On the square of the antipode in a connected filtered Hopf algebra

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Abstract. It is well-known that the antipode S of a commutative or cocommutative Hopf algebra satisfies $S^2 = \operatorname{id}$ (where $S^2 = S \circ S$). Recently, similar results have been obtained by Aguiar, Lauve and Mahajan for connected graded Hopf algebras: Namely, if H is a connected graded Hopf algebra with grading $H = \bigoplus_{n \geq 0} H_n$, then each positive integer n satisfies $\left(\operatorname{id} - S^2\right)^n (H_n) = 0$ and (even stronger) $\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{n-1}\right) (H_n) = 0$. For some specific H's such as the Malvenuto–Reutenauer Hopf algebra FQSym, the exponents can be lowered.

In this note, we generalize these results in several directions: We replace the base field by a commutative ring, replace the Hopf algebra by a coalgebra (actually, a slightly more general object, with no coassociativity required), and replace both id and S^2 by "coalgebra homomorphisms" (of sorts). Specializing back to connected graded Hopf algebras, we show that the exponent n in the identity $\left(\operatorname{id} - S^2\right)^n (H_n) = 0$ can be lowered to $n-1$ (for $n > 1$) if and only if $\left(\operatorname{id} - S^2\right) (H_2) = 0$. (A sufficient condition for this is that every pair of elements of H_1 commutes; this is satisfied, e.g., for FQSym.)

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Consider, for simplicity, a connected graded Hopf algebra H over a field (we will soon switch to more general settings). Let S be the antipode of H. A classical result (e.g., [Sweedl69, Proposition 4.0.1 6)] or [HaGuKi10, Corollary 3.3.11] or [Abe80, Theorem 2.1.4 (vi)] or [Radfor12, Corollary 7.1.11]) says that $S^2 = \operatorname{id}$ when H is commutative or cocommutative. (Here and in the following, powers are composition powers; thus, S^2 means $S \circ S$.) In general, $S^2 = \operatorname{id}$ need not hold. However, in [AguLau14, Proposition 7], Aguiar and Lauve showed that S^2 is still locally unipotent, and more precisely, we have

$$\left(\operatorname{id} - S^2\right)^n (H_n) = 0 \qquad \text{for each } n > 0,$$

where H_n denotes the n-th graded component of H. Later, Aguiar and Mahajan [AguMah17, Lemma 12.50] strengthened this equality to

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{n-1}\right) (H_n) = 0 \qquad \text{for each } n > 0.$$

For specific combinatorially interesting Hopf algebras, even stronger results hold; in particular,

$$\left(\operatorname{id} - S^2\right)^{n-1} (H_n) = 0 \qquad \text{holds for each } n > 1$$

when H is the Malvenuto–Reutenauer Hopf algebra (see [AguLau14, Example 8]).

In this note, we will unify these results and transport them to a much more general setting. First of all, the ground field will be replaced by an arbitrary commutative ring; this generalization is not unexpected, but renders the proof strategy of [AguLau14, Proposition 7] insufficient[1]. Second, we will replace the Hopf algebra by a coalgebra, or rather by a more general structure that does not even require coassociativity. The squared antipode S^2 will be replaced by an arbitrary "coalgebra" endomorphism f (we are using scare quotes because our structure is not really a coalgebra), and the identity map by another such endomorphism e. Finally, the graded components will be replaced by an arbitrary sequence of submodules satisfying certain compatibility relations. We state the general result in Section 2.1 and prove it in Section 3.1. In Sections 2.2–2.4, we progressively specialize this result: first to connected filtered coalgebras with coalgebra endomorphisms (in Section 2.2), then to connected filtered Hopf algebras with S^2 (in Section 2.3), and finally to connected graded Hopf algebras with S^2 (in Section 2.4). The latter specialization covers the results of Aguiar and Lauve. (The Malvenuto–Reutenauer Hopf algebra turns out to be a red herring; any connected graded Hopf algebra H with the property that $ab = ba$ for all $a, b \in H_1$ will do.)

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[1] In fact, the proof in [AguLau14, Proposition 7] relies on the coradical filtration of H and its associated graded structure $\operatorname{gr} H$. If the base ring is a field, then $\operatorname{gr} H$ is a well-defined commutative Hopf algebra (see, e.g., [AguLau14, Lemma 1]), and thus the antipode of H can be viewed as a "deformation" of the antipode of $\operatorname{gr} H$. But the latter antipode does square to id because $\operatorname{gr} H$ is commutative. Unfortunately, this proof does not survive our generalization; in fact, even defining a Hopf algebra structure on $\operatorname{gr} H$ would likely require at least some flatness assumptions.

Remark on alternative versions

This is the regular version of the present note. A more detailed version (with longer proofs) is available at

`http://www.cip.ifi.lmu.de/~grinberg/algebra/antipode-squared-detailed.pdf`

(and is also available as an ancillary file to this preprint on the arXiv).

1. Notations

We will use the notions of coalgebras, bialgebras and Hopf algebras over a commutative ring, as defined (e.g.) in [Abe80, Chapter 2], [GriRei20, Chapter 1], [HaGuKi10, Chapters 2, 3], [Radfor12, Chapters 2, 5, 7] or [Sweedl69, Chapters I–IV]. (In particular, our Hopf algebras are **not** twisted by a $\mathbb{Z}/2$-grading as the topologists' ones are.) We use the same notations for Hopf algebras as in [GriRei20, Chapter 1]. In particular:

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- "Rings" and "algebras" are always required to be associative and have a unity.
- We fix a commutative ring $\mathbf{k}$. The symbols "$\otimes$" and "End" shall always mean "$\otimes_{\mathbf{k}}$" and "$\operatorname{End}_{\mathbf{k}}$", respectively. The unity of the ring $\mathbf{k}$ will be called $1_{\mathbf{k}}$ or just 1 if confusion is unlikely.
- The comultiplication and the counit of a $\mathbf{k}$-coalgebra are denoted by Δ and ϵ.
- "Graded" $\mathbf{k}$-modules mean $\mathbb{N}$-graded $\mathbf{k}$-modules. The base ring $\mathbf{k}$ itself is not supposed to have any nontrivial grading.
- The n-th graded component of a graded $\mathbf{k}$-module V will be called V_n. If $n < 0$, then this is the zero submodule 0.
- A graded $\mathbf{k}$-Hopf algebra means a $\mathbf{k}$-Hopf algebra that has a grading as a $\mathbf{k}$-module, and whose structure maps (multiplication, unit, comultiplication and counit) are graded maps. (The antipode is automatically graded in this case, by [GriRei20, Exercise 1.4.29 (e)].)
- If f is a map from a set to itself, and if $k \in \mathbb{N}$ is arbitrary, then f^k shall denote the map $\underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$. (Thus, $f^1 = f$ and $f^0 = \operatorname{id}$.)

2. Theorems

2.1. The main theorem

We can now state the main result of this note:

Theorem 2.1. Let D be a $\mathbf{k}$-module, and let $(D_1, D_2, D_3, \ldots)$ be a sequence of $\mathbf{k}$-submodules of D. Let $\delta : D \to D \otimes D$ be any $\mathbf{k}$-linear map.

Let $e : D \to D$ and $f : D \to D$ be two $\mathbf{k}$-linear maps such that

$$\operatorname{Ker} \delta \subseteq \operatorname{Ker} (e - f) \qquad \text{and} \tag{1}$$

$$(f \otimes f) \circ \delta = \delta \circ f \qquad \text{and} \tag{2}$$

$$(e \otimes e) \circ \delta = \delta \circ e \qquad \text{and} \tag{3}$$

$$f \circ e = e \circ f. \tag{4}$$

Let p be a positive integer such that

$$(e - f) \left(D_1 + D_2 + \cdots + D_p\right) = 0. \tag{5}$$

Assume furthermore that

$$\delta \left(D_n\right) \subseteq \sum_{i=1}^{n-1} D_i \otimes D_{n-i} \qquad \text{for each } n > p. \tag{6}$$

(Here, the "$D_i \otimes D_{n-i}$" on the right hand side means the image of $D_i \otimes D_{n-i}$ under the canonical map $D_i \otimes D_{n-i} \to D \otimes D$ that is obtained by tensoring the two inclusion maps $D_i \to D$ and $D_{n-i} \to D$ together. When $\mathbf{k}$ is not a field, this canonical map may fail to be injective.)

Then, for any integer $u > p$, we have

$$(e - f)^{u-p} \left(D_u\right) \subseteq \operatorname{Ker} \delta \tag{7}$$

and

$$(e - f)^{u-p+1} \left(D_u\right) = 0. \tag{8}$$

As the statement of this theorem is not very intuitive, some explanations are in order. The reader may think of the D in Theorem 2.1 as a "pre-coalgebra", with δ being its "reduced coproduct". Indeed, the easiest way to obtain a nontrivial example is to fix a connected graded Hopf algebra H, then define D to be either H or the "positive part" of H (that is, the submodule $\bigoplus_{n>0} H_n$ of H), and define δ to be the map $x \mapsto \Delta(x) - x \otimes 1 - 1 \otimes x + \epsilon(x) 1 \otimes 1$ (the so-called *reduced coproduct* of H). From this point of view, $\operatorname{Ker} \delta$ can be regarded as the set of "primitive" elements of D. The maps f and e can be viewed as two commuting "coalgebra endomorphisms" of D (indeed, the assumptions (2) and (3) are essentially saying that f and

e preserve the "reduced coproduct" δ). The submodules $D_1, D_2, D_3, \ldots$ are an analogue of the (positive-degree) graded components of D, while the assumption (6) says that the "reduced coproduct" δ "respects the grading" (as is indeed the case for connected graded Hopf algebras).

We stress that p is allowed to be 1 in Theorem 2.1; in this case, the assumption (5) simplifies to $(e - f)(0) = 0$, which is automatically true by the linearity of $e - f$.

We shall prove Theorem 2.1 in Section 3.1. First, however, let us explore its consequences for coalgebras and Hopf algebras, recovering in particular the results of Aguiar and Lauve promised in the introduction.

2.2. Connected filtered coalgebras

We begin by specializing Theorem 2.1 to the setting of a connected filtered coalgebra. There are several ways to define what a filtered coalgebra is; ours is probably the most liberal:

Definition 2.2. A *filtered* $\mathbf{k}$*-coalgebra* means a $\mathbf{k}$-coalgebra C equipped with an infinite sequence $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ of $\mathbf{k}$-submodules of C satisfying the following three conditions:

- We have
$$C_{\leq 0} \subseteq C_{\leq 1} \subseteq C_{\leq 2} \subseteq \cdots . \tag{9}$$
- We have
$$\bigcup_{n \in \mathbb{N}} C_{\leq n} = C. \tag{10}$$
- We have
$$\Delta(C_{\leq n}) \subseteq \sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n \in \mathbb{N}. \tag{11}$$
  (Here, the "$C_{\leq i} \otimes C_{\leq n-i}$" on the right hand side means the image of $C_{\leq i} \otimes C_{\leq n-i}$ under the canonical map $C_{\leq i} \otimes C_{\leq n-i} \to C \otimes C$ that is obtained by tensoring the two inclusion maps $C_{\leq i} \to C$ and $C_{\leq n-i} \to C$ together. When $\mathbf{k}$ is not a field, this canonical map may fail to be injective.)

The sequence $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ is called the *filtration* of the filtered $\mathbf{k}$-coalgebra C.

A more categorically-minded person might replace the condition $\Delta(C_{\leq n}) \subseteq \sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i}$ in this definition by a stronger requirement (e.g., asking Δ to factor through a linear map $C_{\leq n} \to \bigoplus_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i}$, where the "$\otimes$" signs now signify the actual tensor products rather than their images in $C \otimes C$). However, we have no need for such stronger requirements. Mercifully, all reasonable definitions of filtered **k**-coalgebras agree when **k** is a field.

The condition (10) in Definition 2.2 shall never be used in the following; we merely state it to avoid muddling the meaning of "filtered **k**-coalgebra".

A graded **k**-coalgebra C automatically becomes a filtered **k**-coalgebra; indeed, we can define its filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ by setting

$$C_{\leq n} = \bigoplus_{i=0}^{n} C_i \qquad \text{for all } n \in \mathbb{N}.$$

Definition 2.3. Let C be a filtered **k**-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$. Let $1_{\mathbf{k}}$ denote the unity of the ring **k**.

