A counterexample to the Burman–Kulishov conjecture on Lie elements

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Abstract

Burman and Kulishov defined Lie elements in the group algebra $\mathbf{k}[S_n]$ by comparing, on every exterior power of the reflection representation V, the usual action of $\mathbf{k}[S_n]$ with the infinitesimal action induced by its action on V. They conjectured that the Lie algebra $\mathcal{L}_n$ of all Lie elements is generated by the Kirchhoff differences $1 - (ij)$. We disprove this conjecture for $n = 4$ by exhibiting an explicit counterexample arising from the $(2, 2)$-block of $\mathbf{k}[S_4]$. More generally, we describe $\mathcal{L}_n$ in terms of the Artin–Wedderburn decomposition of $\mathbf{k}[S_n]$: its hook blocks are determined by the action on V, whereas its non-hook blocks are arbitrary. Consequently, the primitive central idempotents of the non-hook blocks yield linearly independent obstructions to the conjecture. We also identify the Lie algebra generated by the Kirchhoff differences in terms of the derived algebra of the Lie algebra generated by transpositions. Along the way, we give an integral, and hence characteristic-free, proof that the exterior powers of V are the hook-shaped Specht modules.

Manifest. This is a negative answer to [BurKul23, Conjecture 2.2], generated by GPT-5.6 Sol (Plus). It includes a proof of Proposition 2 for its expository value. The output was edited and checked by me.[1]

1 The conjecture

Let $\mathbf{k}$ be a field of characteristic 0, let $n \geq 2$, and let V be the reflection representation of the symmetric group S_n. Thus,

$$V = \{(x_1, \ldots, x_n) \in \mathbf{k}^n \mid x_1 + \cdots + x_n = 0\};$$

the group S_n acts on this by permuting the coordinates. For every $m \in \mathbb{N}$, Burman and Kulishov (in [BurKul23]) define $\mathbf{k}$-linear maps

$$\mathcal{G}_m, \mathcal{A}_m : \mathbf{k}[S_n] \longrightarrow \operatorname{End}_{\mathbf{k}} (\Lambda^m V)$$

by extending the following rules linearly from S_n to $\mathbf{k}[S_n]$:

$$\mathcal{G}_m(g)(v_1 \wedge \cdots \wedge v_m) = g(v_1) \wedge \cdots \wedge g(v_m),$$
$$\mathcal{A}_m(g)(v_1 \wedge \cdots \wedge v_m) = \sum_{i=1}^{m} \underbrace{v_1 \wedge \cdots \wedge g(v_i) \wedge \cdots \wedge v_m}_{\substack{\text{the } i\text{-th factor is } g(v_i);\\ \text{all other factors are the respective } v_j}}.$$

Here $\mathcal{G}_0(g) = \mathrm{id}_{\mathbf{k}}$ and $\mathcal{A}_0(g) = 0$. Note that $\mathcal{G}_m(a)$ (for any $a \in \mathbf{k}[S_n]$) is the action of a on the representation $\Lambda^m V$ of S_n.

An element $a \in \mathbf{k}[S_n]$ is called a *Lie element* if

$$\mathcal{G}_m(a) = \mathcal{A}_m(a) \qquad \text{for every } m \in \mathbb{N}. \tag{1}$$

It is enough to require (1) for $0 \leq m \leq n-1$, since all $m \geq n$ satisfy $\Lambda^m V = 0$. The Lie elements form a Lie subalgebra of $\mathbf{k}[S_n]$ (see [BurKul23, Corollary 1.6]); we denote this Lie subalgebra by $\mathcal{L}_n$.

For distinct $i, j \in [n]$, define the *Kirchhoff difference*

$$\varkappa_{ij} = 1 - (ij) \in \mathbf{k}[S_n],$$

where (ij) denotes the transposition in S_n that swaps i with j. We write

$$\mathfrak{h}_n = \operatorname{Lie} \langle \varkappa_{ij} \mid 1 \leq i < j \leq n \rangle \subseteq \mathbf{k}[S_n]$$

for the Lie algebra generated by the Kirchhoff differences. Burman and Kulishov proved that every $\varkappa_{ij}$ is a Lie element and conjectured that the Lie algebra $\mathcal{L}_n$ is generated by the $\varkappa_{ij}$, or equivalently that $\mathcal{L}_n = \mathfrak{h}_n$; see [BurKul23, Conjecture 2.2]. We shall show that this conjecture is false, first structurally and then by an explicit counterexample in $\mathbf{k}[S_4]$.

