Research statement
Darij Grinberg
[detailed version]

I work in the field of *algebraic combinatorics*, centered on (but not limited to) *symmetric functions* and related concepts, such as *combinatorial Hopf algebras*, *Young tableaux* and *trees*. These objects live at the borderlands of algebra and combinatorics, often allowing for viewpoints from both sides and transfer of knowledge from one to the other. I also study adjacent subjects such as invariant theory, Lie algebras, representations, and constructive algebra.

Among my contributions to these disciplines are a new generalization of the dual stable Grothendieck polynomials (which themselves generalize the Schur functions), an antipode formula for quasisymmetric functions, a proof of a conjecture of K. Mészáros on the "subdivision algebra", and a periodicity result on "birational rowmotion" (originally conjectured by D. Einstein and J. Propp) that has seen several uses in dynamical algebraic combinatorics. Details on this and other work can be found below.

Background. The history of *symmetric functions* goes back at least as far as the 17th Century, when Newton and Girard explored the relations between elementary symmetric polynomials and power sums. The next major steps were Cauchy's 1815 introduction of what later came to be known as *Schur functions*, and Jacobi's and Trudi's determinantal formulas for them (1841 and 1864). Symmetric functions found various uses in the algebra of the 19th Century, particularly in *Galois* and *invariant theory*, as well as in *Schubert calculus*. However, their combinatorial meaning was not discovered until the 1930s, when Schur and others connected them to *Young tableaux* and the *representation theory of symmetric and general linear groups*. This connection opened the floodgates, and the research that followed since the mid-20th Century could fill bookshelves (see, e.g., [Stan99, Ch. 7], [Fulton97], [Sagan01]). Symmetric functions were found to form a *Hopf algebra*, which tended to appear in various guises in seemingly unrelated fields such as algebraic topology (as *cohomology* of some classifying spaces) and number theory (as coordinate ring of the *Witt vectors*, e.g., [Hazewi08]). The multiplication of Schur functions turned out to be governed by a combinatorial rule (the *Littlewood-Richardson rule*), formulated in 1934 and first proven in 1974, with applications in theoretical physics. The combinatorics of Young tableaux became a subject of its own, bordering on theoretical computer science (Knuth devoted a section in "The Art of Computer Programming" to it). Symmetric functions have been applied in fields as diverse as *random matrix theory*, *K-theory* (particularly of *Grassmannians*), *group theory* and *quantum groups*. The description of the representations of symmetric and general linear groups using Schur functions has become a mold in which many other representation theories have been shaped.

By now, even as various questions on Schur functions remain unanswered, the focus has broadened to include generalizations and analogues thereof, such as Hall-Littlewood and Macdonald polynomials, factorial Schur functions, Schubert and Grothendieck polynomials, P-partition enumerators, and others.

It is such generalizations that I have been dealing with in much of my research. From a bird's eye view, their theories follow a certain pattern: a family of power series is defined, and analogues of classical properties of the Schur functions (such as symmetry, determinantal formulas à la Jacobi-Trudi, Littlewood-Richardson rule(s), and antipode formulas) are proven for this family. However, this is rarely ever straightforward, as each generalization comes with its additional complications; consequently, these programs are at rather different stages of completion, and some of them (e.g., a full Littlewood-Richardson rule for Schubert polynomials) appear out of reach today. Additionally, each generalization has its own motivation, sometimes stemming from a totally different field.

Combinatorial Hopf algebras are one (although not the only) place where these generalizations live. They are interesting both in their intrinsic properties (e.g., some of them are free as algebras for non-obvious reasons) and for the special elements they contain (such as the above-mentioned generalizations of symmetric functions). They have found applications to algebraic groups, Lie groups, probability and renormalization theory. One of the most frequently seen among these Hopf algebras is the ring of *quasisymmetric functions*, which has been introduced by Gessel and Stanley for combinatorial purposes in the 1970s, but has recently appeared in topology ([BakRic08]) and K-theory ([Morava15], [Oesing18]); this ring has also been involved in much of my past research (e.g., [Grinbe15a], [Grinbe14], [Grinbe17c]).

Overview of selected past results. My results so far, as well as my research plans for the future, live in and around the algebro-combinatorial landscape surveyed above. Among my finished work on symmetric functions and related Hopf algebras, the following are the most relevant:

- In [Grinbe14], I prove a conjecture of Mike Zabrocki on a quasisymmetric analogue of *Bernstein's creation-operator approach* to the Schur functions. The proof relies on a *dendriform algebra* structure on the quasisymmetric functions – a structure I later apply to a combinatorial problem in [Grinbe17c].

- In [Grinbe15a], I reprove and generalize a formula of Malvenuto and Reutenauer for the *antipode of a P-partition enumerator* (which itself extends a classical formula for the antipode of a Schur function).

- In [GaGrLi15], Pavel Galashin, Gaku Liu and I refine the *dual stable Grothendieck polynomials* (a recent generalization of Schur functions motivated by K-theory) to include new parameters, and prove the symmetry of these new power series combinatorially.

- In [BorGri13], James Borger and I explore *positivity* properties of symmetric functions and apply them to *Witt vectors over semirings*.

- In [Grinbe15d], I leverage a universal property to obtain a new construction of the Bernstein homomorphism for commutative connected graded Hopf algebras (which generalizes the internal comultiplication of the quasisymmetric functions).

- The lecture notes [GriRei15] (joint with Victor Reiner) are an introduction to both symmetric functions and combinatorial Hopf algebras.

Some other works of mine are not directly concerned with symmetric functions:

- In the two papers [GriRob14], Tom Roby and I prove the periodicity of *birational rowmotion* on rectangles and some related results.
- In [Grinbe17a], I answer a question of Karola Mészáros on the *subdivision algebra* (a deformation of the *Orlik-Solomon algebra* of the type-A braid arrangement).
- In [GrHuRe17], Jia Huang, Victor Reiner and I study the combinatorics of the *Grothendieck groups of a finite-dimensional Hopf algebra*, revealing a parallel to the *chip-firing game* on directed graphs.
- In [GriPos17], Alexander Postnikov and I demonstrate a property of *reduced expressions* in *Coxeter groups*.
- In [GriOlv18], Peter Olver and I factorize the determinant of a matrix arising from the *n-body problem*. I have since generalized this factorization in [Grinbe19a].
- In [AaGrSc18], Erik Aas, Travis Scrimshaw and I prove two conjectures on *multiline queues* and the *totally asymmetric simple exclusion process* (one coming from physics, one from probability theory).

Current and finished research

Cohomology-like quotients of symmetric polynomials. See [Grinbe18b] for a survey (slightly out of date) and https://www.cip.ifi.lmu.de/~grinberg/algebra/umn2019.pdf for a semi-introductory talk.

It is well-known since the early 20th century that the symmetric functions can be used to describe the cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{k,n})$ of the Grassmannian $\mathrm{Gr}_{k,n}$ (see [Fulton97, Part III] for an introduction). More precisely, this cohomology ring is a quotient of the ring of symmetric functions modulo an ideal generated by all "large" elementary symmetric functions $e_{k+1}, e_{k+2}, e_{k+3}, \ldots$ and the k complete homogeneous symmetric functions $h_{n-k+1}, h_{n-k+2}, \ldots, h_n$. More recently it has been revealed that a deformation of this ideal induces a quotient that is isomorphic to the *quantum* cohomology ring $\mathrm{QH}^*(\mathrm{Gr}_{k,n})$, whose structure constants are the Gromov-Witten invariants (see [Postni05] for a recent perspective). These facts motivate a more general scenario:

Fix integers $n \geq k \geq 0$, and let $\mathcal{S}$ be the ring of symmetric polynomials in k variables $x_1, x_2, \ldots, x_k$ over an arbitrary base ring $\mathbf{k}$. (We can view $\mathcal{S}$ itself as a quotient of the ring of symmetric functions by the elementary symmetric functions $e_{k+1}, e_{k+2}, \ldots$.) Fix k scalars $a_1, a_2, \ldots, a_k \in \mathbf{k}$, and let I be the ideal of $\mathcal{S}$ generated by $h_{n-k+i} - a_i$ for all $i \in \{1, 2, \ldots, k\}$. The quotient algebra $\mathcal{S}/I$ then generalizes both the classical and the quantum cohomology rings.