(a) The filtered **k**-coalgebra C is said to be *connected* if the restriction $\epsilon \mid_{C_{\leq 0}}$ is a **k**-module isomorphism from $C_{\leq 0}$ to **k**.

(b) In this case, the element $\left(\epsilon \mid_{C_{\leq 0}}\right)^{-1} (1_{\mathbf{k}}) \in C_{\leq 0}$ is called the *unity* of C and is denoted by 1_C.

Now, assume that C is a connected filtered **k**-coalgebra.

(c) An element x of C is said to be *primitive* if $\Delta(x) = x \otimes 1_C + 1_C \otimes x$.

(d) The set of all primitive elements of C is denoted by $\operatorname{Prim} C$.

These notions of "connected", "unity" and "primitive" specialize to the commonly established concepts of these names when C is a graded **k**-bialgebra. Indeed, Definition 2.3 **(b)** defines the unity 1_C of C to be the unique element of $C_{\leq 0}$ that gets sent to $1_{\mathbf{k}}$ by the map ϵ; but this property is satisfied for the unity of a graded **k**-bialgebra as well. (We will repeat this argument in more detail later on, in the proof of Proposition 2.10.)

The following property of connected filtered **k**-coalgebras will be crucial for us:

Proposition 2.4. Let C be a connected filtered **k**-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$. Define a **k**-linear map $\delta : C \to C \otimes C$ by setting

$$\delta(c) := \Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C \qquad \text{for each } c \in C.$$

Then:

(a) We have

$$\delta(C_{\leq n}) \subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n > 0.$$

(b) If $f : C \to C$ is a **k**-coalgebra homomorphism satisfying $f(1_C) = 1_C$, then we have $(f \otimes f) \circ \delta = \delta \circ f$.

(c) We have $\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$.

(d) The set $\operatorname{Prim} C$ is a **k**-submodule of C.

(e) We have $\operatorname{Ker} \delta = \mathbf{k} \cdot 1_C + \operatorname{Prim} C$.

We shall prove Proposition 2.4 in Section 3.2. The map δ in Proposition 2.4 is called the *reduced coproduct* of C.

Proposition 2.4 helps us apply Theorem 2.1 to filtered **k**-coalgebras, resulting in the following:

Corollary 2.5. Let C be a connected filtered **k**-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$.

Let $e : C \to C$ and $f : C \to C$ be two **k**-coalgebra homomorphisms such that

$$e(1_C) = 1_C \qquad \text{and}$$
$$f(1_C) = 1_C \qquad \text{and}$$
$$\operatorname{Prim} C \subseteq \operatorname{Ker}(e - f) \qquad \text{and} \tag{12}$$
$$f \circ e = e \circ f. \tag{13}$$

Let p be a positive integer such that

$$(e - f)\left(C_{\leq p}\right) = 0. \tag{14}$$

Then:

(a) For any integer $u > p$, we have

$$(e - f)^{u-p}(C_{\leq u}) \subseteq \operatorname{Prim} C. \tag{15}$$

(b) For any integer $u \geq p$, we have

$$(e - f)^{u-p+1}(C_{\leq u}) = 0. \tag{16}$$

Corollary 2.5 results from an easy (although not completely immediate) application of Theorem 2.1 and Proposition 2.4. The detailed proof can be found in Section 3.3.

Specializing Corollary 2.5 further to the case of $p = 1$, we can obtain a nicer result:

Corollary 2.6. Let C be a connected filtered **k**-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$.

Let $e : C \to C$ and $f : C \to C$ be two **k**-coalgebra homomorphisms such that

$$e(1_C) = 1_C \qquad \text{and}$$
$$f(1_C) = 1_C \qquad \text{and}$$
$$\operatorname{Prim} C \subseteq \operatorname{Ker}(e - f) \qquad \text{and}$$
$$f \circ e = e \circ f.$$

Then:

(a) For any integer $u > 1$, we have

$$(e - f)^{u-1} (C_{\leq u}) \subseteq \operatorname{Prim} C.$$

(b) For any positive integer u, we have

$$(e - f)^{u} (C_{\leq u}) = 0.$$

See Section 3.3 for a proof of this corollary.
The particular case of Corollary 2.6 for $e = \operatorname{id}$ is particularly simple:

Corollary 2.7. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$.
Let $f : C \to C$ be a $\mathbf{k}$-coalgebra homomorphism such that

$$f(1_C) = 1_C \qquad \text{and} \qquad \operatorname{Prim} C \subseteq \operatorname{Ker}(\operatorname{id} - f).$$

Then:
(a) For any integer $u > 1$, we have

$$(\operatorname{id} - f)^{u-1} (C_{\leq u}) \subseteq \operatorname{Prim} C.$$

(b) For any positive integer u, we have

$$(\operatorname{id} - f)^{u} (C_{\leq u}) = 0.$$

Again, the proof of this corollary can be found in Section 3.3.
Note that Corollary 2.7 **(b)** is precisely [Grinbe17, Theorem 37.1 **(a)**].

2.3. Connected filtered bialgebras and Hopf algebras

We shall now apply our above results to connected filtered bialgebras and Hopf algebras. We first define what we mean by these notions:

Definition 2.8. (a) A *filtered* $\mathbf{k}$*-bialgebra* means a $\mathbf{k}$-bialgebra H equipped with an infinite sequence $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ of $\mathbf{k}$-submodules of H satisfying the following five conditions:

- We have
$$H_{\leq 0} \subseteq H_{\leq 1} \subseteq H_{\leq 2} \subseteq \cdots.$$
- We have
$$\bigcup_{n \in \mathbb{N}} H_{\leq n} = H.$$

- We have

$$\Delta\left(H_{\leq n}\right) \subseteq \sum_{i=0}^{n} H_{\leq i} \otimes H_{\leq n-i} \qquad \text{for each } n \in \mathbb{N}.$$

(Here, the "$H_{\leq i} \otimes H_{\leq n-i}$" on the right hand side means the image of $H_{\leq i} \otimes H_{\leq n-i}$ under the canonical map $H_{\leq i} \otimes H_{\leq n-i} \to H \otimes H$ that is obtained by tensoring the two inclusion maps $H_{\leq i} \to H$ and $H_{\leq n-i} \to H$ together.)

- We have $H_{\leq i} H_{\leq j} \subseteq H_{\leq i+j}$ for any $i, j \in \mathbb{N}$. (Here, $H_{\leq i} H_{\leq j}$ denotes the **k**-linear span of the set of all products ab with $a \in H_{\leq i}$ and $b \in H_{\leq j}$.)

- The unity of the **k**-algebra H belongs to $H_{\leq 0}$.

The sequence $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ is called the *filtration* of the filtered **k**-bialgebra H.

(b) A *filtered* **k**-*Hopf algebra* means a filtered **k**-bialgebra H such that the **k**-bialgebra H is a Hopf algebra (i.e., has an antipode) and such that the antipode S of H respects the filtration (i.e., satisfies $S\left(H_{\leq n}\right) \subseteq H_{\leq n}$ for each $n \in \mathbb{N}$).

The $H_{\leq i} H_{\leq j} \subseteq H_{\leq i+j}$ condition in Definition 2.8 **(a)** will not actually be used in what follows. Thus, we could have omitted it; but this would have resulted in a less common (and less well-behaved in other ways) concept of "filtered bialgebra". Likewise, we have included the $S\left(H_{\leq n}\right) \subseteq H_{\leq n}$ condition in Definition 2.8 **(b)**, even though we will never use it.

Every **k**-bialgebra is automatically a **k**-coalgebra. Thus, every filtered **k**-bialgebra is automatically a filtered **k**-coalgebra. This allows the following definition:

Definition 2.9. A filtered **k**-bialgebra H is said to be *connected* if the filtered **k**-coalgebra H is connected.

Thus, if H is a connected filtered **k**-bialgebra, then Definition 2.3 **(b)** defines a "unity" 1_H of H. This appears to cause an awkward notational quandary, since H already has a unity by virtue of being a **k**-algebra (and this latter unity is also commonly denoted by 1_H). Fortunately, this cannot cause any confusion, since these two unities are identical, as the following proposition shows:

Proposition 2.10. Let H be a connected filtered **k**-bialgebra. Then, the unity 1_H defined according to Definition 2.3 **(b)** equals the unity of the **k**-algebra H.

Proof of Proposition 2.10. Both unities in question belong to $H_{\leq 0}$ (indeed, the former does so by its definition, whereas the latter does so because H is a filtered **k**-bialgebra) and are sent to $1_{\mathbf{k}}$ by the map ϵ (indeed, the former does so by its definition, whereas the latter does so by the axioms of a **k**-bialgebra). However, since the map $\epsilon\mid_{H_{\leq 0}}$ is a **k**-module isomorphism (because the filtered **k**-coalgebra H is connected), these two properties uniquely determine these unities. Thus, these two unities are equal. Proposition 2.10 is thus proven. $\square$

In Definition 2.3, we have defined the notion of a "primitive element" of a connected filtered **k**-coalgebra C. In the same way, we can define a "primitive element" of a **k**-bialgebra H (using the unity of the **k**-algebra H instead of 1_C):

Definition 2.11. Let H be a **k**-bialgebra with unity 1_H.
(a) An element x of H is said to be *primitive* if $\Delta(x) = x \otimes 1_H + 1_H \otimes x$.
(b) The set of all primitive elements of H is denoted by $\operatorname{Prim} H$.

When H is a connected filtered **k**-bialgebra, Definition 2.11 **(a)** agrees with Definition 2.3 **(c)**, since Proposition 2.10 shows that the two meanings of 1_H are actually identical. Thus, when H is a connected filtered **k**-bialgebra, Definition 2.11 **(b)** agrees with Definition 2.3 **(d)**. The notation $\operatorname{Prim} H$ is therefore unambiguous.

Next we state some basic properties of the antipode in a Hopf algebra that will be used later on:

Lemma 2.12. Let H be a **k**-Hopf algebra with unity $1_H \in H$ and antipode $S \in \operatorname{End} H$. Then:
(a) The map $S^2 : H \to H$ is a **k**-coalgebra homomorphism.
(b) We have $S(1_H) = 1_H$.
(c) We have $S(x) = -x$ for every primitive element x of H.
(d) We have $S^2(x) = x$ for every primitive element x of H.

This lemma, as well as the remaining claims made in Section 2.3, shall be proved in Section 3.4.

We can now state our main consequence for connected filtered Hopf algebras:

Corollary 2.13. Let H be a connected filtered **k**-Hopf algebra with filtration $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ and antipode S.

Let p be a positive integer such that

$$\left(\operatorname{id} - S^2\right)(H_{\leq p}) = 0. \tag{17}$$

Then:
(a) For any integer $u > p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p}(H_{\leq u}) \subseteq \operatorname{Prim} H \tag{18}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-p}\right)(H_{\leq u}) = 0. \tag{19}$$

(b) For any integer $u \geq p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p+1}(H_{\leq u}) = 0. \tag{20}$$

Specializing this to $p = 1$, we can easily obtain the following:

Corollary 2.14. Let H be a connected filtered **k**-Hopf algebra with filtration $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ and antipode S. Then:

(a) For any integer $u > 1$, we have

$$\left(\operatorname{id} - S^2\right)^{u-1} (H_{\leq u}) \subseteq \operatorname{Prim} H \tag{21}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right) (H_{\leq u}) = 0. \tag{22}$$

(b) For any positive integer u, we have

$$\left(\operatorname{id} - S^2\right)^{u} (H_{\leq u}) = 0. \tag{23}$$

Corollary 2.14 **(b)** has already appeared in [Grinbe17, Theorem 37.7 **(a)**].