2 The structure of the Lie-element algebra

For $m \in \mathbb{N}$, define a linear map

$$D_m : \operatorname{End}_{\mathbf{k}}(V) \longrightarrow \operatorname{End}_{\mathbf{k}}\left(\Lambda^m V\right)$$

by

$$\left(D_m(X)\right)\left(v_1 \wedge \cdots \wedge v_m\right) = \sum_{i=1}^{m} v_1 \wedge \cdots \wedge X(v_i) \wedge \cdots \wedge v_m. \tag{2}$$

Thus, D_m is the differential at the identity of the m-th exterior-power representation $\mathrm{GL}(V) \to \mathrm{GL}(\Lambda^m V)$. Equivalently, D_m is the canonical representation of the Lie algebra $\mathfrak{gl}(V)$ on the m-th exterior power of its natural representation V. In particular, D_m is a Lie algebra morphism.

Lemma 1. *For every $a \in \mathbf{k}[S_n]$ and every $m \in \mathbb{N}$, one has*

$$\mathcal{A}_m(a) = D_m\left(\mathcal{G}_1(a)\right). \tag{3}$$

Proof. Write $a = \sum_{g \in S_n} a_g g$. For $v_1, \ldots, v_m \in V$, the definition of $\mathcal{A}_m$ gives

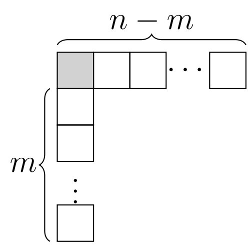
$$\begin{aligned}
\mathcal{A}_m(a)(v_1 \wedge \cdots \wedge v_m) &= \sum_{g \in S_n} a_g \sum_{i=1}^{m} v_1 \wedge \cdots \wedge g(v_i) \wedge \cdots \wedge v_m \\
&= \sum_{i=1}^{m} v_1 \wedge \cdots \wedge \left(\sum_{g \in S_n} a_g g\right)(v_i) \wedge \cdots \wedge v_m \\
&= \sum_{i=1}^{m} v_1 \wedge \cdots \wedge \left(\mathcal{G}_1(a)\right)(v_i) \wedge \cdots \wedge v_m \\
&= D_m\left(\mathcal{G}_1(a)\right)(v_1 \wedge \cdots \wedge v_m).
\end{aligned}$$

This proves (3). ◻

For $0 \le m \le n-1$, define the partition

$$h_m = (n-m, 1^m) \qquad \text{of } n.$$

Thus, $h_0 = (n)$ and $h_1 = (n-1, 1)$ and $h_{n-1} = (1, 1, \ldots, 1)$. The partitions $h_0, h_1, \ldots, h_{n-1}$ are called *hook partitions*, due to their shape:



$$h_m = (n-m, 1^m)$$

Every partition of n that is not one of $h_0, h_1, \dots, h_{n-1}$ will be called a *non-hook partition* (or, for short, a *non-hook*).

For any partition λ of n, let M^λ denote the *Young permutation module* of shape λ (the free $\mathbf{k}$-vector space with basis formed by the tabloids $\{T\}$ of shape λ). We let S^λ denote the corresponding Specht module, that is, the submodule of M^λ spanned by the polytabloids $\mathbf{e}_T$. The standard polytabloids $\mathbf{e}_T$ form a basis of S^λ. Since $\mathbf{k}$ has characteristic 0, the S_n-module S^λ is irreducible.

We record a standard fact (in a characteristic-free version, because we can)[2].

Proposition 2. *Let R be any commutative ring, and let*

$$V_R = \{(x_1, \dots, x_n) \in R^n \mid x_1 + \cdots + x_n = 0\}$$

be the reflection representation of S_n over R (where S_n acts by permuting the coordinates). For $0 \le m \le n-1$, let $S_R^{h_m}$ be the Specht module of shape h_m defined over R. Then, there is an $R[S_n]$-module isomorphism

$$\Lambda_R^m V_R \cong S_R^{h_m}. \tag{4}$$

In particular, this holds over every field, without any restriction on its characteristic.

Proof. We shall first construct the isomorphism over $\mathbb{Z}$ and then apply base change.

Let $U_\mathbb{Z} = \mathbb{Z}^n$ be the natural permutation representation of S_n, with standard basis $\varepsilon_1, \dots, \varepsilon_n$. Thus,

$$V_\mathbb{Z} = \{(x_1, \dots, x_n) \in U_\mathbb{Z} \mid x_1 + \cdots + x_n = 0\}.$$

[2] I (Darij) learned it from Mark Wildon.

