**6th QEDMO 2009, Problem 4 (the Cauchy identity)**

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} (X+k)^k (Y-k)^{n-k} = \sum_{t=0}^{n} \frac{n!}{t!} (X+Y)^t \qquad (1)$$

in the polynomial ring $\mathbb{Z}[X,Y]$.

*Remark:* Here, we denote by $\mathbb{N}$ the set $\{0,1,2,...\}$ (and not the set $\{1,2,3,...\}$, as some authors do).

***Solution by Darij Grinberg***

We start with a very useful lemma:

**Theorem 1.** Let $R$ be a commutative ring with unity. Let $N \in \mathbb{N}$. Then, the equalities

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell = 0 \qquad \text{for every } \ell \in \{0,1,...,N-1\} \qquad (2)$$

and

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^N = (-1)^N N! \qquad (3)$$

are satisfied in $R$.

*Proof of Theorem 1.* We will prove Theorem 1 by induction over $N$:

*Induction base:* If $N=0$, then (2) is true (since there doesn't exist any $\ell \in \{0,1,...,N-1\}$ when $N=0$) and (3) is true (because if $N=0$, then

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^N = \sum_{k=0}^{0} (-1)^k \binom{0}{k} k^0 = \underbrace{(-1)^0}_{=1} \underbrace{\binom{0}{0}}_{=1} \underbrace{0^0}_{=1} = 1$$

and

$$(-1)^N N! = \underbrace{(-1)^0}_{=1} \underbrace{0!}_{=1} = 1,$$

thus

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^N = (-1)^N N!$$

). Hence, if $N=0$, then Theorem 1 holds. This completes the induction base.

*Induction step:* Let $n \in \mathbb{N}$. Assume that Theorem 1 holds for $N=n$. In order to complete the induction step, we must prove that Theorem 1 also holds for $N=n+1$.

We have assumed that Theorem 1 holds for $N = n$. In other words, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^\ell = 0 \qquad \text{for every } \ell \in \{0, 1, ..., n-1\} \tag{4}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^n = (-1)^n n!. \tag{5}$$

Now, every $m \in \mathbb{N}$ satisfies

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^m = \sum_{k=0}^{n+1} (-1)^k \left( \binom{n}{k} + \binom{n}{k-1} \right) k^m$$

$$\left( \text{since } \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \text{ by the recurrence equation of the binomial coefficients} \right)$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} k^m + \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} k^m$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} k^m + \sum_{k=-1}^{n} (-1)^{k+1} \binom{n}{(k+1)-1} (k+1)^m$$

(here we substituted $k+1$ for $k$ in the second sum)

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} k^m + \sum_{k=-1}^{n} \underbrace{(-1)^{k+1}}_{=-(-1)^k} \binom{n}{k} (k+1)^m$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} k^m + \sum_{k=-1}^{n} \left( -(-1)^k \right) \binom{n}{k} (k+1)^m$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} k^m + \sum_{k=0}^{n} \left( -(-1)^k \right) \binom{n}{k} (k+1)^m$$

$$\left( \begin{array}{c} \text{here we replaced the } \sum\limits_{k=-1}^{n} \text{ sign by an } \sum\limits_{k=0}^{n} \text{ sign, since the addend for } k=-1 \text{ is zero} \\ \left(\text{as } \binom{n}{k} = 0 \text{ for } k = -1\right) \end{array} \right)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m + \sum_{k=0}^{n} \left( -(-1)^k \right) \binom{n}{k} (k+1)^m$$

$$\left( \begin{array}{c} \text{here we replaced the } \sum\limits_{k=0}^{n+1} \text{ sign by an } \sum\limits_{k=0}^{n} \text{ sign, since the addend for } k=n+1 \text{ is zero} \\ \left(\text{as } \binom{n}{k} = 0 \text{ for } k = n+1\right) \end{array} \right)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m - \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^m = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(k^m - (k+1)^m\right)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( - \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^\ell \right)$$

$$\left( \begin{array}{c} \text{since the binomial formula yields} \\ (k+1)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} k^\ell = \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^\ell + \underbrace{\binom{m}{m}}_{=1} k^m = \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^\ell + k^m \\ \text{and thus } k^m - (k+1)^m = - \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^\ell \end{array} \right)$$

$$= - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^\ell = - \sum_{\ell=0}^{m-1} \binom{m}{\ell} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^\ell. \qquad (6)$$

If $m \in \{0, 1, ..., n\}$, then this yields

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^m = - \sum_{\ell=0}^{m-1} \binom{m}{\ell} \underbrace{\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^\ell}_{\substack{=0 \text{ by (4), since } 0 \leq \ell \leq m-1 \\ \text{and } m \in \{0,1,...,n\} \\ \text{yield } \ell \in \{0,1,...,n-1\}}} = - \sum_{\ell=0}^{m-1} \binom{m}{\ell} 0 = 0.$$

So we have proved that

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^m = 0 \qquad \text{for every } m \in \{0, 1, ..., n\}.$$

In other words,

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^\ell = 0 \qquad \text{for every } \ell \in \{0, 1, ..., n\}.$$

In other words, (2) holds for $N = n + 1$ (since $N - 1 = n$ for $N = n + 1$).

Also, (6) (applied to $m = n + 1$) yields

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^{n+1} = - \sum_{\ell=0}^{(n+1)-1} \binom{n+1}{\ell} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^\ell.$$

In other words,

$$\begin{aligned}
\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}k^{n+1} &= -\sum_{\ell=0}^{n}\binom{n+1}{\ell}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\ell \\
&= -\left(\sum_{\ell=0}^{n-1}\binom{n+1}{\ell}\underbrace{\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\ell}_{\substack{=0 \text{ by (4), since} \\ \ell\in\{0,1,...,n-1\}}} + \sum_{\ell=n}^{n}\binom{n+1}{\ell}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\ell\right) \\
&= -\left(\underbrace{\sum_{\ell=0}^{n-1}\binom{n+1}{\ell}0}_{=0} + \sum_{\ell=n}^{n}\binom{n+1}{\ell}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\ell\right) \\
&= -\sum_{\ell=n}^{n}\binom{n+1}{\ell}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\ell = -\underbrace{\binom{n+1}{n}}_{=n+1}\underbrace{\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n}_{=(-1)^n n! \text{ by (5)}} \\
&= -(n+1)(-1)^n n! = \underbrace{-(-1)^n}_{=(-1)^{n+1}}\underbrace{(n+1)\,n!}_{=(n+1)!} = (-1)^{n+1}(n+1)!.
\end{aligned}$$

In other words, (3) holds for $N = n+1$.

Hence, we have proved that both (2) and (3) hold for $N = n+1$. In other words, Theorem 1 holds for $N = n+1$. This completes the induction step.

Thus, the induction proof of Theorem 1 is complete.

**Corollary 2.** Let $R$ be a commutative ring with unity. Let $N \in \mathbb{N}$. Let $U \in R$. Then, the equalities

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}(U-k)^\ell = 0 \qquad \text{for every } \ell \in \{0,1,...,N-1\} \tag{7}$$

and

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}(U-k)^N = N! \tag{8}$$

are satisfied in $R$.

*Proof of Corollary 2.* Every $m \in \mathbb{N}$ satisfies

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} (U-k)^m = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \sum_{\ell=0}^{m} \binom{m}{\ell} \underbrace{(-k)^\ell}_{=(-1)^\ell k^\ell} U^{m-\ell}$$

$$\left(\text{since } (U-k)^m = ((-k)+U)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} (-k)^\ell U^{m-\ell} \text{ by the binomial formula}\right)$$

$$= \sum_{k=0}^{N} (-1)^k \binom{N}{k} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell k^\ell U^{m-\ell} = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell U^{m-\ell} \sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell. \tag{9}$$

If $m \in \{0, 1, ..., N-1\}$, then this yields

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} (U-k)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell U^{m-\ell} \underbrace{\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell}_{\substack{=0 \text{ by (2), since } 0 \le \ell \le m \\ \text{and } m \in \{0,1,...,N-1\} \\ \text{yield } \ell \in \{0,1,...,N-1\}}} = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell U^{m-\ell} 0 = 0.$$

So we have proved that

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} (U-k)^m = 0 \qquad \text{for every } m \in \{0, 1, ..., N-1\}.$$

In other words,

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} (U-k)^\ell = 0 \qquad \text{for every } \ell \in \{0, 1, ..., N-1\}.$$

Thus, (7) is proven.