2.4. Connected graded Hopf algebras

Let us now specialize our results even further to connected **graded** Hopf algebras. We have already seen that any graded **k**-coalgebra automatically becomes a filtered **k**-coalgebra. In the same way, any graded **k**-Hopf algebra automatically becomes a filtered **k**-Hopf algebra. Moreover, a graded **k**-Hopf algebra H is connected (in the sense that $H_0 \cong \mathbf{k}$ as **k**-modules) if and only if the filtered **k**-coalgebra H is connected. (This follows easily from [GriRei20, Exercise 1.3.20 (e)].) Thus, our above results for connected filtered **k**-Hopf algebras can be applied to connected graded **k**-Hopf algebras. From Corollary 2.14, we easily obtain the following:

Corollary 2.15. Let H be a connected graded **k**-Hopf algebra with antipode S. Then, for any positive integer u, we have

$$\left(\operatorname{id} - S^2\right)^{u-1} (H_u) \subseteq \operatorname{Prim} H \tag{24}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right) (H_u) = 0 \tag{25}$$

and

$$\left(\operatorname{id} - S^2\right)^{u} (H_u) = 0. \tag{26}$$

We will prove this corollary – as well as all others stated in Section 2.4 – in Section 3.5 further below. We note that Corollary 2.15 is not an immediate consequence of Corollary 2.14, since the condition "u is positive" is weaker than the condition

"$u > 1$" in Corollary 2.14 **(a)**; thus, deriving Corollary 2.15 from Corollary 2.14 requires some extra work to account for the case of $u = 1$.

The equality (25) in Corollary 2.15 yields [AguMah17, Lemma 12.50], whereas the equality (26) yields [AguLau14, Proposition 7]. Next, we apply Corollary 2.13 to the graded setting:

Corollary 2.16. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S. Let p be a positive integer such that all $i \in \{2, 3, \ldots, p\}$ satisfy

$$\left(\operatorname{id} - S^2\right)(H_i) = 0. \tag{27}$$

Then:

(a) For any integer $u > p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p}(H_{\leq u}) \subseteq \operatorname{Prim} H \tag{28}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-p}\right)(H_{\leq u}) = 0. \tag{29}$$

(b) For any integer $u \geq p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p+1}(H_{\leq u}) = 0. \tag{30}$$

The particular case of Corollary 2.16 for $p = 2$ is the most useful, as the condition (27) boils down to the equality $\left(\operatorname{id} - S^2\right)(H_2) = 0$ in this case, and the latter equality is satisfied rather frequently. Here is one sufficient criterion:

Corollary 2.17. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S. Assume that

$$ab = ba \qquad \text{for every } a, b \in H_1. \tag{31}$$

Then:

(a) We have

$$\left(\operatorname{id} - S^2\right)(H_2) = 0.$$

(b) For any integer $u > 2$, we have

$$\left(\operatorname{id} - S^2\right)^{u-2}(H_{\leq u}) \subseteq \operatorname{Prim} H \tag{32}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-2}\right)(H_{\leq u}) = 0. \tag{33}$$

(c) For any integer $u > 1$, we have

$$\left(\operatorname{id} - S^2\right)^{u-1} \left(H_{\leq u}\right) = 0. \tag{34}$$

The equality (34) in Corollary 2.17 generalizes [AguLau14, Example 8]. Indeed, if H is the Malvenuto–Reutenauer Hopf algebra[2], then the condition (31) is satisfied (since H_1 is a free $\mathbf{k}$-module of rank 1 in this case); therefore, Corollary 2.17 **(c)** can be applied in this case, and we recover [AguLau14, Example 8]. Likewise, we can obtain the same result if H is the Hopf algebra WQSym of word quasisymmetric functions[3].

It is worth noticing that the condition (31) is only sufficient, but not necessary for (34). For example, if H is the tensor algebra of a free $\mathbf{k}$-module V of rank ≥ 2, then (34) holds (since H is cocommutative, so that $S^2 = \operatorname{id}$), but (31) does not (since $u \otimes v \neq v \otimes u$ if u and v are two distinct basis vectors of V).

An example of a connected graded Hopf algebra H that does **not** satisfy (34) (and thus does not satisfy (31) either) is not hard to construct:

Example 2.18. Assume that the ring $\mathbf{k}$ is not trivial. Let H be the free $\mathbf{k}$-algebra with three generators a, b, c. We equip this $\mathbf{k}$-algebra H with a grading, by requiring that its generators a, b, c are homogeneous of degrees $1, 1, 2$, respectively. Next, we define a comultiplication Δ on H by setting

$$\Delta(a) = a \otimes 1 + 1 \otimes a;$$
$$\Delta(b) = b \otimes 1 + 1 \otimes b;$$
$$\Delta(c) = c \otimes 1 + a \otimes b + 1 \otimes c$$

(where 1 is the unity of H). Furthermore, we define a counit ϵ on H by setting $\epsilon(a) = \epsilon(b) = \epsilon(c) = 0$. It is straightforward to see that H thus becomes a connected graded $\mathbf{k}$-bialgebra, hence (by [GriRei20, Proposition 1.4.16]) a connected graded $\mathbf{k}$-Hopf algebra. Its antipode S is easily seen to satisfy $S(c) = ab - c$ and $S^2(c) = ba - ab + c \neq c$; thus, $\left(\operatorname{id} - S^2\right)(H_2) \neq 0$. Hence, (34) does not hold for $u = 2$.

The Hopf algebra H in this example is in fact an instance of a general construction of connected graded $\mathbf{k}$-Hopf algebras that are "generic" (in the sense that their structure maps satisfy no relations other than ones that hold in every connected graded $\mathbf{k}$-Hopf algebra). This latter construction will be elaborated upon in future work.

[2]See [Meliot17, §12.1], [HaGuKi10, §7.1] or [GriRei20, §8.1] for the definition of this Hopf algebra. (It is denoted FQSym in [Meliot17] and [GriRei20], and denoted *MPR* in [HaGuKi10].)

[3]See (e.g.) [MeNoTh13, §4.3.2] for a definition of this Hopf algebra.

Remark 2.19. A brave reader might wonder whether the connectedness condition in Corollary 2.15 could be replaced by something weaker – e.g., instead of requiring H to be connected, we might require that the subalgebra H_0 be commutative. However, such a requirement would be insufficient. In fact, let $\mathbf{k} = \mathbb{C}$. Then, for any integer $n > 1$ and any primitive n-th root of unity $q \in \mathbf{k}$, the Taft algebra $H_{n,q}$ defined in [Radfor12, §7.3] can be viewed as a graded Hopf algebra (with $a \in H_0$ and $x \in H_1$) whose subalgebra $H_0 = \mathbf{k}[a] / (a^n - 1)$ is commutative, but whose antipode S does not satisfy $\left(\operatorname{id} - S^2\right)^k (H_1) = 0$ for any $k \in \mathbb{N}$ (since $S^2(x) = q^{-1}x$ and therefore $\left(\operatorname{id} - S^2\right)^k (x) = \left(1 - q^{-1}\right)^k x \neq 0$ because $q^{-1} \neq 1$).

3. Proofs

We shall now prove all statements left unproved above.

3.1. Proof of Theorem 2.1

Proof of Theorem 2.1. We shall prove (7) and (8) by strong induction on u:

Induction step: Fix an integer $n > p$. Assume (as the induction hypothesis) that (7) and (8) hold for all integers $u > p$ satisfying $u < n$. We must prove that (7) and (8) hold for $u = n$. In other words, we must prove that

$$(e - f)^{n-p}(D_n) \subseteq \operatorname{Ker} \delta$$

and

$$(e - f)^{n-p+1}(D_n) = 0.$$

We shall focus on proving the first of these two equalities; the second will then easily follow from (1).

Consider the $\mathbf{k}$-algebras $\operatorname{End} D$ and $\operatorname{End}(D \otimes D)$. (The multiplication in each of these $\mathbf{k}$-algebras is composition of $\mathbf{k}$-linear maps.) Note that $u \otimes v \in \operatorname{End}(D \otimes D)$ for any $u, v \in \operatorname{End} D$.

We have $e, f \in \operatorname{End} D$. Let us define two elements $g \in \operatorname{End} D$ and $h \in \operatorname{End}(D \otimes D)$ by

$$g = e - f \qquad \text{and} \qquad h = e \otimes e - f \otimes f.$$

Then, from $g = e - f$, we obtain

$$\begin{aligned} g \otimes f + e \otimes g &= (e - f) \otimes f + e \otimes (e - f) = e \otimes f - f \otimes f + e \otimes e - e \otimes f \\ &= e \otimes e - f \otimes f = h, \end{aligned}$$

so that

$$h = g \otimes f + e \otimes g.$$

Moreover, (5) rewrites as $g\left(D_1 + D_2 + \cdots + D_p\right) = 0$ (since $g = e - f$). Thus,

$$g\left(D_u\right) = 0 \qquad \text{for all } u \in \{1, 2, \ldots, p\}. \tag{35}$$

Now, recall that the multiplication in the **k**-algebra $\operatorname{End} D$ is composition of maps. Thus, $\alpha\beta = \alpha \circ \beta$ for any $\alpha, \beta \in \operatorname{End} D$. (The same holds for $\operatorname{End}(D \otimes D)$.) Hence, (4) rewrites as $fe = ef$. Therefore, $\underbrace{g}_{=e-f} e = (e-f)\, e = ee - \underbrace{fe}_{=ef} = ee - ef = e \underbrace{(e-f)}_{=g} = eg$ and similarly $gf = fg$. In particular, $ge = eg$ shows that the elements g and e commute. Therefore, for each $i \in \mathbb{N}$ and $j \in \mathbb{N}$, we have

$$g^i e^j = e^j g^i \tag{36}$$

(since powers of commuting elements always commute).

Furthermore, in $\operatorname{End}(D \otimes D)$, we have[4]

$$(g \otimes f)(e \otimes g) = \underbrace{(ge)}_{=eg} \otimes \underbrace{(fg)}_{=gf} = (eg) \otimes (gf) = (e \otimes g)(g \otimes f).$$

In other words, the elements $g \otimes f$ and $e \otimes g$ of $\operatorname{End}(D \otimes D)$ commute. Hence, we can apply the binomial formula to $g \otimes f$ and $e \otimes g$. We thus conclude that each $k \in \mathbb{N}$ satisfies[5]

$$\begin{aligned}
(g \otimes f + e \otimes g)^k &= \sum_{r=0}^{k} \binom{k}{r} \underbrace{(g \otimes f)^r}_{=g^r \otimes f^r} \underbrace{(e \otimes g)^{k-r}}_{=e^{k-r} \otimes g^{k-r}} = \sum_{r=0}^{k} \binom{k}{r} \underbrace{(g^r \otimes f^r)\left(e^{k-r} \otimes g^{k-r}\right)}_{=\left(g^r e^{k-r}\right) \otimes \left(f^r g^{k-r}\right)} \\
&= \sum_{r=0}^{k} \binom{k}{r} \underbrace{\left(g^r e^{k-r}\right)}_{\substack{=e^{k-r} g^r \\ \text{(by (36))}}} \otimes \left(f^r g^{k-r}\right) = \sum_{r=0}^{k} \binom{k}{r} \underbrace{\left(e^{k-r} g^r\right) \otimes \left(f^r g^{k-r}\right)}_{=\left(e^{k-r} \otimes f^r\right)\left(g^r \otimes g^{k-r}\right)} \\
&= \sum_{r=0}^{k} \binom{k}{r} \left(e^{k-r} \otimes f^r\right) \left(g^r \otimes g^{k-r}\right).
\end{aligned} \tag{37}$$

[4] We are here using the fact that

$$(\alpha \otimes \beta)(\gamma \otimes \delta) = (\alpha\gamma) \otimes (\beta\delta) \qquad \text{for all } \alpha, \beta, \gamma, \delta \in \operatorname{End} D.$$

This fact can be verified easily by comparing how the left and the right hand sides transform any given pure tensor $u \otimes v \in D \otimes D$.

[5] We will now again use the fact that

$$(\alpha \otimes \beta)(\gamma \otimes \delta) = (\alpha\gamma) \otimes (\beta\delta) \qquad \text{for all } \alpha, \beta, \gamma, \delta \in \operatorname{End} D,$$

as well as its consequence that

$$(\alpha \otimes \beta)^i = \alpha^i \otimes \beta^i \qquad \text{for all } \alpha, \beta \in \operatorname{End} D \text{ and } i \in \mathbb{N}.$$

For each $k \in \mathbb{N}$ and $r \in \mathbb{N}$, we define a map $h_{k,r} \in \operatorname{End}(D \otimes D)$ by

$$h_{k,r} = \binom{k}{r} \left(e^{k-r} \otimes f^r\right) \left(g^r \otimes g^{k-r}\right). \tag{38}$$

Thus, we can rewrite (37) as follows: Each $k \in \mathbb{N}$ satisfies

$$h^k = \sum_{r=0}^{k} h_{k,r} \tag{39}$$

(because of (38), and because $h = g \otimes f + e \otimes g$).