I have initiated the study of this quotient algebra $\mathcal{S}/I$ in [Grinbe18a] (work in progress). In particular, I have shown that $\mathcal{S}/I$ is a free $\mathbf{k}$-module of rank $\binom{n}{k}$ with three explicit bases. All three bases are indexed by the set $P_{k,n}$ of all partitions that "fit inside" the $k \times (n-k)$-rectangle (that is, partitions that have at most k parts, with each part being $\leq n-k$). If we let $\overline{f}$ denote the projection of a symmetric polynomial $f \in \mathcal{S}$ onto $\mathcal{S}/I$, then these three bases are $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$, $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ and $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$, where s_λ, h_λ and m_λ are Schur polynomials, complete homogeneous symmetric polynomials and monomial symmetric polynomials, respectively.

Furthermore, I have shown that the structure coefficients of the algebra $\mathcal{S}/I$ in the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ satisfy an S_3-symmetry, generalizing those of the Littlewood-Richardson coefficients and of the Gromov-Witten invariants. I have found a Pieri rule for products of the form $\overline{s_\lambda h_i}$ (with $i \leq n-k$) as well as a "rim hook algorithm" for reducing arbitrary Schur polynomials $\overline{s_\mu}$ (with $\mu \notin P_{k,n}$) in the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$. (Both of these are significantly subtler than the original results of Bertram, Ciocan-Fontanine and Fulton [BeCiFu99] they generalize.) Further work on this algebra is ongoing, the goal being to explore what other properties of the cohomology rings generalize to this situation:

- The structure coefficients of $\mathcal{S}/I$ in the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ seem to satisfy a positivity property (more precisely, predictable signs) generalizing the positivity of Gromov-Witten invariants. This is so far a conjecture, which I expect to be highly difficult.

- What further properties of classical and quantum cohomology extend to $\mathcal{S}/I$? Computations have shown that neither the $\mathrm{Gr}_{k,n} \leftrightarrow \mathrm{Gr}_{n-k,n}$ duality nor Postnikov's "curious duality" [Postni05] do (at least not without further restrictions), but others remain to be tested.

- What other bases with combinatorial meaning does $\mathcal{S}/I$ have? For any of the dozens of known bases $(b_\lambda)_\lambda$ of $\mathcal{S}$ (indexed by partitions λ into $\leq k$ parts), we can ask whether $\left(\overline{b_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of $\mathcal{S}/I$. So far, this has been verified for $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$, $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ and $(\overline{m_\lambda})_{\lambda \in P_{k,n}}$ and refuted for $(\overline{p_\lambda})_{\lambda \in P_{k,n}}$ (even when $\mathbf{k} = \mathbb{Q}$).

- How many of the properties survive when $a_1, a_2, \ldots, a_k$ are no longer scalars in $\mathbf{k}$, but symmetric polynomials with $\deg(a_i) < n-k+i$ for all i? For example, the $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$-basis still holds in that generality, while the S_3-symmetry does not.

- Inspired by [CoKrWa09], I have introduced a variant of the ideal I, in which the h_j are replaced by the power-sum symmetric polynomials $p_j = x_1^j + x_2^j + \cdots + x_k^j$. Thus, define an ideal I_p of $\mathcal{S}$ generated by $p_{n-k+i} - a_i$ for all $i \in \{1, 2, \ldots, k\}$. Is $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ again a basis of the quotient $\mathbf{k}$-module $\mathcal{S}/I_p$, assuming that $\mathbf{k}$ is a $\mathbb{Q}$-algebra (a natural assumption when dealing with p_j's)? After I proposed this question for the MIT PRIMES project 2019 (an REU-like research program for talented high-schoolers), a student (Andrew Weinfeld) has come up with a highly elegant proof. Further questions naturally suggest themselves here.

Connections with the splitting algebras of Laksov and Thorup [LakTho12], with equivariant quantum cohomology [BeMiTa18][1] and with the structure constants of the Schubert polynomials [Manive01] is also suspected.

Petrie symmetric functions. See [Grinbe19b] for a (so far unfinished) draft.

Recent algebro-geometric work by Liu and Polo [LiuPol19] got me interested in a family of symmetric functions which I dubbed the *Petrie symmetric functions*[2]. Given a positive integer k, we define a symmetric formal power series

$$G\left(k\right) = \prod_{i=1}^{\infty} \left(1 + x_i + x_i^2 + \cdots + x_i^{k-1}\right) \qquad \text{in the indeterminates } x_1, x_2, x_3, \ldots;$$

this is the sum of all monomials containing no exponents $\geq k$. Also, if k is a positive and m a nonnegative integer, then $G\left(k, m\right)$ shall denote the m-th graded component of $G\left(k\right)$; this is the sum of all monomials of degree m containing no exponents $\geq k$. A conjecture by Liu and Polo [LiuPol19, Remark 1.4.5] can be rephrased as saying that

$$G\left(n, 2n-1\right) = \sum_{i=0}^{n-2} \left(-1\right)^i s_{\left(n-1, n-1-i, 1^{i+1}\right)} \qquad \text{for every integer } n > 1,$$

where s_λ denote the Schur functions as usual. I have proved this conjecture and showed, more generally, that $G\left(k, m\right)$ is always a linear combination $\sum_\lambda \operatorname{pet}_k\left(\lambda\right) s_\lambda$ of Schur functions s_λ in which the coefficients $\operatorname{pet}_k\left(\lambda\right)$ are certain numbers in the set $\{1, -1, 0\}$. These numbers are easiest to describe as determinants of certain Petrie matrices [GorWil74]. One of my next goals (partly achieved) is to find a simple combinatorial rule for $\operatorname{pet}_k\left(\lambda\right)$ in terms of the k-abacus of λ.

More generally, for any partition λ, the product $G\left(k, m\right) \cdot s_\lambda$ is a linear combination of s_μ's with coefficients in $\{1, -1, 0\}$. The coefficients, again, are determinants of Petrie matrices. A more direct combinatorial description would be rather interesting, as it would interpolate between the Pieri rule for $h_m s_\lambda$ and the Murnaghan–Nakayama rule for $p_m s_\lambda$ (since $h_m = G\left(k, m\right)$ for all $k > m$, whereas $p_m = G\left(m+1, m\right) - G\left(m, m\right)$).

Other questions can be asked. It is not hard to see that for any given $k > 1$, the sequence of symmetric functions $\left(G\left(k,1\right), G\left(k,2\right), G\left(k,3\right), \ldots\right)$ is algebraically independent. Thus, the products of these $G\left(k, m\right)$'s form a new basis of the ring of symmetric functions over $\mathbb{Q}$. Can anything be said about this basis?

Refined dual stable Grothendieck polynomials. Dual stable Grothendieck polynomials first appear in the work of Lam and Pylyavskyy [LamPyl07], after having

[1] Discussions with Kaisa Taipale have revealed that the equivariant cohomology ring of $\operatorname{Gr}_{k,n}$ is also isomorphic to $\mathcal{S}/I$ for an appropriate choice of $a_1, a_2, \ldots, a_k$ in $\mathcal{S}$ (not constants, but with $\deg\left(a_i\right) < n - k + i$), but the isomorphism does not map $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ to any (known) geometrically interesting basis of the cohomology ring.

[2] The name is tentative and honors the egyptologist Flinders Petrie, who introduced a class of matrices (now known as *Petrie matrices*) to help with the seriation of artifacts; see, e.g., [Shucha84]. Petrie matrices appear in the evaluation of my symmetric functions, whence the name.

been anticipated by Lenart and Buch. In joint work [GaGrLi15] with Pavel Galashin and Gaku Liu, I have extended their definition and some of their properties to a more general setup, involving an infinite family of new parameters.

A *weak composition* means a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$) such that all but finitely many i satisfy $\alpha_i = 0$.

Consider a skew partition λ/μ. A *reverse plane partition* (short: *rpp*) of shape λ/μ means a filling of the skew Young diagram of λ/μ with positive integers which increase weakly along rows and weakly along columns. (Requiring them to increase strictly along columns would instead yield the definition of a semistandard tableau.) For every rpp T, we let $\operatorname{ircont} T$ be the weak composition whose i-th entry is the number of columns of T which contain the entry i. Moreover, for every rpp T, we let $\operatorname{ceq} T$ be the weak composition whose i-th entry is the number of cells c in the i-th row of T such that the entry of T in cell c equals the entry of T in the cell directly below c (and, in particular, the latter entry exists). Notice that $\operatorname{ceq} T = (0, 0, 0, \ldots)$ if and only if T is a semistandard tableau.