Besides, (9) (applied to $m = N$) yields

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} (U-k)^N = \sum_{\ell=0}^{N} \binom{N}{\ell} (-1)^\ell U^{N-\ell} \sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell$$

$$= \sum_{\ell=0}^{N-1} \binom{N}{\ell} (-1)^\ell U^{N-\ell} \underbrace{\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell}_{\substack{=0 \text{ by (2), since} \\ \ell \in \{0,1,...,N-1\}}} + \sum_{\ell=N}^{N} \binom{N}{\ell} (-1)^\ell U^{N-\ell} \sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell$$

$$= \underbrace{\sum_{\ell=0}^{N-1} \binom{N}{\ell} (-1)^\ell U^{N-\ell} 0}_{=0} + \sum_{\ell=N}^{N} \binom{N}{\ell} (-1)^\ell U^{N-\ell} \sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell$$

$$= \sum_{\ell=N}^{N} \binom{N}{\ell} (-1)^\ell U^{N-\ell} \sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell$$

$$= \underbrace{\binom{N}{N}}_{=1} (-1)^N \underbrace{U^{N-N}}_{=U^0=1} \underbrace{\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^N}_{=(-1)^N N! \text{ by (3)}} = \underbrace{(-1)^N (-1)^N}_{=((-1)\cdot(-1))^N = 1^N = 1} N! = N!,$$

and thus (8) is proven. This completes the proof of Corollary 2.

Now comes a technical lemma:

**Lemma 3.** Let $n \in \mathbb{N}$. Let $R$ be a commutative ring with unity. For every triple $(k,i,j) \in \mathbb{Z}^3$ satisfying $0 \leq k \leq n$, $0 \leq i \leq k$ and $0 \leq j \leq n-k$, let $a_{k,i,j}$ be an element of $R$. Then,

$$\sum_{k=0}^{n} \sum_{i=0}^{k} \sum_{j=0}^{n-k} a_{k,i,j} = \sum_{t=0}^{n} \sum_{i=0}^{t} \sum_{k=0}^{n-t} a_{(n-t+i)-k,i,t-i}.$$

*Proof of Lemma 3.* We have

$$\underbrace{\sum_{k=0}^{n}}_{=\sum_{\substack{k\in\mathbb{Z};\\ 0\leq k\leq n}}} \underbrace{\sum_{i=0}^{k}}_{=\sum_{\substack{i\in\mathbb{Z};\\ 0\leq i\leq k}}} \underbrace{\sum_{j=0}^{n-k}}_{=\sum_{\substack{j\in\mathbb{Z};\\ 0\leq j\leq n-k}}} a_{k,i,j} = \sum_{\substack{k\in\mathbb{Z};\\ 0\leq k\leq n}} \sum_{\substack{i\in\mathbb{Z};\\ 0\leq i\leq k}} \sum_{\substack{j\in\mathbb{Z};\\ 0\leq j\leq n-k}} a_{k,i,j} = \sum_{\substack{(k,i,j)\in\mathbb{Z}^3;\\ 0\leq k\leq n;\\ 0\leq i\leq k;\\ 0\leq j\leq n-k}} a_{k,i,j}.$$

But

$$\left\{(k,i,j) \in \mathbb{Z}^3 \mid 0 \leq k \leq n \text{ and } 0 \leq i \leq k \text{ and } 0 \leq j \leq n-k\right\}$$
$$= \left\{(k,i,j) \in \mathbb{Z}^3 \mid 0 \leq i+j \leq n \text{ and } 0 \leq i \leq i+j \text{ and } 0 \leq n-(i+j)+i-k \leq n-(i+j)\right\}$$

(because for every triple $(k,i,j) \in \mathbb{Z}^3$, the assertions

$$(0 \leq k \leq n \text{ and } 0 \leq i \leq k \text{ and } 0 \leq j \leq n-k)$$

and

$$(0 \leq i+j \leq n \text{ and } 0 \leq i \leq i+j \text{ and } 0 \leq n-(i+j)+i-k \leq n-(i+j))$$

are equivalent[1]).

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[1] In fact,

- if
$$(0 \leq k \leq n \text{ and } 0 \leq i \leq k \text{ and } 0 \leq j \leq n-k),$$
then
$$(0 \leq i+j \leq n \text{ and } 0 \leq i \leq i+j \text{ and } 0 \leq n-(i+j)+i-k \leq n-(i+j))$$
(since if
$$(0 \leq k \leq n \text{ and } 0 \leq i \leq k \text{ and } 0 \leq j \leq n-k),$$
then
$$\begin{aligned}
0 &\leq i+j && (\text{since } 0 \leq i \text{ and } 0 \leq j);\\
i+j &\leq n && (\text{since } i \leq k \text{ and } j \leq n-k \text{ yield } i+j \leq k+(n-k)=n);\\
0 &\leq i; && \\
i &\leq i+j && (\text{since } 0 \leq j);\\
0 &\leq n-(i+j)+i-k && \left(\begin{array}{c}\text{since } (i+j)-i = j \leq n-k \text{ rewrites as}\\ 0 \leq (n-k)-((i+j)-i) = n-(i+j)+i-k\end{array}\right);\\
n-(i+j)+i-k &\leq n-(i+j) && (\text{since } i \leq k \text{ yields } i-k \leq 0)
\end{aligned}$$

);

- conversely, if

$$\left(0 \leq i+j \leq n \text{ and } 0 \leq i \leq i+j \text{ and } 0 \leq n-(i+j)+i-k \leq n-(i+j)\right),$$

then

$$(0 \leq k \leq n \text{ and } 0 \leq i \leq k \text{ and } 0 \leq j \leq n-k)$$

(since if

$$\left(0 \leq i+j \leq n \text{ and } 0 \leq i \leq i+j \text{ and } 0 \leq n-(i+j)+i-k \leq n-(i+j)\right),$$

then

$$\begin{aligned}
&0 \leq k && \left(\begin{array}{c}\text{since } n-(i+j)+i-k \leq n-(i+j) \text{ rewrites as } i-k \leq 0\text{, so that}\\ i \leq k\text{, what, together with } 0 \leq i\text{, yields } 0 \leq k\end{array}\right);\\
&k \leq n && \left(\begin{array}{c}\text{since } i \leq i+j \text{ rewrites as } 0 \leq j\text{, what, together with}\\ 0 \leq n-(i+j)+i-k = n-j-k\text{, yields}\\ 0 \leq j+(n-j-k) = n-k\end{array}\right);\\
&0 \leq i;\\
&i \leq k && (\text{as proven above});\\
&0 \leq j && (\text{as proven above});\\
&j \leq n-k && (\text{since } 0 \leq n-(i+j)+i-k = n-j-k \text{ rewrites as } j \leq n-k)
\end{aligned}$$

).