Subtracting (2) from (3), we obtain[6]

$$(e \otimes e) \circ \delta - (f \otimes f) \circ \delta = \delta \circ e - \delta \circ f = \delta \circ \underbrace{(e - f)}_{=g} = \delta \circ g.$$

Thus,

$$\delta \circ g = (e \otimes e) \circ \delta - (f \otimes f) \circ \delta = \underbrace{(e \otimes e - f \otimes f)}_{=h} \circ \delta = h \circ \delta. \tag{40}$$

Hence, by induction on k, we easily see that

$$\delta \circ g^k = h^k \circ \delta \qquad \text{for each } k \in \mathbb{N}. \tag{41}$$

Our induction hypothesis says that (7) and (8) hold for all integers $u > p$ satisfying $u < n$. In particular, (8) holds for all integers $u > p$ satisfying $u < n$. In other words, for each integer $u > p$ satisfying $u < n$, we have

$$g^{u-p+1}(D_u) = 0 \tag{42}$$

(since $g = e - f$). Hence, it is easy to see that every positive integer $u < n$ and every positive integer $v > u - p$ satisfy

$$g^v(D_u) = 0. \tag{43}$$

(Indeed, if $u > p$, then this follows from (42), because $v \geq u - p + 1$. However, if $u \leq p$, then (43) follows from (35), because $v \geq 1$. Thus, (43) is proved in all possible cases.)

Now, let $k = n - p$. Then, $k > 0$ (since $n > p$), so that $k \in \mathbb{N}$. Furthermore, (41) yields $\delta \circ g^k = h^k \circ \delta$. Thus,

$$\begin{aligned}
\left(\delta \circ g^k\right)(D_n) &= \left(h^k \circ \delta\right)(D_n) = h^k(\delta(D_n)) \\
&\subseteq h^k\left(\sum_{i=1}^{n-1} D_i \otimes D_{n-i}\right) \qquad \text{(by (6))} \\
&= \sum_{i=1}^{n-1} h^k(D_i \otimes D_{n-i}).
\end{aligned} \tag{44}$$

[6]We are using the **k**-linearity of δ here.

We shall now prove that each $i \in \{1,2,\ldots,n-1\}$ and each $r \in \{0,1,\ldots,k\}$ satisfy

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = 0. \tag{45}$$

[*Proof of (45):* Fix $i \in \{1,2,\ldots,n-1\}$ and $r \in \{0,1,\ldots,k\}$. We must prove (45).

We have $i \in \{1,2,\ldots,n-1\}$. Thus, both i and $n-i$ are positive integers that are $< n$. Hence, $n > i$. Also, $\min\{k,i\} > 0$ (since $k > 0$ and $i > 0$).

We have $k = n - p > i - p$ (since $n > i$) and $i > i - p$ (since $p > 0$). In other words, both k and i are $> i - p$. Hence, $\min\{k,i\} > i - p$.

We are in one of the following two cases:

Case 1: We have $r \geq \min\{k,i\}$.

Case 2: We have $r < \min\{k,i\}$.

Let us first consider Case 1. In this case, we have $r \geq \min\{k,i\}$. This entails $r \geq \min\{k,i\} > i - p$. Moreover, the integer r is positive (since $r \geq \min\{k,i\} > 0$). Hence, (43) (applied to $u = i$ and $v = r$) yields $g^r(D_i) = 0$ (since $r > i - p$). Now,

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = \underbrace{g^r(D_i)}_{=0} \otimes g^{k-r}(D_{n-i}) = 0 \otimes g^{k-r}(D_{n-i}) = 0.$$

Thus, (45) is proved in Case 1.

Next, let us consider Case 2. In this case, we have $r < \min\{k,i\}$. In other words, we have $r < k$ and $r < i$. Now, the integer $k - r$ is positive (since $r < k$). Furthermore, from $k = n - p$, we obtain

$$k - r = n - p - \underbrace{r}_{<i} > n - p - i = n - i - p.$$

Hence, (43) (applied to $u = n - i$ and $v = k - r$) yields $g^{k-r}(D_{n-i}) = 0$. Now,

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = g^r(D_i) \otimes \underbrace{g^{k-r}(D_{n-i})}_{=0} = g^r(D_i) \otimes 0 = 0.$$

Thus, (45) is proved in Case 2.

We have now proved (45) in both Cases 1 and 2. Thus, the proof of (45) is complete.]

Using (45), we can easily see that each $i \in \{1,2,\ldots,n-1\}$ and each $r \in \{0,1,\ldots,k\}$ satisfy

$$h_{k,r}(D_i \otimes D_{n-i}) = 0. \tag{46}$$

(Indeed, if $i \in \{1,2,\ldots,n-1\}$ and each $r \in \{0,1,\ldots,k\}$ are arbitrary, then (38) yields

$$\begin{aligned} h_{k,r}(D_i \otimes D_{n-i}) &= \left(\binom{k}{r}\left(e^{k-r} \otimes f^r\right)\left(g^r \otimes g^{k-r}\right)\right)(D_i \otimes D_{n-i}) \\ &= \binom{k}{r}\left(e^{k-r} \otimes f^r\right)\underbrace{\left(\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i})\right)}_{\substack{=0 \\ \text{(by (45))}}} = 0, \end{aligned}$$

and thus (46) is proven.)

Now, each $i \in \{1, 2, \ldots, n-1\}$ satisfies

$$\begin{aligned} h^k \left(D_i \otimes D_{n-i}\right) &= \left(\sum_{r=0}^{k} h_{k,r}\right) \left(D_i \otimes D_{n-i}\right) \qquad \text{(by (39))} \\ &\subseteq \sum_{r=0}^{k} \underbrace{h_{k,r} \left(D_i \otimes D_{n-i}\right)}_{\substack{=0 \\ \text{(by (46))}}} \\ &= 0. \end{aligned} \tag{47}$$

Hence, (44) becomes

$$\left(\delta \circ g^k\right) \left(D_n\right) \subseteq \sum_{i=1}^{n-1} \underbrace{h^k \left(D_i \otimes D_{n-i}\right)}_{\substack{\subseteq 0 \\ \text{(by (47))}}} \subseteq 0.$$

In other words, $\delta \left(g^k \left(D_n\right)\right) \subseteq 0$. Equivalently,

$$g^k \left(D_n\right) \subseteq \operatorname{Ker} \delta.$$

Since $g = e - f$ and $k = n - p$, we can rewrite this as follows:

$$(e - f)^{n-p} \left(D_n\right) \subseteq \operatorname{Ker} \delta. \tag{48}$$

However, we have $\operatorname{Ker} \delta \subseteq \operatorname{Ker} (e - f)$ (by (1)) and therefore $(e - f) (\operatorname{Ker} \delta) = 0$. Thus,

$$(e - f)^{n-p+1} \left(D_n\right) = (e - f) \left(\underbrace{(e - f)^{n-p} \left(D_n\right)}_{\substack{\subseteq \operatorname{Ker} \delta \\ \text{(by (48))}}}\right) \subseteq (e - f) (\operatorname{Ker} \delta) = 0.$$

In other words,

$$(e - f)^{n-p+1} \left(D_n\right) = 0. \tag{49}$$

We have now proved the relations (48) and (49). In other words, (7) and (8) hold for $u = n$. This completes the induction step. Thus, Theorem 2.1 is proven. □

3.2. Proof of Proposition 2.4

Our next goal is to prove Proposition 2.4. We shall work towards this goal by proving a simple lemma:

Lemma 3.1. Let C be any $\mathbf{k}$-coalgebra. Let $a, b, d \in C$ be three elements satisfying $\epsilon(a) = 1$ and $\epsilon(b) = 1$ and $\Delta(d) = d \otimes a + b \otimes d$. Then, $\epsilon(d) = 0$.

We shall later apply Lemma 3.1 to the case when $a = b = 1_C$ (and C is either a connected filtered $\mathbf{k}$-coalgebra or a $\mathbf{k}$-bialgebra, so that 1_C does make sense); however, it is not any harder to prove it in full generality:

Proof of Lemma 3.1. Let γ be the canonical $\mathbf{k}$-module isomorphism $C \otimes \mathbf{k} \to C$, $c \otimes \lambda \mapsto \lambda c$. One of the axioms of a coalgebra says that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow \text{id} & & \downarrow \text{id} \otimes \epsilon \\ C & \xleftarrow[\gamma]{} & C \otimes \mathbf{k} \end{array}$$

is commutative. Thus, $\gamma \circ (\text{id} \otimes \epsilon) \circ \Delta = \text{id}$. Applying both sides of this equality to d, we obtain

$$(\gamma \circ (\text{id} \otimes \epsilon) \circ \Delta)(d) = \text{id}(d) = d.$$

Hence,

$$\begin{aligned} d &= (\gamma \circ (\text{id} \otimes \epsilon) \circ \Delta)(d) = \gamma\left((\text{id} \otimes \epsilon)\left(\underbrace{\Delta(d)}_{= d \otimes a + b \otimes d}\right)\right) \\ &= \gamma\left(\underbrace{(\text{id} \otimes \epsilon)(d \otimes a + b \otimes d)}_{= d \otimes \epsilon(a) + b \otimes \epsilon(d)}\right) = \gamma(d \otimes \epsilon(a) + b \otimes \epsilon(d)) \\ &= \underbrace{\epsilon(a)}_{=1} d + \epsilon(d) b = d + \epsilon(d) b. \end{aligned}$$

Subtracting d from both sides, we obtain $\epsilon(d) b = 0$. Applying the map ϵ to both sides of this equality, we find $\epsilon(\epsilon(d) b) = 0$. In view of

$$\epsilon(\epsilon(d) b) = \epsilon(d) \underbrace{\epsilon(b)}_{=1} = \epsilon(d),$$

this rewrites as $\epsilon(d) = 0$. This proves Lemma 3.1. □

Next, let us define a "reduced identity map" $\overline{\text{id}}$ for any connected filtered $\mathbf{k}$-coalgebra C, and explore some of its properties:

Lemma 3.2. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$. Define a $\mathbf{k}$-linear map $\overline{\text{id}} : C \to C$ by setting

$$\overline{\text{id}}(c) := c - \epsilon(c) 1_C \qquad \text{for each } c \in C.$$

Define a $\mathbf{k}$-linear map $\delta : C \to C \otimes C$ by setting

$$\delta(c) := \Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C \qquad \text{for each } c \in C.$$

Then:

(a) We have $\delta = \left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right) \circ \Delta$.

(b) We have $\overline{\operatorname{id}}(C_{\leq n}) \subseteq C_{\leq n}$ for each $n \in \mathbb{N}$.

(c) We have $\overline{\operatorname{id}}(C_{\leq 0}) = 0$.

Proof of Lemma 3.2. **(a)** Let $c \in C$. Write the tensor $\Delta(c) \in C \otimes C$ in the form

$$\Delta(c) = \sum_{i=1}^{m} c_i \otimes d_i \tag{50}$$

for some $m \in \mathbb{N}$, some $c_1, c_2, \ldots, c_m \in C$ and some $d_1, d_2, \ldots, d_m \in C$.