The coordinate-sum map $U_{\mathbb{Z}} \to \mathbb{Z}$ (which sends each basis vector ε_i to 1) has a $\mathbb{Z}$-linear right inverse (namely, the $\mathbb{Z}$-linear map $\mathbb{Z} \to U_{\mathbb{Z}}$ that sends $1 \mapsto \varepsilon_1$). Thus, $V_{\mathbb{Z}}$ (being the kernel of this map) is a direct summand of $U_{\mathbb{Z}}$ as a $\mathbb{Z}$-module (not as an S_n-representation!); in particular, we can regard $\Lambda^m_{\mathbb{Z}} V_{\mathbb{Z}}$ as a $\mathbb{Z}$-submodule of $\Lambda^m_{\mathbb{Z}} U_{\mathbb{Z}}$.

Let $M^{h_m}_{\mathbb{Z}}$ be the integral Young permutation module of shape h_m. If T is a tableau of shape h_m (with entries $1, 2, \ldots, n$), write

$$c_r = T(r+1, 1) \quad (\text{for } 0 \le r \le m)$$

for the entries in its first column, read from top to bottom (in particular, $c_0 = T(1,1)$). Define a $\mathbb{Z}$-linear map

$$\Phi : M^{h_m}_{\mathbb{Z}} \longrightarrow \Lambda^m_{\mathbb{Z}} U_{\mathbb{Z}}$$

on the tabloid basis by

$$\Phi\left(\{T\}\right) = \varepsilon_{c_1} \wedge \varepsilon_{c_2} \wedge \cdots \wedge \varepsilon_{c_m}. \tag{4a}$$

This is well-defined. Indeed, the rows $2, 3, \ldots, m+1$ of h_m are singletons, so the tabloid $\{T\}$ determines the entries $c_1, \ldots, c_m$. The possibly ambiguous entry c_0 does not occur in (4a). It is also immediate from the definition that Φ is S_n-equivariant.

Let T be any tableau of shape h_m (not necessarily standard), and let

$$\mathbf{e}_T = \sum_{\pi \in C_T} \operatorname{sgn}(\pi) \{\pi T\}$$

be the polytabloid associated with T, where C_T is the column group of T (that is, the group of permutations in S_n that fix each column of T as a set). Note that this column group C_T is the symmetric group $S_{\{c_0, c_1, \ldots, c_m\}}$, since the only column that (possibly) contains more than one entry is the first column. We claim that

$$\Phi(\mathbf{e}_T) = m! \, \left(\varepsilon_{c_1} - \varepsilon_{c_0}\right) \wedge \left(\varepsilon_{c_2} - \varepsilon_{c_0}\right) \wedge \cdots \wedge \left(\varepsilon_{c_m} - \varepsilon_{c_0}\right). \tag{4b}$$

Indeed, the definition of Φ gives

$$\Phi(\mathbf{e}_T) = \sum_{\pi \in C_T} \operatorname{sgn}(\pi) \, \varepsilon_{\pi(c_1)} \wedge \cdots \wedge \varepsilon_{\pi(c_m)}.$$

This is the antisymmetrization of $\varepsilon_{c_1} \wedge \cdots \wedge \varepsilon_{c_m}$ with respect to the $m+1$ elements $c_0, c_1, \ldots, c_m$. To check (4b) explicitly, for each $0 \leq r \leq m$ compare the coefficient of the pure wedge

$$\varepsilon_{c_0} \wedge \cdots \wedge \widehat{\varepsilon_{c_r}} \wedge \cdots \wedge \varepsilon_{c_m}$$

(since, after reordering the wedge factors and removing the wedges that have two equal factors, both sides of (4b) can be expressed as linear combinations of such pure wedges). On each side, this coefficient is $m!(-1)^r$: on the left, the $m!$ relevant permutations π are those satisfying $\pi(c_0) = c_r$; on the right, this follows by expanding the wedge product. This proves (4b).