Hence,

$$\sum_{\substack{(k,i,j)\in\mathbb{Z}^3;\\ 0\leq k\leq n;\\ 0\leq i\leq k;\\ 0\leq j\leq n-k}} a_{k,i,j} = \sum_{\substack{(k,i,j)\in\mathbb{Z}^3;\\ 0\leq i+j\leq n;\\ 0\leq i\leq i+j;\\ 0\leq n-(i+j)+i-k\leq n-(i+j)}} \underbrace{a_{k,i,j}}_{=a_{k,i,(i+j)-i}} = \sum_{\substack{(k,i,j)\in\mathbb{Z}^3;\\ 0\leq i+j\leq n;\\ 0\leq i\leq i+j;\\ 0\leq n-(i+j)+i-k\leq n-(i+j)}} a_{k,i,(i+j)-i}$$

$$= \sum_{i\in\mathbb{Z}} \sum_{\substack{j\in\mathbb{Z};\\ 0\leq i+j\leq n;\\ 0\leq i\leq i+j}} \sum_{\substack{k\in\mathbb{Z};\\ 0\leq n-(i+j)+i-k\leq n-(i+j)}} a_{k,i,(i+j)-i}$$

$$= \sum_{i\in\mathbb{Z}} \sum_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n;\\ 0\leq i\leq t}} \sum_{\substack{k\in\mathbb{Z};\\ 0\leq n-t+i-k\leq n-t}} a_{k,i,t-i} \qquad \text{(here we substituted } t \text{ for } i+j \text{ in the second sum)}$$

$$= \sum_{\substack{(t,i,k)\in\mathbb{Z}^3;\\ 0\leq t\leq n;\\ 0\leq i\leq t;\\ 0\leq n-t+i-k\leq n-t}} a_{k,i,t-i} = \sum_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n}} \sum_{\substack{i\in\mathbb{Z};\\ 0\leq i\leq t}} \sum_{\substack{k\in\mathbb{Z};\\ 0\leq n-t+i-k\leq n-t}} \underbrace{a_{k,i,t-i}}_{\substack{=a_{(n-t+i)-(n-t+i-k),i,t-i},\\ \text{since } k=(n-t+i)-(n-t+i-k)}}$$

$$= \sum_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n}} \sum_{\substack{i\in\mathbb{Z};\\ 0\leq i\leq t}} \sum_{\substack{k\in\mathbb{Z};\\ 0\leq n-t+i-k\leq n-t}} a_{(n-t+i)-(n-t+i-k),i,t-i} = \underbrace{\sum_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n}}}_{=\sum_{t=0}^{n}} \underbrace{\sum_{\substack{i\in\mathbb{Z};\\ 0\leq i\leq t}}}_{=\sum_{i=0}^{t}} \underbrace{\sum_{\substack{k\in\mathbb{Z};\\ 0\leq k\leq n-t}}}_{=\sum_{k=0}^{n-t}} a_{(n-t+i)-k,i,t-i}$$

$$\text{(here we substituted } k \text{ for } n-t+i-k \text{ in the third sum)}$$

$$= \sum_{t=0}^{n} \sum_{i=0}^{t} \sum_{k=0}^{n-t} a_{(n-t+i)-k,i,t-i}.$$

Hence,

$$\sum_{k=0}^{n} \sum_{i=0}^{k} \sum_{j=0}^{n-k} a_{k,i,j} = \sum_{\substack{(k,i,j)\in\mathbb{Z}^3;\\ 0\leq k\leq n;\\ 0\leq i\leq k;\\ 0\leq j\leq n-k}} a_{k,i,j} = \sum_{t=0}^{n} \sum_{i=0}^{t} \sum_{k=0}^{n-t} a_{(n-t+i)-k,i,t-i},$$

and thus Lemma 3 is proven.

Now let us solve the problem: For every triple $(k,i,j)\in\mathbb{Z}^3$ satisfying $0\leq k\leq n$, $0\leq i\leq k$ and $0\leq j\leq n-k$, define an element $a_{k,i,j}\in K[X,Y]$ by

$$a_{k,i,j} = \binom{n}{k}\cdot\binom{k}{i}\cdot\binom{n-k}{j}(-1)^{(n-k)-j}\, k^{n-i-j} X^i Y^j.$$

Then, for every triple $(k,i,j)\in\mathbb{Z}^3$ satisfying $0\leq k\leq n$, $0\leq i\leq k$ and $0\leq j\leq n-k$, we have

$$
\begin{aligned}
a_{k,i,j} &= \binom{n}{k} \cdot \binom{k}{i} \cdot \binom{n-k}{j} (-1)^{(n-k)-j} k^{n-i-j} X^i Y^j \\
&= \underbrace{\binom{n}{k}}_{=\frac{n!}{k!\,(n-k)!}} \cdot \underbrace{\binom{k}{i}}_{=\frac{k!}{i!\,(k-i)!}} \cdot \underbrace{\binom{n-k}{j}}_{\substack{=\frac{(n-k)!}{j!\,((n-k)-j)!} \\ =\frac{(n-k)!}{j!\,((n-j)-k)!}}} \underbrace{(-1)^{(n-k)-j}}_{=(-1)^{(n-j)-k}} k^{n-i-j} X^i Y^j \\
&= \underbrace{\frac{n!}{k!\,(n-k)!} \cdot \frac{k!}{i!\,(k-i)!} \cdot \frac{(n-k)!}{j!\,((n-j)-k)!}}_{=\frac{1}{(n-i-j)!} \cdot \frac{n!}{i!j!} \cdot \frac{(n-i-j)!}{((n-j)-k)!\,(k-i)!}} \cdot (-1)^{(n-j)-k} k^{n-i-j} X^i Y^j \\
&= \frac{1}{(n-i-j)!} \cdot \frac{n!}{i!j!} \cdot \underbrace{\frac{(n-i-j)!}{((n-j)-k)!\,(k-i)!}}_{\substack{=\frac{(n-i-j)!}{((n-j)-k)!\,((n-i-j)-((n-j)-k))!} \\ =\binom{n-i-j}{(n-j)-k}}} \cdot (-1)^{(n-j)-k} k^{n-i-j} X^i Y^j \\
&= \frac{1}{(n-i-j)!} \cdot \frac{n!}{i!j!} \cdot \binom{n-i-j}{(n-j)-k} \cdot (-1)^{(n-j)-k} k^{n-i-j} X^i Y^j .
\end{aligned}
$$

Hence, for every triple $(t,i,k) \in \mathbb{Z}^3$ satisfying $0 \leq t \leq n$, $0 \leq i \leq t$ and $0 \leq k \leq n-t$, we have

$$
\begin{aligned}
& a_{(n-t+i)-k,i,t-i} \\
&= \underbrace{\frac{1}{(n-i-(t-i))!}}_{=\frac{1}{(n-t)!}} \cdot \frac{n!}{i!\,(t-i)!} \cdot \underbrace{\binom{n-i-(t-i)}{(n-(t-i))-((n-t+i)-k)}}_{\substack{=\binom{n-t}{k}\text{, since} \\ n-i-(t-i)=n-t \text{ and} \\ (n-(t-i))-((n-t+i)-k)=k}} \\
&\qquad \cdot \underbrace{(-1)^{(n-(t-i))-((n-t+i)-k)}}_{\substack{=(-1)^k\text{, since} \\ (n-(t-i))-((n-t+i)-k)=k}} \underbrace{((n-t+i)-k)^{n-i-(t-i)}}_{\substack{=((n-t+i)-k)^{n-t}\text{, since} \\ n-i-(t-i)=n-t}} X^i Y^{t-i} \\
&= \frac{1}{(n-t)!} \cdot \frac{n!}{i!\,(t-i)!} \cdot \binom{n-t}{k} \cdot (-1)^k ((n-t+i)-k)^{n-t} X^i Y^{t-i}. \qquad (10)
\end{aligned}
$$