Let γ be the canonical $\mathbf{k}$-module isomorphism $C \otimes \mathbf{k} \to C$, $c \otimes \lambda \mapsto \lambda c$. Let γ' be the canonical $\mathbf{k}$-module isomorphism $\mathbf{k} \otimes C \to C$, $\lambda \otimes c \mapsto \lambda c$. As we know from our above proof of Lemma 3.1, we have

$$\gamma \circ (\operatorname{id} \otimes \epsilon) \circ \Delta = \operatorname{id}. \tag{51}$$

Similarly, we have

$$\gamma' \circ (\epsilon \otimes \operatorname{id}) \circ \Delta = \operatorname{id}. \tag{52}$$

Applying the map $\operatorname{id} \otimes \epsilon$ to both sides of the equality (50), we obtain

$$(\operatorname{id} \otimes \epsilon)(\Delta(c)) = (\operatorname{id} \otimes \epsilon)\left(\sum_{i=1}^{m} c_i \otimes d_i\right) = \sum_{i=1}^{m} c_i \otimes \epsilon(d_i).$$

Applying the map γ to both sides of this equality, we obtain

$$\gamma((\operatorname{id} \otimes \epsilon)(\Delta(c))) = \gamma\left(\sum_{i=1}^{m} c_i \otimes \epsilon(d_i)\right) = \sum_{i=1}^{m} \epsilon(d_i) c_i$$

(by the definition of γ). Comparing this with

$$\gamma((\operatorname{id} \otimes \epsilon)(\Delta(c))) = \underbrace{(\gamma \circ (\operatorname{id} \otimes \epsilon) \circ \Delta)}_{\substack{= \operatorname{id} \\ (\text{by } (51))}}(c) = \operatorname{id}(c) = c,$$

we obtain

$$\sum_{i=1}^{m} \epsilon(d_i) c_i = c. \tag{53}$$

An analogous argument (but using $\epsilon \otimes \operatorname{id}$ instead of $\operatorname{id} \otimes \epsilon$, and using (52) instead of (51)) yields

$$\sum_{i=1}^{m} \epsilon(c_i) d_i = c. \tag{54}$$

Hence, $c = \sum_{i=1}^{m} \epsilon(c_i) d_i$. Applying the map ϵ to both sides of this latter equality, we obtain

$$\epsilon(c) = \epsilon\left(\sum_{i=1}^{m} \epsilon(c_i) d_i\right) = \sum_{i=1}^{m} \epsilon(c_i) \epsilon(d_i) \tag{55}$$

(since the map ϵ is $\mathbf{k}$-linear).

Now, applying the map $\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}$ to both sides of the equality (50), we obtain

$$\begin{aligned}
&\left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right)(\Delta(c)) \\
&= \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right)\left(\sum_{i=1}^{m} c_i \otimes d_i\right) = \sum_{i=1}^{m} \underbrace{\overline{\mathrm{id}}(c_i)}_{\substack{=c_i - \epsilon(c_i)1_C \\ \text{(by the definition of } \overline{\mathrm{id}}\text{)}}} \otimes \underbrace{\overline{\mathrm{id}}(d_i)}_{\substack{=d_i - \epsilon(d_i)1_C \\ \text{(by the definition of } \overline{\mathrm{id}}\text{)}}} \\
&= \sum_{i=1}^{m} \underbrace{(c_i - \epsilon(c_i)1_C) \otimes (d_i - \epsilon(d_i)1_C)}_{=c_i \otimes d_i - c_i \otimes (\epsilon(d_i)1_C) - (\epsilon(c_i)1_C) \otimes d_i + (\epsilon(c_i)1_C) \otimes (\epsilon(d_i)1_C)} \\
&= \sum_{i=1}^{m} \left(c_i \otimes d_i - c_i \otimes (\epsilon(d_i)1_C) - (\epsilon(c_i)1_C) \otimes d_i + (\epsilon(c_i)1_C) \otimes (\epsilon(d_i)1_C)\right) \\
&= \underbrace{\sum_{i=1}^{m} c_i \otimes d_i}_{\substack{=\Delta(c) \\ \text{(by (50))}}} - \underbrace{\sum_{i=1}^{m} c_i \otimes (\epsilon(d_i)1_C)}_{=\left(\sum_{i=1}^{m} \epsilon(d_i)c_i\right) \otimes 1_C} - \underbrace{\sum_{i=1}^{m} (\epsilon(c_i)1_C) \otimes d_i}_{=1_C \otimes \left(\sum_{i=1}^{m} \epsilon(c_i)d_i\right)} + \underbrace{\sum_{i=1}^{m} (\epsilon(c_i)1_C) \otimes (\epsilon(d_i)1_C)}_{=\left(\sum_{i=1}^{m} \epsilon(c_i)\epsilon(d_i)\right)1_C \otimes 1_C} \\
&= \Delta(c) - \underbrace{\left(\sum_{i=1}^{m} \epsilon(d_i) c_i\right)}_{\substack{=c \\ \text{(by (53))}}} \otimes 1_C - 1_C \otimes \underbrace{\left(\sum_{i=1}^{m} \epsilon(c_i) d_i\right)}_{\substack{=c \\ \text{(by (54))}}} + \underbrace{\left(\sum_{i=1}^{m} \epsilon(c_i) \epsilon(d_i)\right)}_{\substack{=\epsilon(c) \\ \text{(by (55))}}} 1_C \otimes 1_C \\
&= \Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C = \delta(c)
\end{aligned}$$

(by the definition of δ). Thus,

$$\delta(c) = \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right)(\Delta(c)) = \left(\left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta\right)(c). \tag{56}$$

Forget that we fixed c. We thus have proved (56) for each $c \in C$. In other words, $\delta = \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta$. This proves Lemma 3.2 **(a)**.

(b) Let $n \in \mathbb{N}$. Definition 2.3 **(b)** yields $1_C \in C_{\leq 0} \subseteq C_{\leq n}$ (by (9)). Now, for each $c \in C_{\leq n}$, the definition of $\overline{\mathrm{id}}$ yields

$$\overline{\mathrm{id}}(c) = \underbrace{c}_{\in C_{\leq n}} - \epsilon(c) \underbrace{1_C}_{\in C_{\leq n}} \in C_{\leq n} - \epsilon(c) C_{\leq n} \subseteq C_{\leq n}.$$

In other words, we have $\overline{\mathrm{id}}(C_{\leq n}) \subseteq C_{\leq n}$. This proves Lemma 3.2 **(b)**.

(c) The filtered $\mathbf{k}$-coalgebra C is connected. In other words, the restriction $\epsilon\mid_{C_{\leq 0}}$ is a $\mathbf{k}$-module isomorphism from $C_{\leq 0}$ to $\mathbf{k}$ (by Definition 2.3 **(a)**). Thus, this restriction $\epsilon\mid_{C_{\leq 0}}$ is injective. Also, Definition 2.3 **(b)** yields $1_C \in C_{\leq 0}$ and $\epsilon(1_C) = 1_{\mathbf{k}}$.

Now, let $c \in C_{\leq 0}$. Set $d = \epsilon(c)\, 1_C$. Then, $d \in C_{\leq 0}$ (since $1_C \in C_{\leq 0}$). From $d = \epsilon(c)\, 1_C$, we obtain

$$\epsilon(d) = \epsilon(\epsilon(c)\, 1_C) = \epsilon(c) \underbrace{\epsilon(1_C)}_{=1_{\mathbf{k}}} = \epsilon(c).$$

In other words, $\left(\epsilon\mid_{C_{\leq 0}}\right)(d) = \left(\epsilon\mid_{C_{\leq 0}}\right)(c)$ (since both d and c belong to $C_{\leq 0}$). Since $\epsilon\mid_{C_{\leq 0}}$ is injective, this entails $d = c$. Therefore, $c = d = \epsilon(c)\, 1_C$. Now, the definition of $\overline{\operatorname{id}}$ yields $\overline{\operatorname{id}}(c) = c - \epsilon(c)\, 1_C = 0$ (since $c = \epsilon(c)\, 1_C$).

Forget that we have fixed c. We thus have shown that $\overline{\operatorname{id}}(c) = 0$ for each $c \in C_{\leq 0}$. In other words, $\overline{\operatorname{id}}(C_{\leq 0}) = 0$. This proves Lemma 3.2 **(c)**. □

Proof of Proposition 2.4. **(a)** Define a $\mathbf{k}$-linear map $\overline{\operatorname{id}} : C \to C$ as in Lemma 3.2.

Now, let $n > 0$ be an integer. Lemma 3.2 **(a)** yields $\delta = \left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right) \circ \Delta$. Thus,

$$\begin{aligned}
&\delta(C_{\leq n}) \\
&= \left(\left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right) \circ \Delta\right)(C_{\leq n}) = \left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right) \underbrace{(\Delta(C_{\leq n}))}_{\substack{\subseteq \sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i} \\ \text{(by (11))}}} \\
&\subseteq \left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right) \left(\sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i}\right) = \sum_{i=0}^{n} \underbrace{\left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right)(C_{\leq i} \otimes C_{\leq n-i})}_{=\overline{\operatorname{id}}(C_{\leq i}) \otimes \overline{\operatorname{id}}(C_{\leq n-i})} \\
&= \sum_{i=0}^{n} \overline{\operatorname{id}}(C_{\leq i}) \otimes \overline{\operatorname{id}}(C_{\leq n-i}) \\
&= \underbrace{\overline{\operatorname{id}}(C_{\leq 0})}_{\substack{=0 \\ \text{(by Lemma 3.2 (c))}}} \otimes \overline{\operatorname{id}}(C_{\leq n}) + \sum_{i=1}^{n-1} \overline{\operatorname{id}}(C_{\leq i}) \otimes \overline{\operatorname{id}}(C_{\leq n-i}) + \overline{\operatorname{id}}(C_{\leq n}) \otimes \underbrace{\overline{\operatorname{id}}(C_{\leq 0})}_{\substack{=0 \\ \text{(by Lemma 3.2 (c))}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addends for } i = 0 \text{ and for } i = n \text{ from the} \\ \text{sum (and these are indeed two distinct addends, since } n > 0)\end{array}\right) \\
&= \underbrace{0 \otimes \overline{\operatorname{id}}(C_{\leq n})}_{=0} + \sum_{i=1}^{n-1} \underbrace{\overline{\operatorname{id}}(C_{\leq i})}_{\substack{\subseteq C_{\leq i} \\ \text{(by Lemma 3.2 (b))}}} \otimes \underbrace{\overline{\operatorname{id}}(C_{\leq n-i})}_{\substack{\subseteq C_{\leq n-i} \\ \text{(by Lemma 3.2 (b))}}} + \underbrace{\overline{\operatorname{id}}(C_{\leq n}) \otimes 0}_{=0} \\
&\subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i}.
\end{aligned}$$

This proves Proposition 2.4 **(a)**.

(b) Let $f : C \to C$ be a $\mathbf{k}$-coalgebra homomorphism satisfying $f(1_C) = 1_C$. Thus, f is a $\mathbf{k}$-coalgebra homomorphism; in other words, f is a $\mathbf{k}$-linear map satisfying $(f \otimes f) \circ \Delta = \Delta \circ f$ and $\epsilon = \epsilon \circ f$.

Let $c \in C$. The definition of δ yields $\delta(c) = \Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C$. Applying the map $f \otimes f$ to both sides of this equality, we obtain

$$\begin{aligned}
&(f \otimes f)(\delta(c)) \\
&= (f \otimes f)(\Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C) \\
&= \underbrace{(f \otimes f)(\Delta(c))}_{=((f\otimes f)\circ\Delta)(c)} - \underbrace{(f \otimes f)(c \otimes 1_C)}_{=f(c)\otimes f(1_C)} - \underbrace{(f \otimes f)(1_C \otimes c)}_{=f(1_C)\otimes f(c)} + \epsilon(c) \underbrace{(f \otimes f)(1_C \otimes 1_C)}_{=f(1_C)\otimes f(1_C)} \\
&= \underbrace{((f \otimes f) \circ \Delta)}_{=\Delta\circ f}(c) - f(c) \otimes \underbrace{f(1_C)}_{=1_C} - \underbrace{f(1_C)}_{=1_C} \otimes f(c) + \underbrace{\epsilon}_{=\epsilon\circ f}(c) \underbrace{f(1_C)}_{=1_C} \otimes \underbrace{f(1_C)}_{=1_C} \\
&= \underbrace{(\Delta \circ f)(c)}_{=\Delta(f(c))} - f(c) \otimes 1_C - 1_C \otimes f(c) + \underbrace{(\epsilon \circ f)(c)}_{=\epsilon(f(c))} 1_C \otimes 1_C \\
&= \Delta(f(c)) - f(c) \otimes 1_C - 1_C \otimes f(c) + \epsilon(f(c)) 1_C \otimes 1_C.
\end{aligned}$$

Comparing this with

$$(\delta \circ f)(c) = \delta(f(c)) = \Delta(f(c)) - f(c) \otimes 1_C - 1_C \otimes f(c) + \epsilon(f(c)) 1_C \otimes 1_C \qquad \text{(by the definition of } \delta\text{)},$$

we obtain $(\delta \circ f)(c) = (f \otimes f)(\delta(c)) = ((f \otimes f) \circ \delta)(c)$.