The right hand side of (4b) belongs to $m!\Lambda^m_{\mathbb{Z}} V_{\mathbb{Z}}$. Since the integral Specht module $S^{h_m}_{\mathbb{Z}}$ is spanned by the polytabloids, we conclude that $\Phi(S^{h_m}_{\mathbb{Z}}) \subseteq m!\Lambda^m_{\mathbb{Z}} V_{\mathbb{Z}}$. Thus, there is a $\mathbb{Z}$-linear map

$$\Psi : S^{h_m}_{\mathbb{Z}} \longrightarrow \Lambda^m_{\mathbb{Z}} V_{\mathbb{Z}}$$

characterized by

$$\Phi(s) = m!\Psi(s) \qquad \text{for every } s \in S^{h_m}_{\mathbb{Z}}. \tag{4c}$$

Indeed, we can simply define Ψ by $\Psi(s) = \Phi(s)/m!$; this is well-defined because of $\Phi(S^{h_m}_{\mathbb{Z}}) \subseteq m!\Lambda^m_{\mathbb{Z}} V_{\mathbb{Z}}$ and since $\Lambda^m_{\mathbb{Z}} U_{\mathbb{Z}}$ is torsion-free. This shows furthermore that the map Ψ is S_n-equivariant (since Φ is). Explicitly, from (4b) and (4c), we conclude that each tableau T satisfies

$$\Psi(\mathbf{e}_T) = (\varepsilon_{c_1} - \varepsilon_{c_0}) \wedge \cdots \wedge (\varepsilon_{c_m} - \varepsilon_{c_0}). \tag{4d}$$

The standard tableaux T of shape h_m all have $c_0 = T(1,1) = 1$ (since $T(1,1)$ is smaller than all other entries of T) and $2 \leq c_1 < c_2 < \cdots < c_m$, and they are in bijection with the m-element subsets $\{c_1 < \cdots < c_m\}$ of $\{2, 3, \ldots, n\}$ (since the entries in the first column determine all the remaining entries). The corresponding polytabloids $\mathbf{e}_T$ form a $\mathbb{Z}$-basis of $S^{h_m}_{\mathbb{Z}}$. On the other hand,

$$\varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_1$$

is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}$, so the pure wedges

$$(\varepsilon_{c_1} - \varepsilon_{c_0}) \wedge \cdots \wedge (\varepsilon_{c_m} - \varepsilon_{c_0})$$

with $2 \leq c_1 < c_2 < \cdots < c_m$ form a $\mathbb{Z}$-basis of $\Lambda^m_{\mathbb{Z}} V_{\mathbb{Z}}$. Thus, (4d) shows that Ψ maps a basis to a basis and is therefore an isomorphism. So we have shown that $\Psi : S_{\mathbb{Z}}^{h_m} \longrightarrow \Lambda_{\mathbb{Z}}^m V_{\mathbb{Z}}$ is an isomorphism of representations of S_n over $\mathbb{Z}$.

Finally, the canonical $\mathbb{Z}$-module homomorphism

$$R \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \to V_R$$

is an isomorphism (since it sends the basis $\varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_1$ of $V_{\mathbb{Z}}$ to the analogous basis of V_R). Since exterior powers commute with base change, we thus obtain

$$\Lambda_R^m V_R \cong \Lambda_R^m (R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}) \cong R \otimes_{\mathbb{Z}} (\Lambda_{\mathbb{Z}}^m V_{\mathbb{Z}}).$$

This isomorphism is furthermore S_n-equivariant. Since we also have an S_n-equivariant isomorphism $S_R^{h_m} \cong R \otimes_{\mathbb{Z}} S_{\mathbb{Z}}^{h_m}$ (because a Specht module has "the same" basis of standard polytabloids over any commutative ring), we thus obtain an S_n-equivariant isomorphism

$$\Psi_R : S_R^{h_m} \longrightarrow \Lambda_R^m V_R$$

by tensoring Ψ with R. This proves (4). $\square$

We now return to our characteristic-0 field $\mathbf{k}$. For every $0 \leq m \leq n-1$, fix an S_n-module isomorphism

$$\theta_m : S^{h_m} \xrightarrow{\sim} \Lambda^m V$$

(this exists by Proposition 2).

Theorem 3. *There is a Lie algebra isomorphism*

$$\mathcal{L}_n \cong \mathfrak{gl}(V) \oplus \bigoplus_{\substack{\lambda \vdash n; \\ \lambda \text{ is a non-hook}}} \mathfrak{gl}(S^\lambda). \tag{5}$$

More precisely, consider the Artin–Wedderburn isomorphism

$$\mathbf{k}[S_n] \xrightarrow{\sim} \bigoplus_{\lambda \vdash n} \operatorname{End}_{\mathbf{k}}(S^\lambda), \qquad a \longmapsto (\rho_\lambda(a))_{\lambda \vdash n} \tag{6}$$