The *refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}$ corresponding to the skew partition λ/μ is defined to be

$$\sum_{T \text{ is an rpp of shape } \lambda/\mu} \mathbf{t}^{\operatorname{ceq} T} \mathbf{x}^{\operatorname{ircont} T} \in (\mathbb{Z}[t_1, t_2, t_3, \ldots])[[x_1, x_2, x_3, \ldots]].$$

Here we are using the notation $\mathbf{x}^\alpha$ for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ whenever $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ is a weak composition, and similarly the notation $\mathbf{t}^\alpha$ stands for $t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \cdots$.

The dual stable Grothendieck polynomials are obtained from the $\widetilde{g}_{\lambda/\mu}$ by setting all t_i equal to 1, whereas the skew Schur functions are obtained by setting all t_i equal to 0. Other specializations of $\widetilde{g}_{\lambda/\mu}$ have not been explored so far, but the parameter space is obviously vast.

In [GaGrLi15], Galashin, Liu and I have shown that the power series $\widetilde{g}_{\lambda/\mu}$ is symmetric (in the $x_1, x_2, x_3, \ldots$).

Two proofs are given in [GaGrLi15], and yet another can be obtained with the methods of [LamPyl07].

Various questions suggest themselves now:

- Do the $\widetilde{g}_{\lambda/\mu}$ satisfy a determinantal formula generalizing (one of) the Jacobi-Trudi identities? The answer appears to be positive, but a proof has not been found so far. Damir Yeliussizov [Yelius16] has found a proof in the $\mu = \varnothing$ case; research on the general case is ongoing.

- Do the $\widetilde{g}_{\lambda/\mu}$ satisfy a Littlewood-Richardson rule? There are several ways to interpret the question, some of which (e.g., expanding the product $\widetilde{g}_{\lambda/\varnothing}\widetilde{g}_{\mu/\varnothing}$ as a linear combination of the $\widetilde{g}_{\nu/\varnothing}$) appear out of reach. I have proven one Littlewood-Richardson rule (expanding $s_\nu \widetilde{g}_{\lambda/\mu}$ in terms of the s_κ) using the results of [GaGrLi15] and an analogue of Stembridge's proof of the classical Littlewood-Richardson rule[3].

[3] See http://www.cip.ifi.lmu.de/~grinberg/algebra/chicago2015.pdf for a statement of this (proof not yet written up) and also of the conjectural Jacobi-Trudi identity.

- It appears that a similar refinement can be done to the (non-dual) stable Grothendieck polynomials $G_{\lambda/\mu}$.

Antipodes and P-partitions. We shall use the notation S for the antipode of a Hopf algebra. It is well-known that any skew Schur function $s_{\lambda/\mu}$ (regarded as a symmetric function) satisfies $S\left(s_{\lambda/\mu}\right) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}$, where ν^t denotes the transpose of a partition ν. Ira Gessel generalized this fact to *P-partition enumerators*. Here, we consider a finite set E with a partial order relation $<_1$ and a total order relation $<_2$. We define an $(E, <_1, <_2)$*-partition* to be a map $f : E \to \{1, 2, 3, \ldots\}$ such that

- $f(u) \leq f(v)$ for any $u \in E$ and $v \in E$ with $u <_1 v$;
- $f(u) < f(v)$ for any $u \in E$ and $v \in E$ with $u <_1 v$ and $v <_2 u$.

We define the P-partition enumerator $F_{(E,<_1,<_2)}$ as the formal power series

$$\sum_{f \text{ is an } (E,<_1,<_2)\text{-partition}} \mathbf{x}_f \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]], \qquad \text{where } \mathbf{x}_f = \prod_{e \in E} x_{f(e)}.$$

The power series $F_{(E,<_1,<_2)}$ is a quasisymmetric function (although its most important particular cases are the skew Schur functions, which are symmetric). Gessel's formula [GriRei15, Corollary 5.27] states that the antipode S of the Hopf algebra of quasisymmetric functions satisfies $S\left(F_{(E,<_1,<_2)}\right) = (-1)^{|E|} F_{(E,>_1,<_2)}$, where $>_1$ is the opposite relation of $<_1$.

In [Grinbe15a], I prove a generalization of this formula in which the elements $e \in E$ are equipped with (positive integer) weights $w(e)$ (and the monomial $\mathbf{x}_f = \prod_{e \in E} x_{f(e)}$ is replaced by $\prod_{e \in E} x_{f(e)}^{w(e)}$), in which the order $<_2$ is no longer required to be total (although not completely arbitrary either), and in which a finite group G is allowed to act on E in a way that preserves $<_1$ and $<_2$. The weights and the partial order $<_2$ are not a new idea (in this generality, the result can be viewed as a restatement of Malvenuto and Reutenauer's [MalReu98, Theorem 3.1]), but the group action appears to be new; it combines Gessel's antipode formula with the ideas of Pólya enumeration.

Dual immaculate creation operators. In [BBSSZ13a], Berg, Bergeron, Saliola, Serrano and Zabrocki introduced a lift of the Schur functions to the ring of noncommutative symmetric functions, and also a "dual" version of this lift in the ring of quasisymmetric functions. The dual version is probably the simpler one to define: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition (i.e., a tuple of positive integers), and let $Y(\alpha)$ be the "Young diagram" of α (this is defined as the set $\left\{(i, j) \in \{1, 2, 3, \ldots\}^2 \mid j \leq \alpha_i\right\}$, even if α is not a partition). An *immaculate tableau* of shape α is a filling of this Young diagram with positive integers such that each row is weakly increasing, and the **first** column is strictly increasing. (This can be viewed as an $(E, <_1, <_2)$-partition, where $(E, <_1)$ is a certain binary tree.) The *dual immaculate function* $\mathfrak{S}^*_\alpha$ corresponding to α is defined to be the sum of $\mathbf{x}_T$ over all immaculate tableaux T of shape α, where $\mathbf{x}_T = \prod_{e \text{ is a cell of } T} x_{T(e)}$. (This is a particular case of the P-partition enumerator $F_{(E,<_1,<_2)}$ defined above.)

Mike Zabrocki has conjectured that an alternative definition of $\mathfrak{S}^*_\alpha$, using a certain notion of quasisymmetric "creation operators" (similar to the Bernstein creation operators that generate the Schur functions, but more intricate), results in the same power series. I proved this in [Grinbe14], by introducing a dendriform algebra structure on the ring of quasisymmetric functions.

This project per se is finished, but it has spawned further research by the "Marne-la-Vallée school" of symmetric functions. Specifically, Novelli, Thibon and Toumazet [NoThTo17] have built upon it to construct a noncommutative generalization of Bell polynomials and study their underlying combinatorics.

Shuffle-compatible permutation statistics. A recent paper of Ira Gessel and Yan Zhuang [GesZhu17] has introduced the concept of *shuffle-compatible permutation statistics*. These are certain combinatorial statistics on the set $\mathfrak{S}$ of all permutations (in a wide sense – i.e., they are maps that assign *something* to a permutation, not always a number); for example, "the set of all descents", "the number of descents", and "the set of all peaks" are shuffle-compatible, but "the number of inversions" is not. The definition of "shuffle-compatible" is combinatorial, but for most permutation statistics it can be restated using quasisymmetric functions. Namely, consider the ring QSym of quasisymmetric functions, and its fundamental basis (F_α) (with α ranging over all compositions). For any permutation π, let $\gamma(\pi)$ denote the descent composition of π. Any map $\operatorname{st} : \{\text{compositions}\} \to X$ gives rise to a submodule $\mathcal{K}_{\operatorname{st}}$ of QSym, which is spanned by $F_\alpha - F_\beta$ where (α, β) ranges over all pairs of compositions of the same size satisfying $\operatorname{st}\alpha = \operatorname{st}\beta$. Also, any map $\operatorname{st} : \{\text{compositions}\} \to X$ gives rise to a map $\{\text{permutations}\} \to X$ which sends each permutation π to $\operatorname{st}(\gamma(\pi))$. The latter map is a permutation statistic, and is shuffle-compatible if and only if $\mathcal{K}_{\operatorname{st}}$ is an ideal of QSym. Thus, shuffle-compatible permutation statistics (provided they only depend on $\gamma(\pi)$) define ideals of QSym (binomial ideals with respect to the fundamental basis), and therefore also quotient algebras of QSym.