Forget that we fixed c. We thus have shown that $(\delta \circ f)(c) = ((f \otimes f) \circ \delta)(c)$ for each $c \in C$. In other words, $\delta \circ f = (f \otimes f) \circ \delta$. This proves Proposition 2.4 **(b)**.

(c) Definition 2.3 **(b)** yields $1_C \in C_{\leq 0}$ and $\epsilon(1_C) = 1_{\mathbf{k}}$.

Let $c \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. Thus, $\delta(c) = 0$ and $\epsilon(c) = 0$. From $\delta(c) = 0$, we obtain

$$\begin{aligned}
0 = \delta(c) &= \Delta(c) - c \otimes 1_C - 1_C \otimes c + \underbrace{\epsilon(c)}_{=0} 1_C \otimes 1_C \qquad \text{(by the definition of } \delta\text{)} \\
&= \Delta(c) - c \otimes 1_C - 1_C \otimes c + \underbrace{0 \cdot 1_C \otimes 1_C}_{=0} \\
&= \Delta(c) - c \otimes 1_C - 1_C \otimes c.
\end{aligned}$$

In other words, $\Delta(c) = c \otimes 1_C + 1_C \otimes c$. In other words, the element c of C is primitive. In other words, $c \in \operatorname{Prim} C$.

Forget that we fixed c. We thus have shown that $c \in \operatorname{Prim} C$ for each $c \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. In other words, $(\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) \subseteq \operatorname{Prim} C$.

Now, let $d \in \operatorname{Prim} C$. Thus, the element d of C is primitive. In other words, $\Delta(d) = d \otimes 1_C + 1_C \otimes d$. Hence, Lemma 3.1 (applied to 1_C and 1_C instead of a and b) yields $\epsilon(d) = 0$ (since $\epsilon(1_C) = 1_{\mathbf{k}} = 1$). Hence, $d \in \operatorname{Ker} \epsilon$.

Furthermore, the definition of δ yields

$$\delta(d) = \underbrace{\Delta(d) - d \otimes 1_C - 1_C \otimes d}_{\substack{=0 \\ (\text{since } \Delta(d) = d \otimes 1_C + 1_C \otimes d)}} + \underbrace{\epsilon(d)}_{=0} 1_C \otimes 1_C = 0.$$

Hence, $d \in \operatorname{Ker} \delta$. Combining this with $d \in \operatorname{Ker} \epsilon$, we obtain $d \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$.

Forget that we fixed d. We thus have shown that $d \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$ for each $d \in \operatorname{Prim} C$. In other words, $\operatorname{Prim} C \subseteq (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. Combining this with $(\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) \subseteq \operatorname{Prim} C$, we obtain $\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. This proves Proposition 2.4 **(c)**.

(d) This follows from Proposition 2.4 **(c)**, since both $\operatorname{Ker} \delta$ and $\operatorname{Ker} \epsilon$ are $\mathbf{k}$-submodules of C.

(e) Definition 2.3 **(b)** yields $1_C \in C_{\leq 0} \subseteq C_{\leq 1}$ (by (9)). Thus,

$$\delta(1_C) \in \delta(C_{\leq 1}) \subseteq \sum_{i=1}^{1-1} C_{\leq i} \otimes C_{\leq 1-i} \qquad \text{(by Proposition 2.4 **(a)**, applied to } n = 1\text{)}$$
$$= (\text{empty sum}) = 0.$$

In other words, $\delta(1_C) = 0$. Hence, $1_C \in \operatorname{Ker} \delta$. Thus, $\mathbf{k} \cdot 1_C \subseteq \operatorname{Ker} \delta$.

Also, Proposition 2.4 **(c)** yields

$$\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) \subseteq \operatorname{Ker} \delta.$$

Hence,

$$\underbrace{\mathbf{k} \cdot 1_C}_{\subseteq \operatorname{Ker} \delta} + \underbrace{\operatorname{Prim} C}_{\subseteq \operatorname{Ker} \delta} \subseteq \operatorname{Ker} \delta + \operatorname{Ker} \delta \subseteq \operatorname{Ker} \delta. \tag{57}$$

Definition 2.3 **(b)** yields $1_C \in C_{\leq 0}$ and $\epsilon(1_C) = 1_{\mathbf{k}}$.

Let $u \in \operatorname{Ker} \delta$. Thus, $\delta(u) = 0$. Set $v = u - \epsilon(u) 1_C$. Thus,

$$\delta(v) = \delta(u - \epsilon(u) 1_C) = \underbrace{\delta(u)}_{=0} - \epsilon(u) \underbrace{\delta(1_C)}_{=0} = 0 - \epsilon(u) 0 = 0,$$

so that $v \in \operatorname{Ker} \delta$. Furthermore, from $v = u - \epsilon(u) 1_C$, we obtain

$$\epsilon(v) = \epsilon(u - \epsilon(u) 1_C) = \epsilon(u) - \epsilon(u) \underbrace{\epsilon(1_C)}_{=1_{\mathbf{k}}} = \epsilon(u) - \epsilon(u) = 0,$$

so that $v \in \operatorname{Ker} \epsilon$. Combining this with $v \in \operatorname{Ker} \delta$, we obtain $v \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) = \operatorname{Prim} C$ (by Proposition 2.4 **(c)**). Now, from $v = u - \epsilon(u) 1_C$, we obtain

$$u = \underbrace{\epsilon(u)}_{\in \mathbf{k}} 1_C + \underbrace{v}_{\in \operatorname{Prim} C} \in \mathbf{k} \cdot 1_C + \operatorname{Prim} C.$$

Forget that we fixed u. We thus have shown that $u \in \mathbf{k} \cdot 1_C + \operatorname{Prim} C$ for each $u \in \operatorname{Ker} \delta$. In other words,

$$\operatorname{Ker} \delta \subseteq \mathbf{k} \cdot 1_C + \operatorname{Prim} C.$$

Combining this with (57), we obtain $\operatorname{Ker} \delta = \mathbf{k} \cdot 1_C + \operatorname{Prim} C$. This proves Proposition 2.4 **(e)**. □

3.3. Proofs of the corollaries from Section 2.2

Proof of Corollary 2.5. We have $(e-f)(1_C) = \underbrace{e(1_C)}_{=1_C} - \underbrace{f(1_C)}_{=1_C} = 1_C - 1_C = 0$. Hence, $1_C \in \operatorname{Ker}(e-f)$, so that $\mathbf{k} \cdot 1_C \subseteq \operatorname{Ker}(e-f)$.

Define the $\mathbf{k}$-linear map $\delta : C \to C \otimes C$ as in Proposition 2.4. The map f is a $\mathbf{k}$-coalgebra homomorphism satisfying $f(1_C) = 1_C$. Thus, Proposition 2.4 **(b)** yields that $(f \otimes f) \circ \delta = \delta \circ f$. The same argument (applied to e instead of f) yields $(e \otimes e) \circ \delta = \delta \circ e$. Moreover, Proposition 2.4 **(e)** yields

$$\operatorname{Ker} \delta = \underbrace{\mathbf{k} \cdot 1_C}_{\subseteq \operatorname{Ker}(e-f)} + \underbrace{\operatorname{Prim} C}_{\substack{\subseteq \operatorname{Ker}(e-f) \\ \text{(by (12))}}} \subseteq \operatorname{Ker}(e-f) + \operatorname{Ker}(e-f) \subseteq \operatorname{Ker}(e-f).$$

However, Proposition 2.4 **(a)** shows that

$$\delta(C_{\leq n}) \subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n > 0.$$

Hence,

$$\delta(C_{\leq n}) \subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n > p.$$

Moreover, (9) yields $C_{\leq 1} + C_{\leq 2} + \cdots + C_{\leq p} \subseteq C_{\leq p}$ (since $C_{\leq p}$ is a $\mathbf{k}$-module). Therefore,

$$(e-f)\left(C_{\leq 1} + C_{\leq 2} + \cdots + C_{\leq p}\right) \subseteq (e-f)\left(C_{\leq p}\right) = 0$$

(by (14)), so that $(e-f)\left(C_{\leq 1} + C_{\leq 2} + \cdots + C_{\leq p}\right) = 0$.

Hence, Theorem 2.1 (applied to $D = C$ and $D_i = C_{\leq i}$) shows that for any integer $u > p$, we have

$$(e-f)^{u-p}(C_{\leq u}) \subseteq \operatorname{Ker} \delta \tag{58}$$

and

$$(e-f)^{u-p+1}(C_{\leq u}) = 0. \tag{59}$$

We are now close to proving Corollary 2.5. Let us begin with part **(a)**:

(a) The map f is a $\mathbf{k}$-coalgebra homomorphism, and thus satisfies $\epsilon \circ f = \epsilon$ (by the definition of a $\mathbf{k}$-coalgebra homomorphism). Similarly, $\epsilon \circ e = \epsilon$. Since the map ϵ is $\mathbf{k}$-linear, we have

$$\epsilon \circ (e-f) = \underbrace{\epsilon \circ e}_{=\epsilon} - \underbrace{\epsilon \circ f}_{=\epsilon} = \epsilon - \epsilon = 0.$$

Now, let $u > p$ be an integer. Thus, $(e-f)^{u-p} = (e-f) \circ (e-f)^{u-p-1}$. Hence,

$$\epsilon \circ (e-f)^{u-p} = \underbrace{\epsilon \circ (e-f)}_{=0} \circ (e-f)^{u-p-1} = 0 \circ (e-f)^{u-p-1} = 0.$$

Therefore, $(e-f)^{u-p}(C_{\leq u}) \subseteq \operatorname{Ker} \epsilon$. Combining this relation with (58), we obtain $(e-f)^{u-p}(C_{\leq u}) \subseteq (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) = \operatorname{Prim} C$ (by Proposition 2.4 **(c)**). This proves Corollary 2.5 **(a)**.

(b) Let $u \geq p$ be an integer. We must prove that $(e-f)^{u-p+1}(C_{\leq u}) = 0$. If $u > p$, then this follows from (59). Thus, for the rest of this proof, we WLOG assume that we don't have $u > p$. Hence, $u = p$ (since we have $u \geq p$). Thus,

$$(e-f)^{u-p+1}(C_{\leq u}) = \underbrace{(e-f)^{p-p+1}}_{=(e-f)^1=e-f}\left(C_{\leq p}\right) = (e-f)\left(C_{\leq p}\right) = 0$$

(by (14)). This proves Corollary 2.5 **(b)**. □

Proof of Corollary 2.6. Define the **k**-linear map $\delta : C \to C \otimes C$ as in Proposition 2.4. Just as we did in the proof of Corollary 2.5, we can show that $\operatorname{Ker} \delta \subseteq \operatorname{Ker}(e-f)$. However, Proposition 2.4 **(a)** (applied to $n = 1$) yields

$$\delta(C_{\leq 1}) \subseteq \sum_{i=1}^{1-1} C_{\leq i} \otimes C_{\leq 1-i} = (\text{empty sum}) = 0.$$

Hence, $C_{\leq 1} \subseteq \operatorname{Ker} \delta \subseteq \operatorname{Ker}(e-f)$. In other words,

$$(e-f)(C_{\leq 1}) = 0. \tag{60}$$

Hence, we can apply Corollary 2.5 to $p = 1$.