(where $\rho_\lambda(a)$ denotes the action of a on S^λ as an element of $\operatorname{End}_{\mathbf{k}}(S^\lambda)$). For each hook partition h_m, write

$$\widetilde{\rho}_{h_m}(a) = \theta_m \rho_{h_m}(a) \theta_m^{-1} \in \operatorname{End}_{\mathbf{k}}(\Lambda^m V). \tag{6a}$$

(This is simply the action of a on $\Lambda^m V$.) Then, the Lie-element algebra $\mathcal{L}_n$ consists of the $a \in \mathbf{k}[S_n]$ satisfying

$$\widetilde{\rho}_{h_0}(a) = 0, \tag{7}$$
$$\widetilde{\rho}_{h_m}(a) = D_m\left(\widetilde{\rho}_{h_1}(a)\right) \qquad (2 \leq m \leq n-1), \tag{8}$$

while all non-hook components $\rho_\lambda(a)$ are arbitrary.

Proof. The group algebra $\mathbf{k}[S_n]$ is split semisimple, so (6) is an isomorphism (of algebras, and thus also of Lie algebras). By the definition of $\widetilde{\rho}_{h_m}$, the map $\mathcal{G}_m$ is precisely the representation map $a \mapsto \widetilde{\rho}_{h_m}(a)$. By (3), the condition $\mathcal{G}_m(a) = \mathcal{A}_m(a)$ is therefore

$$\widetilde{\rho}_{h_m}(a) = D_m\left(\widetilde{\rho}_{h_1}(a)\right).$$

For $m = 0$, this says $\widetilde{\rho}_{h_0}(a) = 0$, since $D_0 = 0$. For $m = 1$, it is tautological, since D_1 is the identity map. For $2 \leq m \leq n-1$, it gives (8).

No non-hook representation occurs among the exterior powers of V. Consequently, the conditions defining $\mathcal{L}_n$ impose no restriction on the non-hook components of a. Thus, an element a of $\mathcal{L}_n$ is uniquely specified by the arbitrary component $X(a) = \widetilde{\rho}_{h_1}(a) \in \mathfrak{gl}(V)$ and arbitrary components $X_\lambda(a) = \rho_\lambda(a) \in \mathfrak{gl}(S^\lambda)$ for the non-hook partitions λ. Its remaining hook components are forced to be $0, D_2(X(a)), \ldots, D_{n-1}(X(a))$. Since every D_m is a Lie algebra morphism, we obtain a Lie algebra isomorphism

$$\begin{aligned} \mathcal{L}_n &\to \mathfrak{gl}(V) \oplus \bigoplus_{\substack{\lambda \vdash n; \\ \lambda \text{ is a non-hook}}} \mathfrak{gl}(S^\lambda), \\ a &\mapsto \left(X(a), (X_\lambda(a))_{\lambda \text{ is a non-hook}}\right). \end{aligned}$$

This proves (5). $\square$

Corollary 4. *Let $f^\lambda = \dim S^\lambda$. Then*

$$\begin{aligned} \dim \mathcal{L}_n &= (n-1)^2 + \sum_{\substack{\lambda \vdash n; \\ \lambda \textit{ is a non-hook}}} (f^\lambda)^2 \\ &= (n-1)^2 + n! - \binom{2n-2}{n-1}. \end{aligned} \tag{9}$$

Proof. The first equality follows from (5). Moreover, every $0 \le m \le n-1$ satisfies

$$f^{h_m} = \binom{n-1}{m}.$$

Hence, the sum of the squares of the dimensions of the hook representations is

$$\sum_{m=0}^{n-1} \binom{n-1}{m}^2 = \binom{2n-2}{n-1}$$

(by Vandermonde's convolution; equivalently, compare the coefficient of t^{n-1} in $(1+t)^{n-1}(1+t)^{n-1} = (1+t)^{2n-2}$). Now use the equality $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$ (which follows from (6) being an isomorphism). $\square$

3 The explicit counterexample for $n = 4$

In $\mathbf{k}[S_4]$, define

$$\begin{aligned} x = {} & 2\,(1 + (12)(34) + (13)(24) + (14)(23)) \\ & - ((123) + (132) + (124) + (142) + (134) + (143) + (234) + (243))\,. \end{aligned} \tag{10}$$

Proposition 5. *The element x is a Lie element, but it does not belong to the Lie algebra generated by the six Kirchhoff differences $\varkappa_{ij}$. Consequently, [BurKul23, Conjecture 2.2] is false already for $n = 4$.*