Now,

$$\sum_{k=0}^{n}\binom{n}{k}(X+k)^{k}(Y-k)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}X^{i}k^{k-i}\sum_{j=0}^{n-k}(-1)^{(n-k)-j}\binom{n-k}{j}Y^{j}k^{(n-k)-j}$$

$$\left(\begin{array}{c}\text{since the binomial formula yields } (X+k)^{k}=\sum_{i=0}^{k}\binom{k}{i}X^{i}k^{k-i} \text{ and}\\ (Y-k)^{n-k}=\sum_{j=0}^{n-k}(-1)^{(n-k)-j}\binom{n-k}{j}Y^{j}k^{(n-k)-j}\end{array}\right)$$

$$=\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{n-k}\binom{n}{k}\cdot\binom{k}{i}\cdot\binom{n-k}{j}(-1)^{(n-k)-j}\underbrace{k^{k-i}k^{(n-k)-j}}_{\substack{=k^{(k-i)+((n-k)-j)}\\ =k^{n-i-j}}}X^{i}Y^{j}$$

$$=\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{n-k}\underbrace{\binom{n}{k}\cdot\binom{k}{i}\cdot\binom{n-k}{j}(-1)^{(n-k)-j}k^{n-i-j}X^{i}Y^{j}}_{=a_{k,i,j}}=\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{n-k}a_{k,i,j}$$

$$=\sum_{t=0}^{n}\sum_{i=0}^{t}\sum_{k=0}^{n-t}a_{(n-t+i)-k,i,t-i} \qquad \text{(by Lemma 3)}$$

$$=\sum_{t=0}^{n}\sum_{i=0}^{t}\sum_{k=0}^{n-t}\frac{1}{(n-t)!}\cdot\frac{n!}{i!\,(t-i)!}\cdot\binom{n-t}{k}\cdot(-1)^{k}((n-t+i)-k)^{n-t}X^{i}Y^{t-i} \qquad \text{(by (10))}$$

$$=\sum_{t=0}^{n}\sum_{i=0}^{t}\frac{1}{(n-t)!}\cdot\frac{n!}{i!\,(t-i)!}\underbrace{\sum_{k=0}^{n-t}(-1)^{k}\binom{n-t}{k}((n-t+i)-k)^{n-t}}_{\substack{=(n-t)!\text{ by (8) (applied to } U=n-t+i \text{ and } N=n-t),\\ \text{since } n\geq t \text{ yields } n-t\in\mathbb{N}}}X^{i}Y^{t-i}$$

$$=\sum_{t=0}^{n}\sum_{i=0}^{t}\frac{1}{(n-t)!}\cdot\frac{n!}{i!\,(t-i)!}(n-t)!X^{i}Y^{t-i}=\sum_{t=0}^{n}\sum_{i=0}^{t}\frac{n!}{i!\,(t-i)!}X^{i}Y^{t-i}.$$

On the other hand,

$$\sum_{t=0}^{n}\frac{n!}{t!}\underbrace{(X+Y)^{t}}_{\substack{=\sum_{i=0}^{t}\binom{t}{i}X^{i}Y^{t-i}\\ \text{by the binomial formula}}}=\sum_{t=0}^{n}\frac{n!}{t!}\sum_{i=0}^{t}\underbrace{\binom{t}{i}}_{=\frac{t!}{i!\,(t-i)!}}X^{i}Y^{t-i}=\sum_{t=0}^{n}\frac{n!}{t!}\sum_{i=0}^{t}\frac{t!}{i!\,(t-i)!}X^{i}Y^{t-i}$$

$$=\sum_{t=0}^{n}\sum_{i=0}^{t}\frac{n!}{t!}\cdot\frac{t!}{i!\,(t-i)!}X^{i}Y^{t-i}=\sum_{t=0}^{n}\sum_{i=0}^{t}\frac{n!}{i!\,(t-i)!}X^{i}Y^{t-i}.$$

Hence,

$$\sum_{k=0}^{n}\binom{n}{k}(X+k)^{k}(Y-k)^{n-k}=\sum_{t=0}^{n}\frac{n!}{t!}(X+Y)^{t},$$

so that (1) is proven. Thus, the problem is solved.

**Remark.** As a consequence of the problem, we can prove a result due to Abel:

**Theorem 4.** Let $n \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} \binom{n}{k} X (X+k)^{k-1} (Y-k)^{n-k} = (X+Y)^n \tag{11}$$

in the quotient field of the polynomial ring $\mathbb{Z}[X,Y]$.

*Proof of Theorem 4.* We have

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (X+k)^k (Y-k)^{(n-1)-k} = \sum_{t=0}^{n-1} \frac{(n-1)!}{t!} (X+Y)^t \tag{12}$$

(by (1), applied to $n-1$ instead of $n$).

By the universal property of the polynomial ring, there exists a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[X,Y] \to \mathbb{Z}[X,Y]$ which maps $X$ to $X+1$ and $Y$ to $Y-1$. Applying this homomorphism to both sides of the equation (12), we obtain

$$\sum_{k=0}^{n-1} \binom{n-1}{k} ((X+1)+k)^k ((Y-1)-k)^{(n-1)-k} = \sum_{t=0}^{n-1} \frac{(n-1)!}{t!} ((X+1)+(Y-1))^t . \tag{13}$$

Now,

$$\sum_{k=0}^{n} \binom{n}{k} k (X+k)^{k-1} (Y-k)^{n-k} = \sum_{k=1}^{n} \binom{n}{k} k (X+k)^{k-1} (Y-k)^{n-k}$$

$$\left( \begin{array}{c} \text{here we replaced the } \sum_{k=0}^{n} \text{ sign by an } \sum_{k=1}^{n} \text{sign, since the addend for } k=0 \text{ is zero} \\ (\text{as } k=0 \text{ for } k=0) \end{array} \right)$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} n \left( \underbrace{X+k}_{=(X+1)+(k-1)} \right)^{k-1} \underbrace{\left( \underbrace{Y-k}_{=(Y-1)-(k-1)} \right)^{n-k}}_{\substack{=((Y-1)-(k-1))^{n-k} \\ =((Y-1)-(k-1))^{(n-1)-(k-1)}}}$$

$$\left( \begin{array}{c} \text{since } \binom{n}{k} k = \binom{n-1}{k-1} n, \text{ because} \\ \binom{n}{k} k = \frac{n!}{k! \cdot (n-k)!} k = \frac{n!}{(k \cdot (k-1)!) \cdot (n-k)!} k = \frac{n!}{(k-1)! \cdot (n-k)!} \\ = \frac{n \cdot (n-1)!}{(k-1)! \cdot (n-k)!} = \frac{(n-1)!}{(k-1)! \cdot (n-k)!} n = \frac{(n-1)!}{(k-1)! \cdot ((n-1)-(k-1))!} n = \binom{n-1}{k-1} n \end{array} \right)$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} n ((X+1)+(k-1))^{k-1} ((Y-1)-(k-1))^{(n-1)-(k-1)}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} n ((X+1)+k)^{k} ((Y-1)-k)^{(n-1)-k}$$

(here we substituted $k$ for $k-1$ in the sum)

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} ((X+1)+k)^{k} ((Y-1)-k)^{(n-1)-k}$$

$$= n \sum_{t=0}^{n-1} \frac{(n-1)!}{t!} ((X+1)+(Y-1))^{t} \qquad \text{(by (13))}$$

$$= \sum_{t=0}^{n-1} \underbrace{\frac{n(n-1)!}{t!}}_{=\frac{n!}{t!}} \left( \underbrace{(X+1)+(Y-1)}_{=X+Y} \right)^{t} = \sum_{t=0}^{n-1} \frac{n!}{t!} (X+Y)^{t}. \tag{14}$$