Therefore, applying Corollary 2.5 **(a)** to $p = 1$, we conclude the following: For any integer $u > 1$, we have $(e-f)^{u-1}(C_{\leq u}) \subseteq \operatorname{Prim} C$. This proves Corollary 2.6 **(a)**. It remains to prove Corollary 2.6 **(b)**:

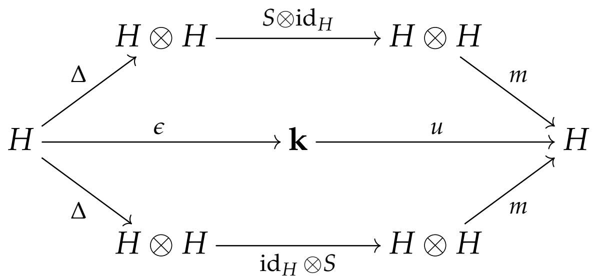
(b) Let u be a positive integer. Thus, $u \geq 1$. Hence, Corollary 2.5 **(b)** (applied to $p = 1$) shows that $(e-f)^{u}(C_{\leq u}) = 0$ (since we know that we can apply Corollary 2.5 to $p = 1$). This proves Corollary 2.6 **(b)**. □

Proof of Corollary 2.7. Clearly, $\operatorname{id} : C \to C$ is a **k**-coalgebra homomorphism such that $\operatorname{id}(1_C) = 1_C$. Furthermore, $f \circ \operatorname{id} = f = \operatorname{id} \circ f$. Hence, we can apply Corollary 2.6 to $e = \operatorname{id}$. As a result, we obtain precisely the claims of Corollary 2.7. □

3.4. Proofs for Section 2.3

Before we prove the claims left unproved in Section 2.3, let us recall the defining property of the antipode of a Hopf algebra (see, e.g., [GriRei20, (1.4.3)]):

Remark 3.3. Let H be a $\mathbf{k}$-Hopf algebra with antipode S. Let 1_H denote the unity of the $\mathbf{k}$-algebra H. Let $m : H \otimes H \to H$ be the $\mathbf{k}$-linear map that sends each pure tensor $x \otimes y \in H \otimes H$ to the product $xy \in H$. Let $u : \mathbf{k} \to H$ be the $\mathbf{k}$-linear map that sends $1_{\mathbf{k}}$ to 1_H. Then, the diagram



commutes. In other words, we have

$$m \circ (S \otimes \mathrm{id}_H) \circ \Delta = u \circ \epsilon \qquad \text{and} \tag{61}$$

$$m \circ (\mathrm{id}_H \otimes S) \circ \Delta = u \circ \epsilon. \tag{62}$$

Proof of Lemma 2.12. All claims in Lemma 2.12 are folklore (see, e.g., [Grinbe17, proof of Lemma 37.8]); we will thus just sketch the proofs.

(a) Let $T : H \otimes H \to H \otimes H$ be the $\mathbf{k}$-linear map that sends each pure tensor $x \otimes y \in H \otimes H$ to $y \otimes x$. This map T is known as the *twist map*. It satisfies $T^2 = \mathrm{id}$. Furthermore, it is easy to see that any two $\mathbf{k}$-linear maps $\alpha, \beta \in \operatorname{End} H$ satisfy

$$(\alpha \otimes \beta) \circ T = T \circ (\beta \otimes \alpha). \tag{63}$$

Furthermore, it is well-known (see, e.g., [Abe80, Theorem 2.1.4 (iii) and (iv)] or [GriRei20, Exercise 1.4.28] or, in an equivalent form, [Radfor12, Proposition 7.1.9 (b)]) that the antipode S of H is a $\mathbf{k}$-coalgebra anti-endomorphism, i.e., that it satisfies

$$\Delta \circ S = T \circ (S \otimes S) \circ \Delta \qquad \text{and} \qquad \epsilon \circ S = \epsilon.$$

Now,

$$\begin{aligned}
\Delta \circ \underbrace{S^2}_{=S \circ S} &= \underbrace{\Delta \circ S}_{=T \circ (S \otimes S) \circ \Delta} \circ S = T \circ (S \otimes S) \circ \underbrace{\Delta \circ S}_{=T \circ (S \otimes S) \circ \Delta} \\
&= T \circ \underbrace{(S \otimes S) \circ T}_{\substack{=T \circ (S \otimes S) \\ \text{(by (63))}}} \circ (S \otimes S) \circ \Delta = \underbrace{T \circ T}_{=T^2 = \mathrm{id}} \circ \underbrace{(S \otimes S) \circ (S \otimes S)}_{=(S \circ S) \otimes (S \circ S)} \circ \Delta \\
&= ((S \circ S) \otimes (S \circ S)) \circ \Delta = \left(S^2 \otimes S^2\right) \circ \Delta \qquad \left(\text{since } S \circ S = S^2\right)
\end{aligned}$$

and

$$\epsilon \circ \underbrace{S^2}_{=S \circ S} = \underbrace{\epsilon \circ S}_{=\epsilon} \circ S = \epsilon \circ S = \epsilon.$$

These two equalities show that S^2 is a $\mathbf{k}$-coalgebra homomorphism (since S^2 is a $\mathbf{k}$-linear map from H to H). This proves Lemma 2.12 **(a)**.

(b) The axioms of a $\mathbf{k}$-bialgebra yield $\epsilon(1_H) = 1_{\mathbf{k}}$ and $\Delta(1_H) = 1_H \otimes 1_H$.

Define the maps m and u as in Remark 3.3. Applying both sides of the equality (61) to 1_H, we obtain

$$(m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(1_H) = (u \circ \epsilon)(1_H) = u\left(\underbrace{\epsilon(1_H)}_{=1_{\mathbf{k}}}\right) = u(1_{\mathbf{k}}) = 1_H$$

(by the definition of u). Hence,

$$1_H = (m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(1_H) = S(1_H)$$

(by a straightforward computation, using $\Delta(1_H) = 1_H \otimes 1_H$). This proves Lemma 2.12 **(b)**.

(c) This is even more well-known than the rest of the lemma (see, e.g., [GriRei20, Proposition 1.4.17]).

Let x be a primitive element of H. Thus, $\Delta(x) = x \otimes 1_H + 1_H \otimes x$ (by the definition of "primitive"). Moreover, the axioms of a $\mathbf{k}$-bialgebra yield $\epsilon(1_H) = 1$. Hence, Lemma 3.1 (applied to $C = H$, $a = 1_H$, $b = 1_H$ and $d = x$) yields $\epsilon(x) = 0$.

Define the maps m and u as in Remark 3.3. Applying both sides of the equality (61) to x, we obtain

$$(m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) = (u \circ \epsilon)(x) = u\left(\underbrace{\epsilon(x)}_{=0}\right) = u(0) = 0.$$

Hence,

$$0 = (m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) = S(x) + x$$

(by a straightforward computation, using $\Delta(x) = x \otimes 1_H + 1_H \otimes x$ and $S(1_H) = 1_H$). Hence, $S(x) = -x$. This proves Lemma 2.12 **(c)**.

(d) Let x be a primitive element of H. Then, Lemma 2.12 **(c)** yields $S(x) = -x$. Applying the map S to this equality, we obtain $S(S(x)) = S(-x) = -\underbrace{S(x)}_{=-x} = -(-x) = x$. Hence, $S^2(x) = S(S(x)) = x$. This proves Lemma 2.12 **(d)**. □

Proof of Corollary 2.13. Lemma 2.12 **(b)** yields $S(1_H) = 1_H$. Hence, it easily follows that $S^2(1_H) = 1_H$. Moreover, Lemma 2.12 **(a)** yields that the map $S^2 : H \to H$ is a $\mathbf{k}$-coalgebra homomorphism. Of course, the map $\mathrm{id} : H \to H$ is a $\mathbf{k}$-coalgebra homomorphism as well, and satisfies $\mathrm{id}(1_H) = 1_H$. Furthermore, every $x \in \operatorname{Prim} H$ is a primitive element of H and therefore satisfies

$$\left(\mathrm{id} - S^2\right)(x) = x - \underbrace{S^2(x)}_{\substack{=x \\ \text{(by Lemma 2.12 (d))}}} = x - x = 0$$

and thus $x \in \operatorname{Ker}\left(\operatorname{id} - S^2\right)$. Hence, we have $\operatorname{Prim} H \subseteq \operatorname{Ker}\left(\operatorname{id} - S^2\right)$. Moreover, $S^2 \circ \operatorname{id} = S^2 = \operatorname{id} \circ S^2$. Furthermore, p is a positive integer and satisfies $\left(\operatorname{id} - S^2\right)\left(H_{\leq p}\right) = 0$ (by (17)). Hence, we can apply Corollary 2.5 to $C = H$ and $C_{\leq i} = H_{\leq i}$ and $e = \operatorname{id}$ and $f = S^2$. Doing so, we immediately obtain

- that (18) holds for any integer $u > p$ (by applying Corollary 2.5 **(a)**), and
- that (20) holds for any integer $u \geq p$ (by applying Corollary 2.5 **(b)**).

It thus remains to prove that (19) holds for any integer $u > p$. So let us do this now.

First, we shall show that $(\operatorname{id} + S)(\operatorname{Prim} H) = 0$.

Indeed, each $x \in \operatorname{Prim} H$ is a primitive element of H and therefore satisfies

$$(\operatorname{id} + S)(x) = x + \underbrace{S(x)}_{\substack{= -x \\ \text{(by Lemma 2.12 (c))}}} = x + (-x) = 0.$$

In other words, we have $(\operatorname{id} + S)(\operatorname{Prim} H) = 0$.

Now, let $u > p$ be any integer. We must prove (19). We have

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-p}\right)\left(H_{\leq u}\right) = (\operatorname{id} + S)\left(\underbrace{\left(\operatorname{id} - S^2\right)^{u-p}\left(H_{\leq u}\right)}_{\substack{\subseteq \operatorname{Prim} H \\ \text{(by (18))}}}\right) \subseteq (\operatorname{id} + S)(\operatorname{Prim} H) = 0.$$

Therefore, $\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-p}\right)\left(H_{\leq u}\right) = 0$. This proves (19). Thus, the proof of Corollary 2.13 is complete. □

Proof of Corollary 2.14. In our above proof of Corollary 2.13, we have already shown that

- we have $S^2(1_H) = 1_H$;
- the map $S^2 : H \to H$ is a $\mathbf{k}$-coalgebra homomorphism;
- we have $\operatorname{Prim} H \subseteq \operatorname{Ker}\left(\operatorname{id} - S^2\right)$.

Hence, we can apply Corollary 2.7 to $C = H$ and $C_{\leq i} = H_{\leq i}$ and $f = S^2$. Doing so, we immediately obtain

- that (21) holds for any integer $u > 1$ (by applying Corollary 2.7 **(a)**), and
- that (23) holds for any positive integer u (by applying Corollary 2.7 **(b)**).

It thus remains to prove that (22) holds for any integer $u > 1$. But this can be deduced from (21) in the same way as we deduced (19) from (18) in our above proof of Corollary 2.13. Thus, the proof of Corollary 2.14 is complete. □

3.5. Proofs for Section 2.4

We shall next focus on proving the claims left unproven in Section 2.4. Before we do so, let us first collect a few basic properties of connected graded Hopf algebras into a lemma for convenience:

Lemma 3.4. Let H be a connected graded $\mathbf{k}$-Hopf algebra with unity 1_H and antipode S. Then:

(a) If n is a positive integer, and if x is an element of H_n, then we have

$$\Delta(x) = 1_H \otimes x + x \otimes 1_H + w \qquad \text{for some } w \in \sum_{k=1}^{n-1} H_k \otimes H_{n-k}.$$

(b) We have $H_1 \subseteq \operatorname{Prim} H$.

(c) We have $S(ab) = ba$ for any $a, b \in H_1$.

Proof of Lemma 3.4. **(a)** This is well-known; see [GriRei20, Exercise 1.3.20 (h)] or [Mancho06, Proposition II.1.1] or [Preiss16, Theorem 2.18] for a proof.

(b) We need to show that each $x \in H_1$ is primitive, i.e., satisfies $\Delta(x) = x \otimes 1_H + 1_H \otimes x$. But this follows easily by applying Lemma 3.4 **(a)** to $n = 1$ (and observing that the sum $\sum_{k=1}^{n-1} H_k \otimes H_{n-k}$ is empty for $n = 1$).

(c) Let $a, b \in H_1$. Then, $a \in H_1 \subseteq \operatorname{Prim} H$ (by Lemma 3.4 **(b)**). In other words, the element a of H is primitive. Hence, $S(a) = -a$ (by Lemma 2.12 **(c)**, applied to $x = a$). Similarly, $S(b) = -b$. However, it is well-known (see, e.g., [GriRei20, Proposition 1.4.10] or [Radfor12, Proposition 7.1.9 (a)]) that the antipode S of H is a $\mathbf{k}$-algebra anti-endomorphism, i.e., that it satisfies $S(1_H) = 1_H$ and

$$S(uv) = S(v) S(u) \qquad \text{for all } u, v \in H. \tag{64}$$

Applying (64) to $u = a$ and $v = b$, we obtain $S(ab) = \underbrace{S(b)}_{=-b} \underbrace{S(a)}_{=-a} = (-b)(-a) = ba$.