Proof. Let $e_{(2,2)}$ be the primitive central idempotent of $\mathbf{k}[S_4]$ corresponding to $S^{(2,2)}$. The character values of $S^{(2,2)}$ on the conjugacy classes of cycle types

$$1^4, \quad 2\,1^2, \quad 2^2, \quad 3\,1, \quad 4$$

are, respectively,

$$2, \quad 0, \quad 2, \quad -1, \quad 0.$$

If we let χ^λ denote the character of the Specht module S^λ, then the central-idempotent formula therefore gives

$$\begin{aligned} e_{(2,2)} &= \frac{\dim S^{(2,2)}}{4!} \sum_{\sigma \in S_4} \chi^{(2,2)}(\sigma^{-1})\sigma \\ &= \frac{x}{12}. \end{aligned} \tag{11}$$

For the three-dimensional reflection representation V of S_4, Proposition 2 yields

$$\Lambda^0 V \cong S^{(4)}, \qquad \Lambda^1 V \cong S^{(3,1)}, \qquad \Lambda^2 V \cong S^{(2,1,1)}, \qquad \Lambda^3 V \cong S^{(1,1,1,1)};$$

none of these S_4-representations contains an $S^{(2,2)}$. Thus, $e_{(2,2)}$ acts as 0 on every $\Lambda^m V$. It follows that

$$\mathcal{G}_m(x) = 0 \qquad \text{for every } m. \tag{12}$$

In particular, $\mathcal{G}_1(x) = 0$, and Lemma 1 yields

$$\mathcal{A}_m(x) = D_m\left(\mathcal{G}_1(x)\right) = D_m(0) = 0 \qquad \text{for every } m. \tag{13}$$

Equations (12) and (13) show that $x \in \mathcal{L}_4$.

It remains to prove that x is not generated by the Kirchhoff differences. Suppose the contrary. Every element of the Lie algebra generated by the $\varkappa_{ij}$ is a linear combination of the $\varkappa_{ij}$ themselves and of nested commutators of bracket-length at least 2. Hence, we could write

$$x = \sum_{1 \le i < j \le 4} c_{ij} \varkappa_{ij} + u, \tag{14}$$

where u is a linear combination of nested commutators of bracket-length at least 2.

Apply the sign representation $\operatorname{sgn} : \mathbf{k}[S_4] \to \mathbf{k}$. Since its target is commutative, it annihilates every commutator. Furthermore,

$$\operatorname{sgn}(\varkappa_{ij}) = 1 - (-1) = 2, \qquad \operatorname{sgn}(x) = 0.$$

The last equality also follows from (11), since $e_{(2,2)}$ annihilates the sign representation. Applying sgn to (14), we obtain

$$\sum_{1 \le i < j \le 4} c_{ij} = 0. \tag{15}$$

Now let

$$\tau_{(2,2)} : \mathbf{k}[S_4] \longrightarrow \mathbf{k}, \qquad a \longmapsto \chi^{(2,2)}(a) = \operatorname{Tr}_{S^{(2,2)}}(a).$$

We have $\tau_{(2,2)}(u) = 0$, since the trace of every commutator is 0. Since $\chi^{(2,2)}((ij)) = 0$, we have

$$\tau_{(2,2)}(\varkappa_{ij}) = \dim S^{(2,2)} - \chi^{(2,2)}((ij)) = 2. \tag{16}$$

Consequently, (14), (15), and (16) would imply

$$\tau_{(2,2)}(x) = 0. \tag{17}$$

On the other hand, (11) shows that x acts as $12\,\mathrm{id}$ on the two-dimensional module $S^{(2,2)}$. Hence,

$$\tau_{(2,2)}(x) = 12 \cdot 2 = 24,$$

contradicting (17). This proves the proposition. □

4 The general obstruction

The preceding counterexample is the smallest instance of a general obstruction.

Proposition 6. *Let $\lambda \vdash n$ be a non-hook partition, and let e_λ be the corresponding primitive central idempotent of $\mathbf{k}[S_n]$. Then*

$$e_\lambda \in \mathcal{L}_n,$$

but e_λ does not belong to the Lie algebra generated by the $\varkappa_{ij}$.

More generally, the cosets in the vector-space quotient $\mathcal{L}_n/\mathfrak{h}_n$ of the elements

$$(e_\lambda)_{\substack{\lambda \vdash n; \\ \lambda \text{ is a non-hook}}}$$

are linearly independent.