Thus,

$$\sum_{k=0}^{n}\binom{n}{k}\underbrace{X}_{=(X+k)-k}(X+k)^{k-1}(Y-k)^{n-k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}((X+k)-k)(X+k)^{k-1}(Y-k)^{n-k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\underbrace{(X+k)(X+k)^{k-1}}_{=(X+k)^{k}}-k(X+k)^{k-1}\right)(Y-k)^{n-k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left((X+k)^{k}-k(X+k)^{k-1}\right)(Y-k)^{n-k}$$

$$=\underbrace{\sum_{k=0}^{n}\binom{n}{k}(X+k)^{k}(Y-k)^{n-k}}_{=\sum_{t=0}^{n}\frac{n!}{t!}(X+Y)^{t}\text{ by (1)}}-\underbrace{\sum_{k=0}^{n}\binom{n}{k}k(X+k)^{k-1}(Y-k)^{n-k}}_{=\sum_{t=0}^{n-1}\frac{n!}{t!}(X+Y)^{t}\text{ by (14)}}$$

$$=\underbrace{\sum_{t=0}^{n}\frac{n!}{t!}(X+Y)^{t}}_{=\sum_{t=0}^{n-1}\frac{n!}{t!}(X+Y)^{t}+\frac{n!}{n!}(X+Y)^{n}}-\sum_{t=0}^{n-1}\frac{n!}{t!}(X+Y)^{t}=\sum_{t=0}^{n-1}\frac{n!}{t!}(X+Y)^{t}+\frac{n!}{n!}(X+Y)^{n}-\sum_{t=0}^{n-1}\frac{n!}{t!}(X+Y)^{t}$$

$$=\underbrace{\frac{n!}{n!}}_{=1}(X+Y)^{n}=(X+Y)^{n}.$$

This proves Theorem 4.

Here is a kind of generalization of the problem:

**Definition.** Let $m\in\mathbb{N}\setminus\{0\}$. Let $n\in\mathbb{N}$. Let $(a_1,a_2,...,a_m)\in\mathbb{Z}^m$. Then, we define an element $\alpha_{a_1,a_2,...,a_m}$ of the quotient field of the polynomial ring $\mathbb{Z}[X_1,X_2,...,X_m]$ by

$$\alpha_{a_1,a_2,...,a_m}=\sum_{\substack{(k_1,k_2,...,k_m)\in\mathbb{N}^m;\\ k_1+k_2+...+k_m=n}}\frac{n!}{k_1!k_2!...k_m!}\prod_{j=1}^{m}(X_j+k_j)^{k_j+a_j}.$$

**Theorem 5.** Let $m\in\mathbb{N}\setminus\{0\}$. Let $n\in\mathbb{N}$. Then,

$$\alpha_{\underbrace{0,0,...,0}_{m\text{ zeroes}}}=\sum_{t=0}^{n}\frac{n!}{t!}\binom{m+n-t-2}{n-t}(n+(X_1+X_2+...+X_m))^{t}\qquad(15)$$

in the polynomial ring $\mathbb{Z}[X_1,X_2,...,X_m]$.

Before we come to a proof of Theorem 5, a lemma:

**Lemma 6.** Let $n \in \mathbb{N}$. Let $\mu \in \mathbb{N} \setminus \{0\}$. Let $(a_1, a_2, ..., a_{\mu+1}) \in \mathbb{Z}^{\mu+1}$. In the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$, we have

$$\alpha_{a_1,a_2,...,a_{\mu+1}}$$
$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j + k_j)^{k_j+a_j} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu + \rho)^{\rho+a_\mu} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho+a_{\mu+1}} .$$

*Proof of Lemma 6.* Clearly,

$$\alpha_{a_1,a_2,...,a_{\mu+1}} = \sum_{\substack{(k_1,k_2,...,k_{\mu+1})\in\mathbb{N}^{\mu+1};\\ k_1+k_2+...+k_{\mu+1}=n}} \frac{n!}{k_1!k_2!...k_{\mu+1}!} \prod_{j=1}^{\mu+1} (X_j + k_j)^{k_j+a_j}$$
$$= \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k_1,k_2,...,k_{\mu+1})\in\mathbb{N}^{\mu+1};\\ k_1+k_2+...+k_{\mu+1}=n;\\ k_1+k_2+...+k_{\mu-1}=\ell}} \frac{n!}{k_1!k_2!...k_{\mu+1}!} \prod_{j=1}^{\mu+1} (X_j + k_j)^{k_j+a_j}$$
$$= \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k_1,k_2,...,k_{\mu+1})\in\mathbb{N}^{\mu+1};\\ k_1+k_2+...+k_{\mu+1}=n;\\ k_1+k_2+...+k_{\mu-1}=\ell;\\ \ell\leq n;\ k_\mu\leq n-\ell}} \frac{n!}{k_1!k_2!...k_{\mu+1}!} \prod_{j=1}^{\mu+1} (X_j + k_j)^{k_j+a_j}$$

(since for any $\ell \in \mathbb{N}$ and for any $(k_1, k_2, ..., k_{\mu+1}) \in \mathbb{N}^{\mu+1}$, the assertions

$$(k_1 + k_2 + ... + k_{\mu+1} = n \text{ and } k_1 + k_2 + ... + k_{\mu-1} = \ell)$$

and

$$(k_1 + k_2 + ... + k_{\mu+1} = n \text{ and } k_1 + k_2 + ... + k_{\mu-1} = \ell \text{ and } \ell \leq n \text{ and } k_\mu \leq n - \ell)$$

are equivalent[2]). Thus,

$$\alpha_{a_1,a_2,...,a_{\mu+1}} = \sum_{\ell\in\mathbb{N}} \sum_{\substack{(k_1,k_2,...,k_{\mu+1})\in\mathbb{N}^{\mu+1};\\ k_1+k_2+...+k_{\mu+1}=n;\\ k_1+k_2+...+k_{\mu-1}=\ell;\\ \ell\leq n;\ k_\mu\leq n-\ell}} \frac{n!}{k_1!k_2!...k_{\mu+1}!} \prod_{j=1}^{\mu+1} (X_j+k_j)^{k_j+a_j}$$

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\leq n-\ell}} \sum_{\substack{k_{\mu+1}\in\mathbb{N};\\ k_1+k_2+...+k_{\mu+1}=n}} \underbrace{\frac{n!}{k_1!k_2!...k_{\mu+1}!}}_{=\frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot k_\mu!k_{\mu+1}!}} \underbrace{\prod_{j=1}^{\mu+1} (X_j+k_j)^{k_j+a_j}}_{\substack{=\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j+a_j}\\ \cdot(X_\mu+k_\mu)^{k_\mu+a_\mu}(X_{\mu+1}+k_{\mu+1})^{k_{\mu+1}+a_{\mu+1}}}}$$

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\leq n-\ell}} \sum_{\substack{k_{\mu+1}\in\mathbb{N};\\ k_1+k_2+...+k_{\mu+1}=n}}$$

$$\underbrace{\frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot k_\mu!k_{\mu+1}!}}_{=\frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!}\cdot\frac{(n-\ell)!}{k_\mu!k_{\mu+1}!}} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j} \cdot (X_\mu+k_\mu)^{k_\mu+a_\mu} (X_{\mu+1}+k_{\mu+1})^{k_{\mu+1}+a_{\mu+1}}$$

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\leq n-\ell}} (X_\mu+k_\mu)^{k_\mu+a_\mu} \sum_{\substack{k_{\mu+1}\in\mathbb{N};\\ k_1+k_2+...+k_{\mu+1}=n}} \frac{(n-\ell)!}{k_\mu!k_{\mu+1}!} (X_{\mu+1}+k_{\mu+1})^{k_{\mu+1}+a_{\mu+1}}$$

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\leq n-\ell}} (X_\mu+k_\mu)^{k_\mu+a_\mu} \sum_{\substack{k_{\mu+1}\in\mathbb{N};\\ \ell+k_\mu+k_{\mu+1}=n}} \frac{(n-\ell)!}{k_\mu!k_{\mu+1}!} (X_{\mu+1}+k_{\mu+1})^{k_{\mu+1}+a_{\mu+1}}$$

$$\left(\begin{array}{c}\text{since } k_1+k_2+...+k_{\mu-1}=\ell \text{ yields}\\ k_1+k_2+...+k_{\mu+1}=(k_1+k_2+...+k_{\mu-1})+k_\mu+k_{\mu+1}=\ell+k_\mu+k_{\mu+1}\end{array}\right)$$