In [Grinbe17c], I prove that the "exterior peak set" statistic is shuffle-compatible (as conjectured by Gessel and Zhuang [GesZhu17]). I further introduce a strengthening of the notion of shuffle-compatibility that separates between left and right shuffles, leading to quotients of QSym as *dendriform* algebras (where the dendriform structure on QSym is the one from [Grinbe14]). I prove that the exterior peak set statistic is left- and right-shuffle-compatible; similar results are proved (and disproved) for some other statistics (e.g., the left peak set is left- and right-shuffle-compatible, whereas the right peak set is not). While [Grinbe17c] is self-contained, the underlying ideas originate in my previous research of P-partitions and the dendriform structure of QSym ([Grinbe14], [Grinbe15a]).

A followup to [Grinbe17c] is in the making, exploring even stronger compatibility properties.

Peak quasisymmetric functions. A question that arose in the just-mentioned study of shuffle-compatibility [Grinbe17c, Question 2.51] has opened my eyes to a new basis of the ring of quasisymmetric functions.

Consider a commutative $\mathbb{Q}$-algebra $\mathbf{k}$. If $n \in \mathbb{N}$ and if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of n (that is, a tuple of positive integers with $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$), then we define the formal power series

$$\eta_\alpha = \sum_{\substack{1 \leq g_1 \leq g_2 \leq \cdots \leq g_n; \\ g_i = g_{i+1} \text{ for each } i \in E(\alpha)}} 2^{|\{g_1, g_2, \ldots, g_n\}|} x_{g_1} x_{g_2} \cdots x_{g_n} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]],$$

where $E(\alpha)$ denotes the set $\{1, 2, \ldots, n-1\} \setminus \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid 0 < i < \ell\}$. This η_α belongs to the $\mathbf{k}$-algebra QSym of quasisymmetric functions.

If we let α range over all *odd compositions* (i.e., compositions $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ whose entries α_i are all odd), then the η_α form a basis of a certain Hopf subalgebra of QSym, known as *Stembridge's Hopf algebra* or the *odd subalgebra* Π_- *of* QSym; see [Stembr97], [Hsiao07] (where they are called "peak functions") and [AgBeSo14]. (Roughly speaking, they are a quasisymmetric analogue of Schur's Q-functions.)

I have taken up the study of η_α for arbitrary compositions α. It is easy to see that these η_α form a basis of the whole QSym. Moreover, their products $\eta_\alpha \eta_\beta$ are given by a novel kind of shuffle, which allows adding k consecutive entries of α together with $k \pm 1$ consecutive entries of β; for example, for $\alpha = (a, b)$ and $\beta = (c, d)$ the formula is

$$\begin{aligned} \eta_{(a,b)} \eta_{(c,d)} &= \eta_{(a,b,c,d)} + \eta_{(a,c,b,d)} + \eta_{(a,c,d,b)} + \eta_{(c,a,b,d)} + \eta_{(c,a,d,b)} + \eta_{(c,d,a,b)} \\ &\quad - \eta_{(a,b+c+d)} - \eta_{(c,a+b+d)} - \eta_{(a+b+c,d)} - \eta_{(a+c+d,b)}. \end{aligned}$$

Further properties of this new basis of QSym remain to be uncovered (and the existing results to be exposed).

Multiline queues. The notion of a *multiline queue* arose in the work of P. Ferrari and J. Martin as a combinatorial tool for the computation of the steady state of the *totally asymmetric simple exclusion process* (a classical random process with multiple applications in physics, biology and traffic analysis).

In joint work with Erik Aas and Travis Scrimshaw [AaGrSc18], I prove several conjectures on multiline queues: the "commutativity conjecture" of Arita, Ayyer, Mallick, and Prolhac [AAMP11], and the enumerative conjectures of Aas and Linusson [AasLin18] on multiline queues with a given bottom row. Our methods reduce these to the classical Lascoux-Schützenberger action of the symmetric group and to a Lindström-Gessel-Viennot lattice path count. This reveals multiline queues as tableau-like objects in disguise (we actually construct a bijection between them and semistandard tableaux in [AaGrSc18], albeit only in a special case); generating functions for multiline queues thus can be viewed as analogues of Schur functions (or, rather, Schur polynomials).

Recent work by Corteel, Mandelshtam and Williams [CoMaWi18], independently of [AaGrSc18], makes a similar connection, although in a different setting and with different weights for the generating function. Their generating functions reveal themselves as Macdonald polynomials (both of the symmetric and nonsymmetric kinds). A connection between the two results seems possible.

Greedoids from ultrametric distance functions. We are now leaving the realm of symmetric (and adjacent) functions for a wider view of algebraic combinatorics. The following problem arose as a way to crystallize the combinatorial undercurrent of Manjul Bhargava's notion of "P-orderings" (see [Bharga97, §2] for the general case, or [Bharga00, §4] and [Bharga09, §2] for elementary expositions of the integer case), but has since widened and revealed links to some existing combinatorial structures (greedoids and phylogenetic trees). Some of the results have been written up in a joint preprint with Fedor Petrov [GriPet19]; the rest will fill another paper of my own.

Let E be a finite set. Assume that a *weight* $w(e) \in \mathbb{R}$ has been assigned to each $e \in E$, and a *distance* $d(e, f) = d(f, e)$ has been assigned to any two distinct $e, f \in E$, such that the distances satisfy the ultrametric triangle inequality $d(a, b) \leqslant \max\{d(a, c), d(b, c)\}$. Any subset F of E has a *perimeter*, defined as the sum of the weights of and pairwise distances between all elements of F. Given some $m \in \mathbb{N}$, how do we find an m-element subset of E with maximum perimeter?

A natural approach (an example of a *greedy algorithm*) is to start with the empty set, and then adjoin new elements one by one, maximizing the perimeter at each step. We show that this does indeed construct a maximum-perimeter m-element subset of E, and any maximum-perimeter m-element subset of E can be constructed this way. Moreover, if we define a collection $\mathcal{F}$ of subsets of E by

$$\mathcal{F} = \{F \subseteq E \mid F \text{ is a maximum-perimeter } m\text{-element subset of } E \text{ for } m = |F|\},$$

then $\mathcal{F}$ is an instance of what is known as a *greedoid*: a combinatorial structure generalizing (and closely connected to) matroids. Greedoids have been extensively studied in [KoLoSc91], and many classes of greedoids are known. In [GriPet19], Petrov and I show that $\mathcal{F}$ is a *strong greedoid* (also known as a *Gauss greedoid*). As a consequence, for any fixed $m \in \mathbb{N}$, the maximum-perimeter m-element subsets of E are the bases of a matroid.

An even better result I have obtained very recently (still unpublished) is that $\mathcal{F}$ is a *Gaussian elimination greedoid* (a greedoid analogue of the notion of a representable matroid). More precisely, over any field with at least $|E|$ many elements, $\mathcal{F}$ can be represented by a vector family.

Our perimeter maximization problem, and the greedy algorithm solving it, generalize Bhargava's "P-orderings" introduced in [Bharga97, §2]. A related problem in mathematical biology has been studied by Moulton, Semple and Steel [MoSeSt06]: to find a set of m leaves of a given phylogenetic tree that maximizes *phylogenetic diversity* (defined as the perimeter of the subtree that connects these m leaves). Again, the solutions form a strong greedoid ([MoSeSt06, Theorem 3.2]). While the triples (E, w, d) considered by Petrov and myself are closely related to phylogenetic trees[4], the two problems are nevertheless distinct, as phylogenetic diversity is not perimeter. The exact relation between these problems is worthy of understanding.

Simplicial complexes from digraphs. Consider a directed graph with edge set E, and fix two vertices s and t. A subset F of E will be called *path-free* if there is no path

[4]To some extent, they can be translated into each other. For one direction, see [GriPet19, Example 2.9].

from s to t using only edges from F. Is the number of path-free subsets of E odd or even? Lukas Katthän, Joel Lewis and I have shown that it is even if the graph has a "useless edge" (i.e., an edge that appears in no path from s to t) or a (directed) cycle; otherwise it is odd. Behind this curious puzzle is some nontrivial theory: The path-free subsets of E form a *simplicial complex*, whose Euler characteristic is always 0 or ± 1 and whose homotopy type (always contractible or homotopic to a sphere) we have determined. These results (and a bit more) are on their way to publication.