Proof. Proposition 2 shows that every exterior power $\Lambda^m V$ of V is isomorphic to the Specht module of a hook shape. Thus, since λ is a non-hook, e_λ annihilates every exterior power $\Lambda^m V$. That is, $\mathcal{G}_m(e_\lambda) = 0$ for every m. In particular, $\mathcal{G}_1(e_\lambda) = 0$, so Lemma 1 gives

$$\mathcal{A}_m(e_\lambda) = D_m\left(\mathcal{G}_1(e_\lambda)\right) = 0 = \mathcal{G}_m(e_\lambda).$$

Hence, $e_\lambda \in \mathcal{L}_n$.

It remains to prove the linear-independence claim (since $e_\lambda \notin \mathfrak{h}_n$ will then immediately follow). Suppose that

$$y = \sum_{\substack{\lambda \vdash n; \\ \lambda \text{ is a non-hook}}} b_\lambda e_\lambda \tag{18a}$$

belongs to the Lie algebra generated by the $\varkappa_{ij}$. We can then write

$$y = \sum_{1 \le i < j \le n} c_{ij} \varkappa_{ij} + u, \tag{18b}$$

where u is a linear combination of nested commutators of bracket-length at least 2. Every non-hook central idempotent annihilates the sign representation, so $\operatorname{sgn}(y) = 0$. Applying the sign representation to (18b) thus yields

$$\sum_{1 \le i < j \le n} c_{ij} = 0, \tag{19}$$

since $\operatorname{sgn}(u) = 0$ (because sgn sends any commutator to 0).

Fix a non-hook partition $\mu \vdash n$. Taking the trace in S^μ in (18a), we find

$$\operatorname{Tr}_{S^\mu}(y) = \sum_{\substack{\lambda \vdash n; \\ \lambda \text{ is a non-hook}}} b_\lambda \operatorname{Tr}_{S^\mu}(e_\lambda) = b_\mu f^\mu,$$

where $f^\mu = \dim S^\mu$ (since e_μ acts as the identity on S^μ, whereas every e_λ with $\lambda \neq \mu$ acts as zero on S^μ). Thus,

$$\begin{aligned} b_\mu f^\mu &= \operatorname{Tr}_{S^\mu}(y) \\ &= \sum_{i<j} c_{ij} \operatorname{Tr}_{S^\mu}(\varkappa_{ij}) + \operatorname{Tr}_{S^\mu}(u) \qquad (\text{by (18b)}) \\ &= \sum_{i<j} c_{ij} \operatorname{Tr}_{S^\mu}(\varkappa_{ij}) \qquad (\text{since } \operatorname{Tr}_{S^\mu} \text{ of any commutator is } 0) \\ &= \sum_{i<j} c_{ij} \left(f^\mu - \chi^\mu((12))\right) \end{aligned}$$

(since each $i < j$ satisfies $\operatorname{Tr}_{S^\mu}(\varkappa_{ij}) = \operatorname{Tr}_{S^\mu}(1 - (ij)) = \operatorname{Tr}_{S^\mu}(1) - \operatorname{Tr}_{S^\mu}((ij)) = \chi^\mu(1) - \chi^\mu((ij)) = f^\mu - \chi^\mu((12))$ due to the conjugacy-invariance of χ^μ). This rewrites as

$$\begin{aligned} b_\mu f^\mu &= \left(f^\mu - \chi^\mu((12))\right) \sum_{i<j} c_{ij} \\ &= 0 \qquad (\text{by (19)}). \end{aligned}$$

Since $\mathbf{k}$ has characteristic 0, the scalar f^μ is nonzero in $\mathbf{k}$, and therefore $b_\mu = 0$. Since μ was arbitrary, all the b_λ vanish. This proves the linear-independence claim and thus completes the proof. □

Corollary 7. *Let $p(n)$ be the number of all partitions of n. There are at least $p(n) - n$ linearly independent obstructions to the Burman–Kulishov conjecture.*

Proof. There are precisely n hook partitions of n, namely $h_0, h_1, \dots, h_{n-1}$. Thus, there are $p(n) - n$ non-hook partitions, and Proposition 6 applies. $\square$

5 What the Kirchhoff differences do generate

The Lie algebra generated by the Kirchhoff differences can be expressed in terms of the well-studied Lie algebra generated by the transpositions. Let

$$\mathfrak{g}_n = \mathrm{Lie}\,\langle (ij) \mid 1 \le i < j \le n \rangle \subseteq \mathbf{k}[S_n],$$

let $\mathfrak{g}_n' = [\mathfrak{g}_n, \mathfrak{g}_n]$, and set

$$N = \binom{n}{2}, \qquad T = \sum_{1 \le i < j \le n} (ij).$$

Recall that $\mathfrak{h}_n$ denotes the Lie algebra generated by the $\varkappa_{ij}$.