[2]Because if $(k_1+k_2+...+k_{\mu+1}=n \text{ and } k_1+k_2+...+k_{\mu-1}=\ell)$, then $(\ell\leq n \text{ and } k_\mu\leq n-\ell)$ (since $(k_1+k_2+...+k_{\mu+1}=n \text{ and } k_1+k_2+...+k_{\mu-1}=\ell)$ yields $\ell\leq n$ (since $n-\ell = (k_1+k_2+...+k_{\mu+1})-(k_1+k_2+...+k_{\mu-1}) = \underbrace{k_\mu}_{\in\mathbb{N}}+\underbrace{k_{\mu+1}}_{\in\mathbb{N}}\in\mathbb{N}$) and $k_\mu\leq n-\ell$ (since $n-\ell = (k_1+k_2+...+k_{\mu+1})-(k_1+k_2+...+k_{\mu-1}) = k_\mu+\underbrace{k_{\mu+1}}_{\substack{\geq 0,\text{ since}\\ k_{\mu+1}\in\mathbb{N}}}\geq k_\mu$)).

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\leq n-\ell}} (X_\mu+k_\mu)^{k_\mu+a_\mu} \frac{(n-\ell)!}{k_\mu!\,((n-\ell)-k_\mu)!} (X_{\mu+1}+(n-\ell)-k_\mu)^{(n-\ell)-k_\mu+a_{\mu+1}}$$

$$\left(\begin{array}{c} \text{since } \ell+k_\mu+k_{\mu+1}=n \text{ is equivalent to } k_{\mu+1}=(n-\ell)-k_\mu, \text{ and therefore} \\ \sum_{\substack{k_{\mu+1}\in\mathbb{N};\\ \ell+k_\mu+k_{\mu+1}=n}} \frac{(n-\ell)!}{k_\mu!k_{\mu+1}!} (X_{\mu+1}+k_{\mu+1})^{k_{\mu+1}+a_{\mu+1}} = \sum_{\substack{k_{\mu+1}\in\mathbb{N};\\ k_{\mu+1}=(n-\ell)-k_\mu}} \frac{(n-\ell)!}{k_\mu!k_{\mu+1}!} (X_{\mu+1}+k_{\mu+1})^{k_{\mu+1}+a_{\mu+1}} \\ = \frac{(n-\ell)!}{k_\mu!\,((n-\ell)-k_\mu)!} (X_{\mu+1}+(n-\ell)-k_\mu)^{(n-\ell)-k_\mu+a_{\mu+1}} \\ \text{(since } k_\mu\leq n-\ell \text{ yields } (n-\ell)-k_\mu\in\mathbb{N}) \end{array}\right)$$

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{\rho\in\mathbb{N};\\ \rho\leq n-\ell}} (X_\mu+\rho)^{\rho+a_\mu} \underbrace{\frac{(n-\ell)!}{\rho!\,((n-\ell)-\rho)!}}_{=\binom{n-\ell}{\rho}} (X_{\mu+1}+(n-\ell)-\rho)^{(n-\ell)-\rho+a_{\mu+1}}$$

(here we have renamed $k_\mu$ as $\rho$ in the third sum)

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=\ell}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{\rho\in\mathbb{N};\\ \rho\leq n-\ell}} \binom{n-\ell}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+(n-\ell)-\rho)^{(n-\ell)-\rho+a_{\mu+1}}$$

$$= \sum_{\substack{\ell\in\mathbb{N};\\ \ell\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}+(n-\ell)=n}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot(n-\ell)!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{\rho\in\mathbb{N};\\ \rho\leq n-\ell}} \binom{n-\ell}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+(n-\ell)-\rho)^{(n-\ell)-\rho+a_{\mu+1}}$$

(since $k_1+k_2+...+k_{\mu-1}=\ell$ is equivalent to $k_1+k_2+...+k_{\mu-1}+(n-\ell)=n$)

$$= \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\leq n}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}+k_\mu=n}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{\rho\in\mathbb{N};\\ \rho\leq k_\mu}} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho+a_{\mu+1}}$$

(here we substituted $k_\mu$ for $n-\ell$ in the first sum)

$$= \sum_{\substack{(k_1,k_2,...,k_{\mu-1},k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_{\mu-1}+k_\mu=n;\\ k_\mu\leq n}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j}$$

$$\cdot \sum_{\substack{\rho\in\mathbb{N};\\ \rho\leq k_\mu}} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho+a_{\mu+1}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n;\\ k_\mu\leq n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j} \cdot \underbrace{\sum_{\substack{\rho\in\mathbb{N};\\ \rho\leq k_\mu}}}_{=\sum_{\rho=0}^{k_\mu}} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho+a_{\mu+1}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n;\\ k_\mu\leq n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho+a_{\mu+1}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j+a_j} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho+a_\mu} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho+a_{\mu+1}}$$

(since for any $(k_1, k_2, ..., k_\mu) \in \mathbb{N}^\mu$, the assertions $(k_1 + k_2 + ... + k_\mu = n$ and $k_\mu \leq n)$ and $k_1 + k_2 + ... + k_\mu = n$ are equivalent[3]). This proves Lemma 6.

*Proof of Theorem 5.* We will prove Theorem 5 by induction over $m$:

*Induction base:* If $m = 1$, then (15) is true (since if $m = 1$, then

$$\alpha_{\underbrace{0,0,...,0}_{m \text{ zeroes}}} = \sum_{\substack{(k_1,k_2,...,k_m)\in\mathbb{N}^m;\\ k_1+k_2+...+k_m=n}} \frac{n!}{k_1!k_2!...k_m!} \prod_{j=1}^{m} (X_j+k_j)^{k_j+0}$$

$$= \sum_{\substack{(k_1)\in\mathbb{N}^1;\\ k_1=n}} \underbrace{\frac{n!}{k_1!}}_{\substack{=1, \text{ since}\\ k_1=n \text{ yields } k_1!=n!}} \underbrace{\prod_{j=1}^{1} (X_j+k_j)^{k_j+0}}_{\substack{=(X_1+k_1)^{k_1+0}\\ =(X_1+k_1)^{k_1}\\ =(X_1+n)^n,\\ \text{since } k_1=n}} = \sum_{\substack{(k_1)\in\mathbb{N}^1;\\ k_1=n}} (X_1+n)^n = (X_1+n)^n$$

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[3]Because if $k_1 + k_2 + ... + k_\mu = n$, then $k_\mu \leq n$ (since $(k_1, k_2, ..., k_\mu) \in \mathbb{N}^\mu$ yields $k_i \geq 0$ for every $i \in \{1, 2, ..., \mu\}$).

and

$$\sum_{t=0}^{n}\frac{n!}{t!}\underbrace{\binom{m+n-t-2}{n-t}}_{=\binom{1+n-t-2}{n-t}=\binom{n-t-1}{n-t}}\left(n+\underbrace{(X_1+X_2+...+X_m)}_{=X_1}\right)^t$$

$$=\sum_{t=0}^{n}\frac{n!}{t!}\binom{n-t-1}{n-t}(n+X_1)^t$$

$$=\sum_{t=0}^{n-1}\frac{n!}{t!}\underbrace{\binom{n-t-1}{n-t}}_{=0,\text{ since }n-t\geq n-t-1\geq 0}(n+X_1)^t+\sum_{t=n}^{n}\underbrace{\frac{n!}{t!}}_{\substack{=1,\text{ since }t=n\\ \text{yields }t!=n!}}\underbrace{\binom{n-t-1}{n-t}}_{\substack{=\binom{n-n-1}{n-n},\\ \text{since }t=n}}\underbrace{(n+X_1)^t}_{\substack{=(n+X_1)^n,\\ \text{since }t=n}}$$

$$=\underbrace{\sum_{t=0}^{n-1}\frac{n!}{t!}\cdot 0\cdot(n+X_1)^t}_{=0}+\sum_{t=n}^{n}1\cdot\underbrace{\binom{n-n-1}{n-n}}_{=\binom{-1}{0}=1}\cdot(n+X_1)^n=0+\underbrace{\sum_{t=n}^{n}1\cdot 1\cdot(n+X_1)^n}_{=1\cdot 1\cdot(n+X_1)^n=(n+X_1)^n}$$

$$=(n+X_1)^n=(X_1+n)^n,$$

what yields (15)). Hence, if $m=1$, then Theorem 5 holds. This completes the induction base.