Critical groups in representation theory.[5] The concept of a *critical group* originated from the theory of *chip-firing* on digraphs (also known as the *sandpile model*). The group can be defined as the cokernel (over $\mathbb{Z}$) of the digraph's *reduced Laplacian* (the submatrix of its Laplacian obtained by removing a certain row and a certain column); from this point of view, the theory of chip-firing can be recast as a study of this Laplacian acting on the integer lattice. In particular, it has been observed by Gabrielov, Benkart, Klivans, Reiner and others (e.g., [BeKlRe16]) that the reduced Laplacian of a strongly connected digraph is a *nonsingular M-matrix* (an integer matrix whose off-diagonal entries are nonpositive, and whose inverse has nonnegative entries; several equivalent definitions exist), and that most of the theory of chip-firing can be recovered from this property. Hence, wherever one encounters a nonsingular M-matrix, one can construct a "chip-firing theory" with similar properties to that arising from a digraph.

Benkart, Klivans, Reiner, and Gaetz ([BeKlRe16], [Gaetz16]) have recently used this strategy to build a "chip-firing theory" from a faithful representation V of a finite group G (over $\mathbb{C}$). The analogue of the Laplacian here is the matrix $L_V = nI - M_V$, where I is the identity matrix, n is the dimension of V, and M_V is the matrix ("*McKay matrix* of V") that represents tensoring by V on the Grothendieck ring of G. Explicitly, if the irreducible representations of G are $S_1, S_2, \ldots, S_{\ell+1}$, then M_V is an $(\ell+1) \times (\ell+1)$-matrix with (i,j)-th entry $[S_i \otimes V : S_j]$. The analogue of the reduced Laplacian is the submatrix of L_V obtained by removing the row and the column corresponding to the trivial representation. They have shown that this reduced Laplacian is a nonsingular M-matrix (it is here that the faithfulness of V shows its relevance), and computed the order of the critical group (the cokernel of the reduced Laplacian) in terms of the character of V.

In a paper [GrHuRe17] with Reiner and Huang, we extend this construction to an arbitrary representation V of a *finite-dimensional Hopf algebra* A over any algebraically closed field. Instead of faithfulness, we now need a slightly subtler property of V (which we call "*tensor-richness*": every simple A-module appears in a composition series of some $V^{\otimes k}$) to ensure that our "reduced Laplacian" is a nonsingular M-matrix. We fully describe the critical group of the regular representation and some further examples; moreover, we express the order of the critical group in the case when $A = \mathbb{F}_p[G]$ for some finite group G (in terms of Brauer characters of G). We have no formula for the order of the critical group of a general V over a general A. Some further questions left to explore are

[5] See http://www.cip.ifi.lmu.de/~grinberg/algebra/madison17.pdf for slides of a talk on this subject, which might clarify the below.

- the extent to which theory generalizes even further to objects in tensor categories (the basic facts do);
- the behavior of the critical groups under restriction and induction.

Birational rowmotion. Given a finite poset P, we can form another poset $\widehat{P}$ by adjoining a global minimum (called 0) and a global maximum (called 1) to P. Given a field (or semifield) $\mathbb{K}$, we consider the set $\mathbb{K}^{\widehat{P}}$ of all labelings of the elements of $\widehat{P}$ by elements of $\mathbb{K}$. On this set, David Einstein and James Propp have defined a birational equivalence, which they call *birational rowmotion*, and which generalizes the notion of rowmotion on the order ideals of P. Einstein and Propp have experimentally observed that, for various special classes of posets P, this birational equivalence has finite order (i.e., a certain power of it is the identity). In [GriRob14], Tom Roby and I prove these observations and some others. The most prominent case is that when P is a "rectangle" (i.e., a product of two chains with p and q elements, respectively); in this case, the order of birational rowmotion is $p + q$. This generalizes Schützenberger's classical result that the *promotion* operator ∂ on the semistandard Young tableaux of a given rectangular shape with entries in $\{1, 2, \ldots, n\}$ satisfies $\partial^n = \mathrm{id}$.

(The generalization is not trivial; translating from tableaux to labellings of a rectangle poset P involves an intermediate step through *Gelfand-Tsetlin triangles*, and the sidelengths of the rectangular partition are not those of P.)

Another class of finite posets P for which birational rowmotion has finite order are graded forests (i.e., forests where all leaves are at the same depth).

The proof of this result on rectangles is inspired by Volkov's proof of the type-AA Zamolodchikov conjecture [Volkov06]; we also analyze some other posets like graded forests (by an inductive argument) and some triangle-shaped posets (via a "folding" reduction to the rectangular case). One case – that of "trapezoidal" posets, which can be seen as a type-B analogue of rectangles – is still unresolved, and a conjecture by Nathan Williams connects it to the rectangular case in a remarkable way [GriRob14, §19], which (if correct) generalizes a conjecture by Elizalde on noncrossing families of Dyck paths [Elizal14].

Further developments have happened in the last few years[6]. For one, Max Glick has found a more direct relation between birational rowmotion in the case of a rectangle and the type-AA Zamolodchikov Y-system, whereas Alexander Postnikov has suggested a connection to the octahedral recurrence which still remains to be fully understood. Richard Stanley suggested a generalization of the case of graded forests, which is currently being written up by others. James Propp has conjectured some "homomesies" (algebraic identities holding for each orbit under birational rowmotion), some of which I have proven. Tom Roby and Gregg Musiker have recently found explicit formulas for iterates of birational rowmotion on rectangles [MusRob17], which are inspired by [GriRob14] but

[6] For a list of papers inspired by [GriRob14], see the citations listed at Google Scholar, e.g., for the second half of [GriRob14]: https://scholar.google.com/scholar?cites=445460616416961588&as_sdt=5,24&sciodt=0,24&hl=en .

are independent of it (and provide new proofs for its main results). Their formulas could help prove the remaining homomesies.

Among the questions that I am planning to study are the following:

- The finite order of birational rowmotion in the above-mentioned "trapezoidal" case needs to be proven.
- The finite order of birational rowmotion proven for rectangles seems to hold even if $\mathbb{K}$ is replaced by a (noncommutative) semifield, up to a certain conjugation[7]; Tom Roby and I have proven this for the most part, but the correct theoretical underpinnings of the noncommutative setting still need to be found.

Roby and I have proven this based on the Gelfand–Retakh theory of noncommutative Plücker coordinates, but foundational work is still in progress. There should be material for at least three smaller papers here.

Subdivision algebras. Fix a commutative ring $\mathbf{k}$, two elements $\alpha \in \mathbf{k}$ and $\beta \in \mathbf{k}$ and a positive integer n. Let $\mathcal{X}$ be the polynomial ring over $\mathbf{k}$ in the $n(n-1)/2$ indeterminates $x_{i,j}$ for all $1 \leq i < j \leq n$. Consider the ideal $\mathcal{J}$ of $\mathcal{X}$ generated by all polynomials of the form $x_{i,j}x_{j,k} - x_{i,k}(x_{i,j} + x_{j,k} + \beta) - \alpha$ for $1 \leq i < j < k \leq n$.

In [Grinbe17a], I find a (combinatorially described) basis of the $\mathbf{k}$-module $\mathcal{X}/\mathcal{J}$, and show that a certain "straightening-and-substitution" map $\mathcal{X}/\mathcal{J} \to \mathbf{k}$ is well-defined. The latter settles (and generalizes) a conjecture by Karola Mészáros. The quotient algebra $\mathcal{X}/\mathcal{J}$, in the case when $\alpha = 0$, relates to various seemingly disjoint areas of combinatorics: It was first introduced by Mészáros in [Meszar09] as a commutative analogue of Anatol Kirillov's quasi-classical Yang-Baxter algebra; it then was used by Laura Escobar and Karola Mészáros [EscMes15, §5] in the study of Grothendieck polynomials; finally, for $\beta = 1$ and $\alpha = 0$, it can be embedded as a subalgebra ($x_{i,j} \mapsto \dfrac{y_j}{y_i - y_j}$) into the localization of the polynomial ring $\mathbf{k}[y_1, y_2, \ldots, y_n]$ at the multiplicative subset generated by all differences $y_i - y_j$ (with $i < j$), which is a well-known object from algebraic geometry. In particular, $\mathcal{X}/\mathcal{J}$ can be regarded as a deformation of the (large) Orlik-Terao algebra of the braid arrangement. This suggests further questions, which I plan to study, such as the action of the symmetric group on $\mathcal{X}/\mathcal{J}$, and whether the Orlik-Terao algebras of other hyperplane arrangements can be similarly deformed.[8]

Recent discussions with Victor Reiner and Nicholas Early have revealed a connection between $\mathcal{X}/\mathcal{J}$ and the Whitehouse representations of symmetric group [Whiteh97], which we are in the process of exploring.

