Algebraic combinatorics related to the free Lie algebra

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Errata and addenda by Darij Grinberg

These errata mostly concern the proof of Solomon's Mackey formula (= Proposition 4.3) and the lemma (Lemma 4.4) used in it.

I will refer to the results appearing in the article "Algebraic combinatorics related to the free Lie algebra" by the numbers under which they appear in this article.

7. Errata and addenda

1. **page 2, Proposition 1.1:** Replace "$\{x_1 \ldots, x_n\}$" by "$\{x_1, \ldots, x_n\}$" (a comma was missing).

2. **page 3:** It is worth mentioning that the permutations in $\mathcal{X}_m$ are also known as the *V-permutations* (since their first-decreasing-then-increasing plot resembles the letter "V") or as the *valley permutations*.

3. **page 5:** It is worth mentioning that the "defect set" $\mathrm{D}(\sigma)$ is more commonly known as the *descent set* of σ, and is usually denoted by $\operatorname{Des} \sigma$.

4. **page 6, (10):** The equivalence of the three statements in (10) is not obvious enough to be left unexplained. Of course, the first two statements ($\pi \in \mathcal{X}_n$ and $D(\pi) = \{1, 2, \ldots, r\}$) are easily seen to be equivalent, but the tricky part is to see that the third statement is also equivalent to them. This relies on the following fact:

   Fact: Let $j_1, j_2, \ldots, j_r$ be elements of $\{1, 2, \ldots, n\}$ satisfying $j_1 > j_2 > \cdots > j_r$. Then, the product

   $$(j_1 \ldots 1)(j_2 \ldots 1) \cdots (j_r \ldots 1)$$

   is the permutation in S_n whose one-line notation (i.e., list of values at $1, 2, \ldots, n$) begins with the numbers $j_1, j_2, \ldots, j_r$ in this order and ends with the remaining $n - r$ elements of $\{1, 2, \ldots, n\}$ in increasing order. (In particular, this product belongs to $\mathcal{X}_n$.)

   This fact is (essentially) Exercise 4 in *Math 4990 Fall 2017 (Darij Grinberg) homework set 7*, where I give a detailed proof. (Note that the order in which permutations are multiplied differs between my writeup and the Blessenohl/Laue paper. Thus, even though the cycles appear in the order of decreasing length in the former and increasing length in the latter, the products are the same.)

5. **page 16, proof of Proposition 4.2:** It is helpful to restate the definition of M_ℓ as follows: The set M_ℓ consists of all permutations $\mu \in S_\ell$ that satisfy $(i+1)\,\mu \geq i\mu - 1$ for each $i < \ell$.

6. **page 17, proof of Proposition 4.2:** Replace "put $k := min\left\{j \mid (i+1)\,\rho \leq j \leq i\rho,\ j\rho^{-1} \geq i\right\} - 1$" (which is just a complicated way to say "put $k := i$", clearly against the authors' intent) by "let k be the largest j satisfying $i\rho \geq j \geq (i+1)\,\rho$ and $j\rho^{-1} \geq i+1$". To see why this choice of k works, we first note that it satisfies $k \neq i\rho$ (since $k\rho^{-1} \geq i+1$, but $(i\rho)\,\rho^{-1} = i$ is not $\geq i+1$), hence $k < i\rho$ (since $i\rho \geq k$), and thus $(k+1)\,\rho^{-1} < i+1$ (since otherwise, k would not be the **largest** j with $j\rho^{-1} \geq i+1$). Hence, in order to prove that $k\rho^{-1} - (k+1)\,\rho^{-1} > 1$, it suffices to show that we cannot have the situation where $k\rho^{-1} = i+1$ and $(k+1)\,\rho^{-1} = i$ (since in all other cases, $k\rho^{-1} - (k+1)\,\rho^{-1} > 1$ follows from $k\rho^{-1} \geq i+1$ and $(k+1)\,\rho^{-1} < i+1$). But this situation is indeed impossible, since it entails $i\rho = k+1$ and $(i+1)\,\rho = k$ in contradiction to $i\rho - (i+1)\,\rho \geq 2$.