*Induction step:* Let $\mu\in\mathbb{N}\setminus\{0\}$. Assume that Theorem 5 holds for $m=\mu$. In order to complete the induction step, we must prove that Theorem 5 also holds for $m=\mu+1$.

We have assumed that Theorem 5 holds for $m=\mu$. In other words, we have

$$\alpha_{\underbrace{0,0,...,0}_{\mu\text{ zeroes}}}=\sum_{t=0}^{n}\frac{n!}{t!}\binom{\mu+n-t-2}{n-t}(n+(X_1+X_2+...+X_\mu))^t \tag{16}$$

in the polynomial ring $\mathbb{Z}[X_1,X_2,...,X_\mu]$ for every $n\in\mathbb{N}$.

Now, let $n \in \mathbb{N}$. In the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$, we have

$$\alpha_{\underbrace{0,0,...,0}_{\mu+1 \text{ zeroes}}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu; \\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j + k_j)^{k_j+0} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu + \rho)^{\rho+0} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho+0}$$

$$\left(\text{by Lemma 6, applied to } (a_1, a_2, ..., a_{\mu+1}) = \left(\underbrace{0,0,...,0}_{\mu+1 \text{ zeroes}}\right)\right)$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu; \\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j + k_j)^{k_j} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu + \rho)^{\rho} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho}. \tag{17}$$

Fix some $k_\mu \in \mathbb{N}$. In the polynomial ring $\mathbb{Z}[X, Y]$, we have

$$\sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X + \rho)^{\rho} (Y - \rho)^{k_\mu-\rho} = \sum_{k=0}^{k_\mu} \binom{k_\mu}{k} (X + k)^{k} (Y - k)^{k_\mu-k}$$

$$\text{(here we have renamed } \rho \text{ as } k \text{ in the sum)}$$

$$= \sum_{t=0}^{k_\mu} \frac{k_\mu!}{t!} (X + Y)^t \qquad \text{(by (1), applied to } k_\mu \text{ instead of } n). \tag{18}$$

By the universal property of the polynomial ring, there exists a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[X, Y] \to \mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$ which maps $X$ to $X_\mu$ and $Y$ to $X_{\mu+1} + k_\mu$. Applying this homomorphism to both sides of the equation (18), we obtain

$$\sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu + \rho)^{\rho} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho} = \sum_{t=0}^{k_\mu} \frac{k_\mu!}{t!} (X_\mu + X_{\mu+1} + k_\mu)^t$$

in the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$. Hence, (17) becomes

$$\alpha_{\underbrace{0,0,...,0}_{\mu+1\text{ zeroes}}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot \underbrace{\sum_{t=0}^{k_\mu}}_{\substack{=\sum\limits_{\substack{t\in\mathbb{N};\\ t\le k_\mu}}}} \frac{k_\mu!}{t!} (X_\mu + X_{\mu+1} + k_\mu)^t$$

$$= \underbrace{\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}}}_{\substack{=\sum\limits_{k_\mu\in\mathbb{N}} \sum\limits_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_\mu=n}} \\ =\sum\limits_{k_\mu\in\mathbb{N}} \sum\limits_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=n-k_\mu}} \\ \text{(since } k_1+k_2+...+k_\mu=n \text{ is} \\ \text{equivalent to } k_1+k_2+...+k_{\mu-1}=n-k_\mu)}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot \underbrace{\sum_{t=0}^{k_\mu}}_{\substack{=\sum\limits_{\substack{t\in\mathbb{N};\\ t\le k_\mu}}}} \frac{k_\mu!}{t!} (X_\mu + X_{\mu+1} + k_\mu)^t$$

$$= \sum_{k_\mu\in\mathbb{N}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=n-k_\mu}} \sum_{\substack{t\in\mathbb{N};\\ t\le k_\mu}} \underbrace{\frac{n!}{k_1!k_2!...k_\mu!} \cdot \frac{k_\mu!}{t!}}_{=\frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot t!}} \cdot \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + k_\mu)^t$$

$$= \sum_{k_\mu\in\mathbb{N}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=n-k_\mu}} \sum_{\substack{t\in\mathbb{N};\\ t\le k_\mu}} \frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot t!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + k_\mu)^t$$

$$= \underbrace{\sum_{t\in\mathbb{N}} \sum_{\substack{(k_1,k_2,...,k_{\mu-1})\in\mathbb{N}^{\mu-1};\\ k_1+k_2+...+k_{\mu-1}=n-t}} \sum_{\substack{k_\mu\in\mathbb{N};\\ k_\mu\le t}}}_{\substack{=\sum\limits_{\substack{(k_1,k_2,...,k_{\mu-1},k_\mu,t)\in\mathbb{N}^{\mu+1};\\ k_1+k_2+...+k_{\mu-1}=n-t;\\ k_\mu\le t}} \\ =\sum\limits_{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu} \sum\limits_{\substack{t\in\mathbb{N};\\ k_1+k_2+...+k_{\mu-1}=n-t;\\ k_\mu\le t}}}} \underbrace{\frac{n!}{k_1!k_2!...k_{\mu-1}!\cdot k_\mu!}}_{=\frac{n!}{k_1!k_2!...k_\mu!}} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + t)^{k_\mu}$$

(here we have renamed $k_\mu$ and $t$ as $t$ and $k_\mu$ in the sums)

$$= \sum_{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu} \underbrace{\sum_{\substack{t\in\mathbb{N};\\ k_1+k_2+...+k_{\mu-1}=n-t;\\ k_\mu\le t}}}_{\substack{=\sum\limits_{\substack{t\in\mathbb{Z};\\ t\ge 0;\\ k_1+k_2+...+k_{\mu-1}=n-t;\\ k_\mu\le t}}}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot \left(X_\mu + X_{\mu+1} + \underbrace{t}_{=(n+k_\mu)-(n+k_\mu-t)}\right)^{k_\mu}$$

$$= \sum_{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu} \sum_{\substack{t\in\mathbb{Z};\\ t\geq 0;\\ k_1+k_2+...+k_{\mu-1}=n-t;\\ k_\mu\leq t}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + (n+k_\mu) - (n+k_\mu - t))^{k_\mu}$$

$$= \sum_{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu} \sum_{\substack{t\in\mathbb{Z};\\ 0\leq n+k_\mu-t\leq n;\\ k_1+k_2+...+k_\mu=n+k_\mu-t}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + (n+k_\mu) - (n+k_\mu - t))^{k_\mu}$$