Hopf Algebras in Combinatorics. The lecture notes [GriRei15] by Victor Reiner and myself are, predominantly, an expository work, providing probably the first introduction into the subject with a strong emphasis on the combinatorics. Nevertheless, a

[7] See the end of my 2014 talk in Vienna (http://www.cip.ifi.lmu.de/~grinberg/algebra/vienna2014.pdf) for an example of this noncommutative phenomenon.

[8] On a more speculative note: The relations defining $\mathcal{X}/\mathcal{J}$ (for $\alpha = 0$) also bear an uncanny resemblance to the axioms of a Rota-Baxter algebra: see https://mathoverflow.net/questions/286510 . Whether there is anything behind this similarity remains to be seen.

few of my exercises contain minor innovations; these are Exercise 2.9.4(f) (the "third comultiplication" Δ_r on the symmetric functions), Exercise 2.9.11(c), Exercise 6.1.32, and Exercise 6.1.33. The first two are concerned with symmetric functions, the latter two with certain extensions of the concept of Lyndon words. The proof of [GriRei15, Theorem 6.2.2] also appears to be new.

A related preprint I have provisionally completed in 2016 is an approach to the quasisymmetric Bernstein homomorphism (and the internal comultiplication of QSym) using universal properties [Grinbe15d]. This work is semi-expository, as the results were known at least to some of the experts (e.g., Aguiar) and could be obtained with some work from published claims; but the proofs (using universal properties and a change of base ring that is almost too simple to be true) appear to be new, and my preprint might clear some hurdles for newcomers on this rough terrain. Questions yet remain (chiefly about how far the results can be generalized).

SageMath. I have contributed some functionality to the open-source SageMath computer algebra system (in particular, parts of its support for symmetric functions), and reviewed some others' code.

Research plans, work in progress, future directions

The Reiner-Saliola-Welker conjecture. Let $n \in \mathbb{N}$, and consider the group algebra $\mathbb{C}\mathfrak{S}_n$ of the n-th symmetric group $\mathfrak{S}_n$. For any $k \in \{1, 2, \ldots, n\}$, we define an element RSW_k of $\mathbb{C}\mathfrak{S}_n$ as $\sum_{w \in \mathfrak{S}_n} (\mathrm{noninv}_k\, w) \cdot w$, where $\mathrm{noninv}_k\, w$ is the number of all k-element subsets I of $\{1, 2, \ldots, n\}$ such that $w\mid_I$ is strictly increasing.

A surprisingly difficult result by Reiner, Saliola and Welker ([ReSaWe11, Theorem 1.1]) states that the elements $\mathrm{RSW}_1, \mathrm{RSW}_2, \ldots, \mathrm{RSW}_n$ commute pairwise. They further conjecture that each of these elements (viewed as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}\mathfrak{S}_n$, given by left multiplication) has integer spectrum (i.e., all its eigenvalues are integers). This suggests the existence of a combinatorially meaningful joint eigenbasis for these operators (similar to, e.g., the seminormal basis for $\mathbb{C}\mathfrak{S}_n$). Indeed, an eigenbasis for RSW_1 has been found recently by Dieker and Saliola [DieSal15], which led to a proof of the fact that the eigenvalues of RSW_1 are integers between 0 and n^2. The proof is a tour-de-force of tensor algebra, representation theory of symmetric groups and Young tableau theory, and naturally suggests further questions (for instance, it appears to contain homological arguments in disguise).

The element RSW_1 of $\mathbb{C}\mathfrak{S}_n$ is known as the "random-to-random operator" on $\mathfrak{S}_n$, due to the following probabilistic interpretation: Imagine a shelf with n books labelled by $1, 2, \ldots, n$. In one step, we take out a randomly chosen book from the shelf, and put it back at a randomly chosen position[9]. The transition matrix of this Markov chain is the representing matrix of RSW_1. Thus, the Dieker-Saliola result claims that this Markov chain has integer eigenvalues; from this viewpoint, it appears surprising that such a simple-looking result has not been proven until 2015, and not without such difficulties!

[9] I.e., we place it in one of the n gaps (either between two books or at the beginning or at the end of the shelf) with equal probability. We do not model gaps between different books as intervals of different sizes (although that, too, might lead to interesting questions).

The Markov chain just described is reminiscent of a simpler and better-known Markov chain: the *Tsetlin library*. Here, one puts the book back at the beginning of the shelf rather than at a random point. This chain is, indeed, closely related, and has a number of similar properties. It, too, has integer eigenvalues, and also corresponds to the first element of a sequence $\mathrm{R2T}_1, \mathrm{R2T}_2, \ldots, \mathrm{R2T}_n$ of pairwise commuting elements of $\mathbb{C}\mathfrak{S}_n$ [10]. These elements $\mathrm{R2T}_k$ not only commute, but also (unlike the RSW_k) span a subalgebra of $\mathbb{C}\mathfrak{S}_n$, and their products can be explicitly expanded; this is a particular case of the famous "Solomon's Mackey formula" for the descent algebra of $\mathfrak{S}_n$.

The symmetric group algebra $\mathbb{C}\mathfrak{S}_n$ is a Hopf algebra (as any group algebra is) and thus has an antipode S. It is not hard to see that $\mathrm{RSW}_k = \dfrac{1}{(n-k)!} \mathrm{R2T}_k \, S\left(\mathrm{R2T}_k\right)$ for every k.

In [Grinbe15b], I describe the kernel of the action of the random-to-top operator $\mathrm{R2T}_1$ on the tensor algebra (or, more precisely, the kernels of two of its actions – one "unsigned" and one "signed") over fields of arbitrary characteristic (and, in the "signed" case, even over arbitrary commutative rings). While the methods used do not directly apply to diagonalizing $\mathrm{R2T}_k$ and RSW_k (which seems out of reach in positive characteristic), they might provide some valuable insights. I hope to explore the algebra of the RSW_k and $\mathrm{R2T}_k$ more thoroughly. Questions of interest are:

- While the integrality of the eigenvalues of RSW_1 has been proved, the question still stands for $\mathrm{RSW}_2, \mathrm{RSW}_3, \ldots, \mathrm{RSW}_n$.
- The proof of the commutativity of the $\mathrm{RSW}_1, \mathrm{RSW}_2, \ldots, \mathrm{RSW}_n$ given in [ReSaWe11, Theorem 1.1] is unsatisfactory (neither slick nor intuitive – its author called it "horrendous"); a better one needs to be found.
- It also feels that the eigenvalues of RSW_1 (at least the fact that they are integers between 0 and n^2) should have a simpler proof than the one in [DieSal15].

Carlitz-Witt vectors and function-field symmetric functions. The "field with one element" ($\mathbb{F}_1$) stands for the meta-mathematical idea that the ring $\mathbb{Z}$ has deep similarities with the polynomial rings $\mathbb{F}_q[T]$ over finite fields; that the combinatorics of sets has analogies with the linear algebra of $\mathbb{F}_q$-vector spaces; that the symmetric group is, in some sense, the "$q = 1$" version of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$. No fully explicatory mathematical foundation for these analogies is known, but they have been highly useful as heuristics many times, and a vast number of objects have been translated from one world to the other. Some basic examples can be found in [Cohn04]; another is the theory of Carlitz polynomials [Conrad15].

The ring of symmetric functions is deeply connected with integer partitions (e.g., almost all of its well-known bases are indexed by partitions); these correspond to conjugacy classes of permutations in symmetric groups. This raises the question of finding an "$\mathbb{F}_q$-analogue" of this ring which is similarly connected to "$\mathbb{F}_q[T]$-partitions" (i.e., sequences

[10] The notation R2T stands for "random-to-top", which relates to how these elements are defined.