This proves Lemma 3.4 **(c)**. □

Proof of Corollary 2.15. As we know, the graded $\mathbf{k}$-Hopf algebra H automatically becomes a filtered $\mathbf{k}$-Hopf algebra with filtration $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ defined by setting

$$H_{\leq n} := \bigoplus_{i=0}^{n} H_i \qquad \text{for all } n \in \mathbb{N}.$$

This filtered $\mathbf{k}$-Hopf algebra H is connected, since $H_{\leq 0} = H_0$. Thus, Corollary 2.14 can be applied.

Let u be a positive integer. Then, the definition of $H_{\leq u}$ yields $H_{\leq u} = \bigoplus_{i=0}^{u} H_i$, so that $H_u \subseteq H_{\leq u}$.

Now, we must prove the three relations (24), (25) and (26). The third one is the easiest: From $H_u \subseteq H_{\leq u}$, we obtain

$$\left(\operatorname{id} - S^2\right)^u (H_u) \subseteq \left(\operatorname{id} - S^2\right)^u (H_{\leq u}) = 0$$

(by Corollary 2.14 **(b)**) and therefore $\left(\operatorname{id} - S^2\right)^u (H_u) = 0$. This proves (26).

We shall now focus on proving (24). Indeed, if $u > 1$, then (24) follows from

$$\begin{aligned}\left(\operatorname{id} - S^2\right)^{u-1} (H_u) &\subseteq \left(\operatorname{id} - S^2\right)^{u-1} (H_{\leq u}) \qquad (\text{since } H_u \subseteq H_{\leq u}) \\ &\subseteq \operatorname{Prim} H \qquad (\text{by (21), since } u > 1).\end{aligned}$$

Thus, in order to complete the proof of (24), we only need to prove it for $u = 1$. In other words, we need to prove that $\left(\operatorname{id} - S^2\right)^0 (H_1) \subseteq \operatorname{Prim} H$. But this follows from

$$\underbrace{\left(\operatorname{id} - S^2\right)^0}_{=\operatorname{id}} (H_1) = \operatorname{id} (H_1) = H_1 \subseteq \operatorname{Prim} H \qquad (\text{by Lemma 3.4 \textbf{(b)}}).$$

This completes our proof of (24).

Now, it remains to prove (25). But we can deduce (25) from (24) in the same way as we deduced (19) from (18) in our above proof of Corollary 2.13. This completes the proof of Corollary 2.15. □

Proof of Corollary 2.16. Let 1_H denote the unity of the $\mathbf{k}$-algebra H.

As in the proof of Corollary 2.15, we know that the graded $\mathbf{k}$-Hopf algebra H automatically becomes a filtered $\mathbf{k}$-Hopf algebra, and this filtered $\mathbf{k}$-Hopf algebra H is connected.

Now, we shall show that

$$\left(\operatorname{id} - S^2\right) (H_{\leq p}) = 0. \tag{65}$$

[*Proof of (65):* In our above proof of Corollary 2.13, we have already shown that $S^2 (1_H) = 1_H$. This easily entails $\left(\operatorname{id} - S^2\right) (1_H) = 0$. However, since the graded Hopf algebra H is connected, it is easy to see that $H_0 = \mathbf{k} \cdot 1_H$. Hence, from $\left(\operatorname{id} - S^2\right) (1_H) = 0$, we obtain $\left(\operatorname{id} - S^2\right) (H_0) = 0$ (since $\operatorname{id} - S^2$ is a $\mathbf{k}$-linear map).

Let $x \in H_1$. Then, $x \in H_1 \subseteq \operatorname{Prim} H$ (by Lemma 3.4 **(b)**). In other words, the element x of H is primitive. Hence, Lemma 2.12 **(d)** yields $S^2 (x) = x$. In other words, $\left(\operatorname{id} - S^2\right) (x) = 0$.

Forget that we fixed x. We thus have shown that $\left(\operatorname{id} - S^2\right) (x) = 0$ for each $x \in H_1$. In other words, we have $\left(\operatorname{id} - S^2\right) (H_1) = 0$.

Now, the definition of $H_{\leq p}$ yields $H_{\leq p} = \bigoplus_{i=0}^{p} H_i = \sum_{i=0}^{p} H_i$. Applying the map

$\operatorname{id} - S^2$ to both sides of this equality, we obtain

$$\left(\operatorname{id} - S^2\right)\left(H_{\leq p}\right) = \left(\operatorname{id} - S^2\right)\left(\sum_{i=0}^{p} H_i\right) = \sum_{i=0}^{p} \left(\operatorname{id} - S^2\right)\left(H_i\right)$$
$$= \underbrace{\left(\operatorname{id} - S^2\right)\left(H_0\right)}_{=0} + \underbrace{\left(\operatorname{id} - S^2\right)\left(H_1\right)}_{=0} + \sum_{i=2}^{p} \underbrace{\left(\operatorname{id} - S^2\right)\left(H_i\right)}_{\substack{=0 \\ \text{(by (27))}}}$$
$$= 0 + 0 + \sum_{i=2}^{p} 0 = 0.$$

This proves (65).]

Hence, we can apply Corollary 2.13. As a result, we obtain precisely the claims of Corollary 2.16. □

Proof of Corollary 2.17. **(a)** Let 1_H denote the unity of the $\mathbf{k}$-algebra H. Define the maps m and u as in Remark 3.3.

Let $x \in H_2$. Then, Lemma 3.4 **(a)** (applied to $n = 2$) yields that we have

$$\Delta\left(x\right) = 1_H \otimes x + x \otimes 1_H + w \qquad \text{for some } w \in H_1 \otimes H_1$$

(since $\sum_{k=1}^{2-1} H_k \otimes H_{2-k} = H_1 \otimes H_1$). Consider this w.

Now, w is a tensor in $H_1 \otimes H_1$. Write this tensor in the form

$$w = \sum_{i=1}^{k} a_i \otimes b_i \tag{66}$$

for some $k \in \mathbb{N}$, some $a_1, a_2, \ldots, a_k \in H_1$ and some $b_1, b_2, \ldots, b_k \in H_1$.

The elements $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ belong to H_1 and therefore are primitive (since Lemma 3.4 **(b)** yields $H_1 \subseteq \operatorname{Prim} H$). Hence, each $i \in \{1, 2, \ldots, k\}$ satisfies

$$S\left(a_i\right) = -a_i \tag{67}$$

(by Lemma 2.12 **(c)**, applied to a_i instead of x) and

$$\begin{aligned} S\left(a_i b_i\right) &= b_i a_i \qquad \text{(by Lemma 3.4 **(c)**, applied to } a = a_i \text{ and } b = b_i\text{)} \\ &= a_i b_i \end{aligned} \tag{68}$$

(by (31), applied to $a = b_i$ and $b = a_i$).

Applying the map $S \otimes \operatorname{id}_H : H \otimes H \to H \otimes H$ to both sides of the equality (66), we obtain

$$(S \otimes \mathrm{id}_H)(w) = (S \otimes \mathrm{id}_H)\left(\sum_{i=1}^{k} a_i \otimes b_i\right) = \sum_{i=1}^{k} \underbrace{S(a_i)}_{\substack{=-a_i \\ \text{(by (67))}}} \otimes \underbrace{\mathrm{id}_H(b_i)}_{=b_i} = \sum_{i=1}^{k} (-a_i) \otimes b_i$$
$$= -\underbrace{\sum_{i=1}^{k} a_i \otimes b_i}_{\substack{=w \\ \text{(by (66))}}} = -w. \qquad (69)$$

The Hopf algebra H is graded. Hence, its counit ϵ is a graded map from H to $\mathbf{k}$. In other words, $\epsilon(H_i) \subseteq \mathbf{k}_i$ for each $i \in \mathbb{N}$. Thus, $\epsilon(H_2) \subseteq \mathbf{k}_2 = 0$ (since the graded $\mathbf{k}$-module $\mathbf{k}$ is concentrated in degree 0). Therefore, $\epsilon(x) = 0$ (since $x \in H_2$).

Lemma 2.12 **(b)** yields $S(1_H) = 1_H$.

Applying both sides of the equality (61) to x, we obtain

$$(m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) = (u \circ \epsilon)(x) = u\left(\underbrace{\epsilon(x)}_{=0}\right) = u(0) = 0.$$

Therefore,

$$\begin{aligned}
0 &= (m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) \\
&= m\left((S \otimes \mathrm{id}_H)\left(\underbrace{\Delta(x)}_{=1_H \otimes x + x \otimes 1_H + w}\right)\right) \\
&= m\left(\underbrace{(S \otimes \mathrm{id}_H)(1_H \otimes x + x \otimes 1_H + w)}_{=S(1_H)\otimes \mathrm{id}_H(x) + S(x) \otimes \mathrm{id}_H(1_H) + (S \otimes \mathrm{id}_H)(w)}\right) \\
&= m\left(\underbrace{S(1_H)}_{=1_H} \otimes \underbrace{\mathrm{id}_H(x)}_{=x} + S(x) \otimes \underbrace{\mathrm{id}_H(1_H)}_{=1_H} + \underbrace{(S \otimes \mathrm{id}_H)(w)}_{\substack{=-w \\ \text{(by (69))}}}\right) \\
&= m(1_H \otimes x + S(x) \otimes 1_H - w) \\
&= \underbrace{1_H x}_{=x} + \underbrace{S(x) \cdot 1_H}_{=S(x)} - m(w) \qquad \text{(by the definition of } m\text{)} \\
&= x + S(x) - m(w).
\end{aligned}$$

Solving this equality for $S(x)$, we obtain

$$S(x) = m(w) - x. \qquad (70)$$

Applying the map $m : H \otimes H \to H$ to both sides of the equality (66), we obtain

$$m(w) = m\left(\sum_{i=1}^{k} a_i \otimes b_i\right) = \sum_{i=1}^{k} a_i b_i \tag{71}$$

(by the definition of m). Applying the map S to both sides of this equality, we obtain

$$\begin{aligned} S(m(w)) &= S\left(\sum_{i=1}^{k} a_i b_i\right) = \sum_{i=1}^{k} \underbrace{S(a_i b_i)}_{\substack{=a_i b_i \\ \text{(by (68))}}} = \sum_{i=1}^{k} a_i b_i \\ &= m(w) \qquad \text{(by (71))}. \end{aligned} \tag{72}$$

Now, applying the map S to both sides of the equality (70), we obtain

$$\begin{aligned} S(S(x)) &= S(m(w) - x) = \underbrace{S(m(w))}_{\substack{=m(w) \\ \text{(by (72))}}} - \underbrace{S(x)}_{\substack{=m(w)-x \\ \text{(by (70))}}} \\ &= m(w) - (m(w) - x) = x. \end{aligned}$$

Now,

$$\left(\mathrm{id} - S^2\right)(x) = \underbrace{\mathrm{id}(x)}_{=x} - \underbrace{S^2(x)}_{=S(S(x))=x} = x - x = 0.$$

Forget that we fixed x. We thus have shown that $\left(\mathrm{id} - S^2\right)(x) = 0$ for each $x \in H_2$. In other words, $\left(\mathrm{id} - S^2\right)(H_2) = 0$. This proves Corollary 2.17 **(a)**.

Now we know that $\left(\mathrm{id} - S^2\right)(H_2) = 0$ (by Corollary 2.17 **(a)**). In other words, all $i \in \{2, 3, \ldots, 2\}$ satisfy $\left(\mathrm{id} - S^2\right)(H_i) = 0$ (since the only $i \in \{2, 3, \ldots, 2\}$ is 2). Hence, we can apply Corollary 2.16 to $p = 2$. Doing so, we obtain precisely the claims of parts **(b)** and **(c)** of Corollary 2.17. (To be precise: Corollary 2.17 **(b)** follows by applying Corollary 2.16 **(a)**, whereas Corollary 2.17 **(c)** follows by applying Corollary 2.16 **(b)**.) □

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