi}} + \pi c_\pi = Ta_1^{q^{\deg \pi}} + T\pi a_\pi$
  $\overset{c_1 = Ta_1}{\iff} (Ta_1)^{q^{\deg \pi}} + \pi c_\pi = Ta_1^{q^{\deg \pi}} + T\pi a_\pi$
  $\iff \pi c_\pi = T\pi a_\pi - \left(T^{q^{\deg \pi}} - T\right) a_1^{q^{\deg \pi}}$
  $\overset{\text{naturality}}{\iff} c_\pi = Ta_\pi - \dfrac{T^{q^{\deg \pi}} - T}{\pi} a_1^{q^{\deg \pi}}.$
  The fraction on the RHS is a polynomial due to a known fact from Galois theory (namely:
  $T^{q^k} - T = \prod_{\gamma \in \mathbb{F}_q[T]_+ \text{ irreducible; } \deg \gamma | k} \gamma$).

- There is also a second construction of $W_q(A)$, using Carlitz polynomials, yielding an isomorphic $\mathbb{F}_q[T]$-algebra. (See the preprint.)

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- Power series? This appears to require a notion of power series where the exponents are polynomials in $\mathbb{F}_q[T]$. Product ill-defined due to lack of actual "positivity". Seems too much to wish...
- $\mathrm{Alg}(\Lambda, A)$? Well, we can try brute force: Remember how Λ was reconstructed from W, and do something similar to "reconstruct" a representing object from W_q. We'll come back to this shortly.

Surprise: $\mathcal{F}$-modules, 1

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- First, a surprise...
- We aren't using the whole $\mathbb{F}_q[T]$-algebra structure on A ! (This is unlike the $\mathbb{Z}$-case, where it seems that we use the commutative ring A in full.)

Surprise: $\mathcal{F}$-modules, 2

- Let $\mathcal{F}$ be the noncommutative ring

$$\mathbb{F}_q \langle F, T \mid FT = T^q F \rangle .$$

This is an $\mathbb{F}_q$-vector space with basis $\left(T^i F^j\right)_{(i,j)\in\mathbb{N}^2}$, and is an Ore polynomial ring. It shares many properties of usual univariate polynomials (see papers of Ore).

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- Actually,

$$\mathcal{F} \cong \left(\mathbb{F}_q [T] [X]_{q-\text{lin}} , +, \circ\right) ,$$

  where $\mathbb{F}_q [T] [X]_{q-\text{lin}}$ are the polynomials in X over $\mathbb{F}_q [T]$ where X occurs only with exponents q^k, and where $\circ$ is composition of polynomials.

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- What matters to us:
  Each commutative $\mathbb{F}_q[T]$-algebra canonically becomes a (left) $\mathcal{F}$-module by having
  - T act as multiplication by T, and
  - F act as the Frobenius (i.e., taking q-th powers).

  Thus, we have a functor $\mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{Mod}_{\mathcal{F}}$.

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  Thus, we have a functor $\mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{Mod}_{\mathcal{F}}$.
- There are other sources of $\mathcal{F}$-modules too (cf. Jacobson on "commutative restricted Lie algebras").

- Let A be a (left) $\mathcal{F}$-module.
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- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

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$\mathbb{F}_q[T]$-Witt vectors of an $\mathcal{F}$-module

- Let A be a (left) $\mathcal{F}$-module.
  We abbreviate a family $(a_N)_{N \in \mathbb{F}_q[T]_+} \in A^{\mathbb{F}_q[T]_+}$ as $\mathbf{a}$.
- For each $N \in \mathbb{F}_q[T]_+$, define a map $w_N : A^{\mathbb{F}_q[T]_+} \to A$ by

$$w_N(\mathbf{a}) = \sum_{D \mid N} D F^{\deg(N/D)} a_D,$$

  where the sum is over all **monic** divisors D of N.
- Let $w : A^{\mathbb{F}_q[T]_+} \to A^{\mathbb{F}_q[T]_+}$ be the map given by

$$w(\mathbf{a}) = (w_N(\mathbf{a}))_{N \in \mathbb{F}_q[T]_+}.$$

- We define $W_q(A)$ to be the $\mathcal{F}$-module
  - that equals $A^{\mathbb{F}_q[T]_+}$ as a set, but
  - which is functorial in A (that is, we are really defining a functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$), and
  - whose $\mathcal{F}$-module structure is such that w is a natural (in A) homomorphism of $\mathcal{F}$-modules from $W_q(A)$ to $A^{\mathbb{F}_q[T]_+}$.

- Now, back to the question: We have found two functors

$$W_q : \mathbf{CRing}_{\mathbb{F}_q[T]} \to \mathbf{CRing}_{\mathbb{F}_q[T]} \qquad \text{and}$$
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What are their representing objects? Call them $\Lambda'_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$.

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- I shall focus on $\Lambda_{\mathcal{F}}$, since it is smaller.

Tinfoil, 2: "Re"constructing $\Lambda_{\mathcal{F}}$

- Proceed in the same way as when we reconstructed Λ from the functor W, but now reconstruct the representing object $\Lambda_{\mathcal{F}}$ of the functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$:
  - The functor $\mathrm{Forget} \circ W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Set}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-module (by Yoneda).
  - The functor $W_q : \mathbf{Mod}_{\mathcal{F}} \to \mathbf{Mod}_{\mathcal{F}}$ determines $\Lambda_{\mathcal{F}}$ as an $\mathcal{F}$-$\mathcal{F}$-bimodule.
  - There is an additional comonad structure on W_q, which determines a "plethysm" on $\Lambda_{\mathcal{F}}$, but I know nothing about it.

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- So what is this $\Lambda_{\mathcal{F}}$?
- I don't really know.

Tinfoil, 3: Some computations inside $\Lambda_{\mathcal{F}}$

- At least, we can compute in $\Lambda_{\mathcal{F}}$ (in theory):
- The left $\mathcal{F}$-module $\Lambda_{\mathcal{F}}$ has a basis $(w_N)_{N \in \mathbb{F}_q[T]_+}$, similarly to the generating set $(w_n)_{n \in \mathbb{N}_+}$ of the commutative ring Λ.
- The left $\mathcal{F}$-module $\Lambda_{\mathcal{F}}$ has an "almost-basis" $(p_N)_{N \in \mathbb{F}_q[T]_+}$, similarly to the "almost-generating set" $(p_n)_{n \in \mathbb{N}_+}$ of the commutative ring Λ.

  Here, "almost-basis" means "basis after localizing so that elements of $\mathbb{F}_q\left[T\right]_+$ become invertible". (Noncommutative localization, but a harmless case thereof.)

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  Here, "almost-basis" means "basis after localizing so that elements of $\mathbb{F}_q[T]_+$ become invertible". (Noncommutative localization, but a harmless case thereof.)
- The right $\mathcal{F}$-module structure is easily expressed on the p_N's (just as the second comultiplication of Λ is easily expressed on the p_n's):

$$p_N f = f p_N \qquad \text{for all } f \in \mathcal{F} \text{ and } N \in \mathbb{F}_q[T]_+ .$$

- You can thus express qf for each $f \in \mathcal{F}$ and $q \in \Lambda_{\mathcal{F}}$ by recursion (all fractions will turn out polynomial at the end), but nothing really explicit.

- Here are some of these expressions:

$$w_\pi T = T w_\pi - \frac{T^{q^{\deg \pi}} - T}{\pi} F^{\deg \pi} w_1;$$
$$w_\pi F = \pi^{q-1} F w_\pi$$

for any irreducible $\pi \in \mathbb{F}_q [T]_+$.

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And thank you!