$(p_1, p_2, p_3, \ldots)$ of monic polynomials in $\mathbb{F}_q[T]$ such that $\cdots \mid p_3 \mid p_2 \mid p_1$ and such that all but finitely many p_i are $= 1$), or, equivalently, to conjugacy classes of matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. Partial results towards the construction of such an analogue can be found in my work-in-progress [Grinbe15c]. My approach to finding such an analogue takes a detour through the notion of *Witt vectors*, which are an affine group whose coordinate ring is the symmetric functions ([Hazewi08, §10]). An $\mathbb{F}_q$-analogue of the Witt vectors (the *Carlitz-Witt vectors*, as I call them due to their use of Carlitz polynomials) is not hard to construct, and its coordinate ring can then be regarded as an $\mathbb{F}_q$-analogue of the symmetric functions. However, the combinatorics of this $\mathbb{F}_q$-analogue still remains to be understood, as the theory of Witt vectors reflects but little of the combinatorics of symmetric functions.

This is not directly related to Hall algebras (see [Dycker15] for a novel categorical viewpoint by Dyckerhoff and Kapranov, which I am highly interested in), which can also be regarded as an $\mathbb{F}_q$-analogue of symmetric functions, but which is a different sort of $\mathbb{F}_q$-analogue. (Hall algebras, at least in the simplest case, still have bases indexed by integer partitions; only their structure constants rely on $\mathbb{F}_q$.)

Other projects. Further ongoing research of mine includes a sequel to [GriPos17] that generalizes the main result to non-reduced expressions (I have been able to prove the first half of that generalization).

I have been engaging in (proof-carrying) expository writing for over a decade now; besides the already mentioned lecture notes [GriRei15], I have written a detailed elementary introduction to determinants ([Grinbe16]) and various smaller notes available on my website, as well as many answers on MathOverflow and math.stackexchange. In Spring 2019, I have developed an undergraduate course on abstract algebra and written a set of lecture notes for it [Grinbe19c] (currently finished and detailed in its first 2 chapters; the rest will likely be finished during my next abstract algebra course). I am currently in the process of creating lecture notes on enumerative combinatorics for my Drexel class (Math 222).

Further work I am currently doing or planning includes:

- noncommutative generalizations of Abel-Hurwitz type identities [Grinbe17b] (with plans to generalize further, possibly including Strehl's [Strehl92] results);
- RSK and dual RSK algorithms for cylindric and periodic tableaux (continuing Neyman's work [Neyman14], which resulted from a PRIMES project under my mentorship).

One other area of interest that I am planning to explore is formal proving in Coq/ssreflect. Coq is a proof assistant for constructive mathematics that has proven its worth in logic and software formalization, and ssreflect is a "sublanguage" that makes it particularly suitable to formalization of complex mathematical proofs (such as those of the four-color conjecture and the Feit-Thompson theorem, developed at the MSR-Inria joint centre by a team under Georges Gonthier). Recent work (mostly) by Florent Hivert (the coq-combi project) has created a library of formally verified results from algebraic combinatorics

(Littlewood-Richardson rule, hook-length formula and more) in ssreflect, and one of my long-term goals is to extend this library. Another promising proof assistant is Lean, and it would be interesting to port Coq's combinatorics libraries to it.

References

[AaGrSc18] Erik Aas, Darij Grinberg, Travis Scrimshaw, *Multiline queues with spectral parameters*, arXiv:1810.08157.

[AasLin18] Erik Aas, Svante Linusson, *Continuous multi-line queues and TASEP*, to appear in Ann. Inst. H. Poincaré Comb. Phys. Interact., arXiv:1501.04417.

[AgBeSo14] Marcelo Aguiar, Nantel Bergeron, Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, 2014, updated version of a paper published in: Compositio Mathematica **142** (2006), issue 1, pp. 1–30.

[ApaZei16] Moa Apagodu, Doron Zeilberger, *Using the "Freshman's Dream" to Prove Combinatorial Congruences*, Amer. Math. Monthly, **124**, No. 7 (2017), pp. 597–608, arXiv:1606.03351v2.

[AAMP11] Chikashi Arita, Arvind Ayyer, Kirone Mallick, Sylvain Prolhac, *Recursive structures in the multispecies TASEP*, J. Phys. A, **44**(33):335004, 2011, arXiv:1104.3752.

[BakRic08] Andrew Baker, Birgit Richter, *Quasisymmetric functions from a topological point of view*, Mathematica Scandinavica, **103**, No. 2 (2008), pp. 208–242, arXiv:math/0605743v4.

[BBSSZ13a] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions*, Canadian Journal of Mathematics **66** (2014), pp. 525–565. Also available as arXiv preprint arXiv:1208.5191v3.

[BeCiFu99] Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, Journal of Algebra **219** (1999), pp. 728–746. For a preprint, see `https://arxiv.org/abs/alg-geom/9705024v1` .

[BeKlRe16] Georgia Benkart, Caroline Klivans, and Victor Reiner, *Chip firing on Dynkin diagrams and McKay quiver*, arXiv:1601.06849v2, Mathematische Zeitschrift **290** (Oct. 2018), pp. 615–648, doi:10.1007/s00209-017-2034-5.

[BeMiTa18] Anna Bertiger, Elizabeth Milićević, Kaisa Taipale, *Equivariant quantum cohomology of the Grassmannian via the rim hook rule*, Algebraic Combinatorics **1** (2018) no. 3, p. 327–352.

[Bharga97] Manjul Bhargava, *P-orderings and polynomial functions on arbitrary subsets of Dedekind rings*, J. reine. angew. Math. **490** (1997), 101–127.

[Bharga00] Manjul Bhargava, *The Factorial Function and Generalizations*, The American Mathematical Monthly, Vol. **107**, No. 9 (Nov., 2000), pp. 783–799.

[Bharga09] Manjul Bhargava, *On P-orderings, rings of integer-valued polynomials, and ultrametric analysis*, Journal of the American Mathematical Society Vol. 22, No. 4 (October 2009), pp. 963–993.

[BorGri13] James Borger, Darij Grinberg, *Boolean Witt vectors and an integral Edrei-Thoma theorem*, arXiv:1311.5031, Selecta Mathematica, **22**(2), pp. 595–629 (doi:10.1007/s00029-015-0198-6).

[Cohn04] Henry Cohn, *Projective geometry over* $\mathbb{F}_1$ *and the Gaussian binomial coefficients*, Amer. Math. Monthly **111**, No. 6 (2004), pp. 487–495, arXiv:math/0407093v1.

[CoKrWa09] Aldo Conca, Christian Krattenthaler, Junzo Watanabe, *Regular Sequences of Symmetric Polynomials*, Rend. Sem. Mat. Univ. Padova **121** (2009), pp. 179–199.

[Conrad15] Keith Conrad, *Carlitz extensions*, 16 September 2015. `http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/carlitz.pdf`

[CoMaWi18] Sylvie Corteel, Olya Mandelshtam, Lauren Williams, *From multiline queues to Macdonald polynomials via the exclusion process*, arXiv:1811.01024.

[DieSal15] A. B. Dieker, Franco Saliola, *Spectral analysis of random-to-random Markov chains*, Advances in Mathematics **323** (2018), pp. 427–485, arXiv:1509.08580v3.

[Dycker15] Tobias Dyckerhoff, *Hall Algebras - Bonn, Wintersemester 14/15*, 5 February 2015, `https://web.archive.org/web/20150601115158/http://www.math.uni-bonn.de/people/dyckerho/notes.pdf`.

[Elizal14] Sergi Elizalde, *Bijections for pairs of non-crossing lattice paths and walks in the plane*, European Journal of Combinatorics **49**, October 2015, pp. 25–41, arXiv:1406.1709v1.

[EscMes15] Laura Escobar, Karola Mészáros, *Subword complexes via triangulations of root polytopes*, arXiv:1502.03997.

[Fulton97] William Fulton, *Young Tableaux*, London Mathematical Society Student Texts 35, Cambridge University Press 1997.

[Gaetz16] Christian Gaetz, *Critical groups of group representations*, Lin. Alg. Appl. **508** (2016), pp. 91–99, arXiv:1606.00798.

[GaGrLi15] Pavel Galashin, Darij Grinberg, Gaku Liu, *Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions*, arXiv:1509.03803v2, Electronic Journal of Combinatorics **23**(3) (2016), #P3.14.

[GesZhu17] Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, Advances in Mathematics, **332**, 9 July 2018, pp. 85–141. Also available at arXiv:1706.00750v3.

[GorWil74] Manfred Gordon, E. Martin Wilkinson, *Determinants of Petrie matrices*, Pacific Journal of Mathematics **51**, no. 2, 1974.

[Grinbe11] Darij Grinberg, *Poincaré-Birkhoff-Witt type results for inclusions of Lie algebras*, diploma thesis.