Proposition 8. *One has*

$$\mathfrak{h}_n = \mathbf{k}(N \cdot 1 - T) \oplus \mathfrak{g}_n'. \tag{20}$$

Proof. The sign representation annihilates $\mathfrak{g}_n'$ but sends $N \cdot 1 - T$ to $2N \neq 0$. Hence, the sum in (20) is direct. It remains to prove that this sum equals $\mathfrak{h}_n$.

First,

$$N \cdot 1 - T = \sum_{1 \le i < j \le n} \varkappa_{ij} \in \mathfrak{h}_n. \tag{21}$$

For a transposition s, we furthermore have

$$(1 - s) - \frac{N \cdot 1 - T}{N} = -\left(s - \frac{T}{N}\right). \tag{22}$$

Let

$$p : \mathbf{k}[S_n] \longrightarrow Z(\mathbf{k}[S_n]), \qquad p(a) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma a \sigma^{-1}$$

be the averaging projection onto the center. Since the transpositions form a single conjugacy class, $p(s) = T/N$ for every transposition s. Moreover, the Lie algebra $\mathfrak{g}_n$ is reductive with center $\mathbf{k}T$, and $\mathfrak{g}'_n = \ker(p|_{\mathfrak{g}_n})$; see [Marin07, Proposition 1]. Hence, each centered transposition $s - T/N$ belongs to $\mathfrak{g}'_n$. Conversely, replacing each transposition s with $s - T/N$ does not change any bracket, since T is central. Thus, the centered transpositions $s - T/N$ generate $\mathfrak{g}'_n$. It now follows from (22) that $\mathfrak{g}'_n \subseteq \mathfrak{h}_n$. Combined with (21), this yields $\mathbf{k}(N \cdot 1 - T) \oplus \mathfrak{g}'_n \subseteq \mathfrak{h}_n$.

It remains to prove the reverse inclusion. Recall that $\mathfrak{h}_n$ is spanned by the Kirchhoff differences $1 - s$ (for s a transposition) and their nested commutators. But every nested commutator involving the elements $1 - s$ is the same, up to the expected signs, as the corresponding nested commutator involving the transpositions s, and hence belongs to $\mathfrak{g}'_n$. Meanwhile, the differences $1 - s$ themselves are linear combinations of $N \cdot 1 - T$ and the centered transpositions (which belong to $\mathfrak{g}'_n$), so they also belong to $\mathbf{k}(N \cdot 1 - T) \oplus \mathfrak{g}'_n$. This proves (20). $\square$

Marin completely determined $\mathfrak{g}'_n$ in [Marin07, Theorem A]. In particular, the hook representations contribute a single factor $\mathfrak{sl}(V)$, acting on the other hooks through the exterior-power maps D_m. A pair of distinct conjugate non-hook partitions contributes a single special-linear factor, diagonally embedded in the two Wedderburn blocks, while a self-conjugate non-hook partition contributes an orthogonal or symplectic factor. This is generally much smaller than the independent full general-linear factors in (5).

For $n = 4$, Theorem 3 and Proposition 8, together with Marin's decomposition, give

$$\begin{aligned} \mathcal{L}_4 &\cong \mathfrak{gl}_3(\mathbf{k}) \oplus \mathfrak{gl}_2(\mathbf{k}), & \dim \mathcal{L}_4 &= 9 + 4 = 13, \\ \mathfrak{h}_4 &\cong \mathbf{k} \oplus \mathfrak{sl}_3(\mathbf{k}) \oplus \mathfrak{sl}_2(\mathbf{k}), & \dim \mathfrak{h}_4 &= 1 + 8 + 3 = 12. \end{aligned}$$

Thus, the explicit element x of (10) represents the unique missing direction modulo $\mathfrak{h}_4$. For $n = 2, 3$, there are no non-hook partitions, and the conjectured equality does hold.

References

[BurKul23] Yurii Burman and Valeriy Kulishov, *Lie elements and the matrix-tree theorem*, Moscow Math. J. **23** (2023), no. 1, 47–58.

[Marin07] Ivan Marin, *L'algèbre de Lie des transpositions*, J. Algebra **310** (2007), 742–774; arXiv:math/0502119.