7. **page 18:** What is called a "decomposition of n" here is more usually called a *composition* of n.

8. **page 18, last line:** It is also worth pointing out that the standard partition $P^q = \left(P_1^q, \ldots, P_\ell^q\right)$ of a composition $q = (q_1, \ldots, q_\ell) \models n$ can be defined as follows: the set P_1^q consists of the smallest q_1 elements of $\{1, 2, \ldots, n\}$; the set P_2^q consists of the next-smallest q_2 elements of $\{1, 2, \ldots, n\}$; and so on.

9. **page 20, proof of Lemma 4.4:** "This implies, by (40), that $x_1 < x_2$": This could use a bit more explanation. Namely, $\rho \in \mathcal{S}^r(M) \subseteq \mathcal{S}^r$. Thus, (40) shows that $\rho\mid_{P_i^r}$ is increasing. Hence, if we had $x_1 \geq x_2$, then we would have $x_1\rho \geq x_2\rho$ (since x_1 and x_2 both belong to P_i^r), which would contradict $x_1\rho < x_2\rho$. Thus, we cannot have $x_1 \geq x_2$, so we must have $x_1 < x_2$.

10. **page 20, proof of Lemma 4.4:** Here is a bit more detail on the derivation of (44): Applying the bijection ρ to both sides of (43), we see that

$$P_i^r\rho \cap P_j^q = R_{i,j}\rho \qquad \text{for all } i \leq k,\ j \leq \ell \text{ and } \rho \in \mathcal{S}^r(M). \tag{43'}$$

Now, taking the union over all i (and recalling that $\bigcup_i \left(P_i^r\rho\right) = \{1, 2, \ldots, n\}$), we obtain (44).

11. **page 20, proof of Lemma 4.4:** Let me explain why "$\rho\sigma\mid_{R_{ij}}$ is the composition of two increasing functions". Indeed, the function $\rho\mid_{R_{ij}}$ is increasing, because (40) shows that $\rho\mid_{P_i^r}$ is increasing (since $\rho \in \mathcal{S}^r(M) \subseteq \mathcal{S}^r$) and because $R_{ij} \subseteq P_i^r$. Furthermore, the function $\sigma\mid_{R_{ij}\rho}$ is increasing, because (40) shows that $\sigma\mid_{P_j^q}$ is increasing (since $\sigma \in \mathcal{S}^q$) and because (43′) shows

that $R_{ij}\rho = P_i^r \rho \cap P_j^q \subseteq P_j^q$. Now, $\rho\sigma \mid_{R_{ij}}$ is the composition of these two increasing functions $\rho \mid_{R_{ij}}$ and $\sigma \mid_{R_{ij}\rho}$.

12. **pages 20–21, proof of Lemma 4.4:** The explanation for why (48) holds in the case $j_1 = j_2$ is a bit laconic. Let me give a more detailed one:

    Assume that $j_1 = j_2$. Then, x_1 and x_2 belong to the same R_{ij} (namely, $R_{ij_1} = R_{ij_2}$), and thus we have $x_1\tau < x_2\tau$ (as $\tau \in \mathcal{S}^{w(M)}$ and $x_1 < x_2$). Moreover, each $h \in \{1,2\}$ satisfies $\underbrace{x_h}_{\in R_{i,j_h}} \underbrace{\rho}_{=\tau\sigma^{-1}} \in \underbrace{R_{i,j_h}\tau}_{\subseteq P_{j_h}^q \sigma} \sigma^{-1} \subseteq P_{j_h}^q$. In view of $j_1 = j_2$, this means that both $x_2\rho$ and $x_1\rho$ belong to the same P_j^q. Hence, if we had $x_2\rho \leq x_1\rho$, then (from $\sigma \in \mathcal{S}^q$) we would obtain $x_2\rho\sigma \leq x_1\rho\sigma$, that is, $x_2\tau \leq x_1\tau$ (since $\rho\sigma = \tau$), which would contradict $x_1\tau < x_2\tau$. Hence, we cannot have $x_2\rho \leq x_1\rho$. Thus, we must have $x_1\rho < x_2\rho$. This proves (48).