[Grinbe14] Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, arXiv:1410.0079v6, published in the Canadian Journal of Mathematics, **69**(2017), pp. 21–53.

[Grinbe15a] Darij Grinberg, *Double posets and the antipode of QSym*, arXiv:1509.08355v3, published in the Electronic Journal of Combinatorics **24**(2) (2017), #P2.22.

[Grinbe15b] Darij Grinberg, *The signed random-to-top operator on tensor space*, arXiv:1505.01201.

[Grinbe15c] Darij Grinberg, *Do the symmetric functions have a function-field analogue?* (`http://www.cip.ifi.lmu.de/~grinberg/algebra/schur-ore.pdf`), work in progress.

[Grinbe15d] Darij Grinberg, *The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality*, arXiv:1604.02969.

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 2015–2018 (github repository).

[Grinbe17a] Darij Grinberg, *t-unique reductions for Mészáros's subdivision algebra*, arXiv:1704.00839, SIGMA **14** (2018), 078, 34 pages.

[Grinbe17b] Darij Grinberg, *Noncommutative Abel-like identities*, work in progress.

[Grinbe17c] Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*, arXiv:1806.04114, Electronic Journal of Combinatorics **25** (2018), Paper #P4.17.

[Grinbe17d] Darij Grinberg, *On binomial coefficients modulo squares of primes*, arXiv:1712.02095, Integers **19** (2019), A14.

[Grinbe18a] Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, work in progress.

[Grinbe18b] Darij Grinberg, *A quotient of the ring of symmetric functions generalizing quantum cohomology (extended abstract)*, Séminaire Lotharingien de Combinatoire **82B** (2019), Article #86, FPSAC 2019 proceedings.

[Grinbe19a] Darij Grinberg, *A double Sylvester determinant*, arXiv:1901.11109.

[Grinbe19b] Darij Grinberg, *Petrie symmetric functions*, work in progress (very rough outline).

[Grinbe19c] Darij Grinberg, *Introduction to Modern Algebra*, 2019– (github repository).

[GriPet19] Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.

[GrHuRe17] Darij Grinberg, Jia Huang, Victor Reiner, *Critical groups for Hopf algebra modules*, arXiv:1704.03778, accepted in Mathematical Proceedings of the Cambridge Philosophical Society.

[GriOlv18] Darij Grinberg, Peter J. Olver, *The n body matrix and its determinant*, arXiv:1802.02900, SIAM Journal on Applied Algebra and Geometry, **3**(1), 2019, pp. 67–86.

[GriPos17] Darij Grinberg, Alexander Postnikov, *Proof of a conjecture of Bergeron, Ceballos and Labbé*, arXiv:1603.03138, New York Journal of Mathematics **23** (2017), pp. 1581–1610.

[GriRei15] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, May 11, 2018, lecture notes, arXiv:1409.8356v5.

[GriRob14] Darij Grinberg, Tom Roby, *Iterative properties of birational rowmotion*, arXiv:1402.6178v6. A modified version of this paper was published in two parts: Electronic Journal of Combinatorics **23**, issue 1, #P1.33, 2016 (part 1 of 2) and Electronic Journal of Combinatorics **22**, issue 3, #P3.40, 2015 (part 2 of 2).

[Hazewi08] Michiel Hazewinkel, *Witt vectors. Part 1*, arXiv:0804.3888v1.

[Hsiao07] Samuel K. Hsiao, *Structure of the peak Hopf algebra of quasisymmetric functions*, 2007.

[Kirill16] Anatol N. Kirillov, *On Some Quadratic Algebras I $\frac{1}{2}$: Combinatorics of Dunkl and Gaudin Elements, Schubert, Grothendieck, Fuss-Catalan, Universal Tutte and Reduced Polynomials*, SIGMA **12** (2016), 002, 172 pages, arXiv:1502.00426v3.

[KoLoSc91] Bernhard Korte, László Lovász, Rainer Schrader, *Greedoids*, Algorithms and Combinatorics #4, Springer 1991.

[LLPT95] D. Laksov, A. Lascoux, P. Pragacz, and A. Thorup, *The LLPT Notes*, edited by A. Thorup, 1995, `http://www.math.ku.dk/noter/filer/sympol.pdf` .

[LakTho12] Dan Laksov, Anders Thorup, *Splitting algebras and Schubert calculus*, Indiana Univ. Math. J. **61**, No. 3 (2012), pp. 1253–1312. `https://www.jstor.org/stable/24904081` (also `https://doi.org/10.1512/iumj.2012.61.4672`).

[LamPyl07] Thomas Lam, Pavlo Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmanians*, arXiv:0705.2189v1. An updated version was later published in: International Mathematics Research Notices, Vol. 2007, Article ID rnm125, 48 pages. doi:10.1093/imrn/rnm125.

[LiuPol19] Linyuan Liu, Patrick Polo, *On the cohomology of line bundles over certain flag schemes II*, arXiv:1908.08432v1.

[Macdon95] Ian G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford University Press 1995.

[MalReu98] Claudia Malvenuto, Christophe Reutenauer, *Plethysm and conjugation of quasi-symmetric functions*, Discrete Mathematics **193** (1998), Issues 1–3, pp. 225–233.

[Manive01] Laurent Manivel, *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*, translated from the French by John R. Swallow, American Mathematical Society 2001.

[Meszar09] Karola Mészáros, *Root polytopes, triangulations, and the subdivision algebra. I*, Trans. Amer. Math. Soc. **363** (2011), pp. 4359–4382.

[Morava15] Jack Morava, *Homotopy-theoretically enriched categories of noncommutative motives*, Research in the Mathematical Sciences, 2015, **2**:8, arXiv:1402.3693.

[MoSeSt06] Vincent Moulton, Charles Semple, Mike Steel, *Optimizing phylogenetic diversity under constraints*, Journal of Theoretical Biology **246** (2007), pp. 186–194.

[MusRob17] Gregg Musiker, Tom Roby, *Paths to Understanding Birational Rowmotion on Products of Two Chains*, arXiv:1801.03877, to appear in Algebraic Combinatorics.

[Neyman14] Eric Neyman, *Cylindric Young Tableaux and their Properties*, arXiv:1410.5039v2.

[NoThTo17] Jean-Christophe Novelli, Jean-Yves Thibon, Frédéric Toumazet, *Noncommutative Bell polynomials and the dual immaculate basis*, Algebraic Combinatorics **1** (2018), no. 5, pp. 653–676.

[Oesing18] Jakob Oesinghaus, *Quasi-symmetric functions and the Chow ring of the stack of expanded pairs*, arXiv:1806.10700.

[Postni05] Alexander Postnikov, *Affine approach to quantum Schubert calculus*, Duke Mathematical Journal **128**, No. 3, pp. 473–509. `https://math.mit.edu/~apost/papers/affine_approach.pdf`

[ReSaWe11] Victor Reiner, Franco Saliola, Volkmar Welker, *Spectra of Symmetrized Shuffling Operators*, arXiv:1102.2460v2.

[Sagan01] Bruce Sagan, *The Symmetric Group*, 2nd edition, Springer 2001.

[Shucha84] Alan Shuchat, *Matrix and Network Models in Archaeology*, Mathematics Magazine **57**, No. 1 (Jan., 1984), pp. 3–14.

[Stan99] Richard Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press, 1999.

[Stembr97] John Stembridge, *Enriched P-partitions*, Trans. AMS **349** (1997), pp. 763–788.

[Strehl92] Volker Strehl, *Identities of Rothe-Abel-Schläfli-Hurwitz-type*, Discrete Mathematics **99**, Issues 1–3, 2 April 1992, pp. 321–340.

[Volkov06] Alexandre Yu. Volkov, *On the Periodicity Conjecture for Y-systems*, Commun. Math. Phys. **276** (2007), pp. 509–517, DOI 10.1007/s00220-007-0343-y. A preprint of this paper is also available under the name *On Zamolodchikov's Periodicity Conjecture* as arXiv:hep-th/0606094v1.

[Whiteh97] Sarah Whitehouse, *The Eulerian representations of Σ_n as restrictions of representations of Σ_{n+1}*, Journal of Pure and Applied Algebra **115**, Issue 3, 14 March 1997, pp. 309–320.

[Yelius16] Damir Yeliussizov, *Duality and deformations of stable Grothendieck polynomials*, arXiv:1601.01581v2, Journal of Algebraic Combinatorics **45** (2017), issue 1, pp. 295–344.