(since for every triple $t \in \mathbb{Z}$ and every $(k_1, k_2, ..., k_\mu) \in \mathbb{N}^\mu$, the assertions

$$(t \geq 0 \text{ and } k_1 + k_2 + ... + k_{\mu-1} = n - t \text{ and } k_\mu \leq t)$$

and

$$(0 \leq n + k_\mu - t \leq n \text{ and } k_1 + k_2 + ... + k_\mu = n + k_\mu - t)$$

are equivalent[4]). Thus,

$$\alpha_{\underbrace{0,0,...,0}_{\mu+1 \text{ zeroes}}}$$

$$= \sum_{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu} \sum_{\substack{t\in\mathbb{Z};\\ 0\leq n+k_\mu-t\leq n;\\ k_1+k_2+...+k_\mu=n+k_\mu-t}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + (n+k_\mu) - (n+k_\mu-t))^{k_\mu}$$

$$= \underbrace{\sum_{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu} \sum_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n;\\ k_1+k_2+...+k_\mu=t}}}_{=\sum\limits_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n}} \sum\limits_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=t}}} \underbrace{\frac{n!}{k_1!k_2!...k_\mu!}}_{=\frac{n!}{t!}\cdot\frac{t!}{k_1!k_2!...k_\mu!}} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot \left(\underbrace{X_\mu + X_{\mu+1} + (n+k_\mu) - t}_{=X_\mu+X_{\mu+1}+n-t+k_\mu}\right)^{k_\mu}$$

(here we substituted $n+k_\mu-t$ for $t$ in the second sum)

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[4] In fact,

- if
$$(t\geq 0 \text{ and } k_1+k_2+...+k_{\mu-1}=n-t \text{ and } k_\mu\leq t),$$
then
$$(0\leq n+k_\mu-t\leq n \text{ and } k_1+k_2+...+k_\mu=n+k_\mu-t)$$
(since if
$$(t\geq 0 \text{ and } k_1+k_2+...+k_{\mu-1}=n-t \text{ and } k_\mu\leq t),$$
then
$$0\leq n+k_\mu-t \qquad \left(\text{since } n+k_\mu-t = \underbrace{(n-t)}_{=k_1+k_2+...+k_{\mu-1}} + k_\mu = \underbrace{k_1}_{\in\mathbb{N}} + \underbrace{k_2}_{\in\mathbb{N}} + ... + \underbrace{k_{\mu-1}}_{\in\mathbb{N}} + \underbrace{k_\mu}_{\in\mathbb{N}} \in \mathbb{N}\right);$$
$$n+k_\mu-t\leq n \qquad (\text{since } k_\mu\leq t \text{ yields } n+k_\mu-t\leq n+t-t=n);$$
$$k_1+k_2+...+k_\mu=n+k_\mu-t$$
$$\left(\text{since } n+k_\mu-t = \underbrace{(n-t)}_{=k_1+k_2+...+k_{\mu-1}} + k_\mu = k_1+k_2+...+k_{\mu-1}+k_\mu = k_1+k_2+...+k_\mu\right)$$
);
- conversely, if
$$(0\leq n+k_\mu-t\leq n \text{ and } k_1+k_2+...+k_\mu=n+k_\mu-t),$$
then
$$(t\geq 0 \text{ and } k_1+k_2+...+k_{\mu-1}=n-t \text{ and } k_\mu\leq t),$$
(since if
$$(0\leq n+k_\mu-t\leq n \text{ and } k_1+k_2+...+k_\mu=n+k_\mu-t),$$
then
$$t\geq 0 \qquad (\text{since } n+k_\mu-t\leq n \text{ yields } n+k_\mu\leq n+t, \text{ thus } k_\mu\leq t, \text{ thus } t\geq k_\mu\geq 0 \text{ (since } k_\mu\in\mathbb{N}));$$
$$k_1+k_2+...+k_{\mu-1}=n-t$$
$$\left(\begin{array}{c}\text{since } (n-t)+k_\mu = n+k_\mu-t = k_1+k_2+...+k_\mu = (k_1+k_2+...+k_{\mu-1})+k_\mu,\\ \text{thus } n-t=k_1+k_2+...+k_{\mu-1}\end{array}\right);$$
$$k_\mu\leq t \qquad (\text{as proven above})$$
).

$$= \underbrace{\sum_{\substack{t\in\mathbb{Z};\\ 0\leq t\leq n}}}_{=\sum_{t=0}^{n}} \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=t}} \frac{n!}{t!}\cdot\frac{t!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j}\cdot(X_\mu+X_{\mu+1}+n-t+k_\mu)^{k_\mu}$$

$$= \sum_{t=0}^{n} \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=t}} \frac{n!}{t!}\cdot\frac{t!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j}\cdot(X_\mu+X_{\mu+1}+n-t+k_\mu)^{k_\mu}$$

$$= \sum_{t=0}^{n} \frac{n!}{t!}\cdot \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=t}} \frac{t!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j}\cdot(X_\mu+X_{\mu+1}+n-t+k_\mu)^{k_\mu}. \tag{19}$$

On the other hand, in the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu]$, we have

$$\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \underbrace{\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j}\cdot(X_\mu+k_\mu)^{k_\mu}}_{=\prod_{j=1}^{\mu}(X_j+k_j)^{k_j}=\prod_{j=1}^{\mu}(X_j+k_j)^{k_j+0}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu}(X_j+k_j)^{k_j+0} = \alpha_{\underbrace{0,0,...,0}_{\mu\text{ zeroes}}}$$

$$= \sum_{t=0}^{n}\frac{n!}{t!}\binom{\mu+n-t-2}{n-t}(n+(X_1+X_2+...+X_\mu))^t \qquad \text{(by (16))}$$

$$= \sum_{\rho=0}^{n}\frac{n!}{\rho!}\binom{\mu+n-\rho-2}{n-\rho}\left(n+\underbrace{(X_1+X_2+...+X_\mu)}_{=(X_1+X_2+...+X_{\mu-1})+X_\mu}\right)^\rho$$

(here we renamed $t$ as $\rho$ in the sum)

$$= \sum_{\rho=0}^{n}\frac{n!}{\rho!}\binom{\mu+n-\rho-2}{n-\rho}(n+(X_1+X_2+...+X_{\mu-1})+X_\mu)^\rho. \tag{20}$$

Let $\tau\in\mathbb{Z}$. By the universal property of the polynomial ring, there exists a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[X_1, X_2, ..., X_\mu]\to\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$ which maps $X_i$ to $X_i$ (for every $i\in\{1,2,...,\mu-1\}$) and $X_\mu$ to $X_\mu+X_{\mu+1}+\tau$. Applying this homomorphism to both sides of the equation (20), we obtain

$$\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j}\cdot(X_\mu+X_{\mu+1}+\tau+k_\mu)^{k_\mu}$$

$$= \sum_{\rho=0}^{n}\frac{n!}{\rho!}\binom{\mu+n-\rho-2}{n-\rho}(n+(X_1+X_2+...+X_{\mu-1})+(X_\mu+X_{\mu+1}+\tau))^\rho. \tag{21}$$

Now, (19) becomes

$$\alpha_{\underbrace{0,0,...,0}_{\mu+1 \text{ zeroes}}}$$

$$= \sum_{t=0}^{n} \frac{n!}{t!} \cdot \underbrace{\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=t}} \frac{t!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j} \cdot (X_\mu + X_{\mu+1} + n - t + k_\mu)^{k_\mu}}_{\substack{=\sum_{\rho=0}^{t} \frac{t!}{\rho!}\binom{\mu+t-\rho-2}{t-\rho}(t+(X_1+X_2+...+X_{\mu-1})+(X_\mu+X_{\mu+1}+(n-t)))^\rho \\ \text{by (21), applied to } t \text{ and } n-t \text{ instead of } n \text{ and } \tau}}$$

$$= \sum_{t=0}^{n} \frac{n!}{t!} \cdot \sum_{\rho=0}^{t} \frac{t!}{\rho!}\binom{\mu+t-\rho-2}{t-\rho}\left(\underbrace{t+(X_1+X_2+...+X_{\mu-1})+(X_\mu+X_{\mu+1}+(n-t))}_{=n+(X_1+X_2+...+X_{\mu-1})+(X_\mu+X_{\mu+1})}\right)^\rho$$

$$= \sum_{t=0}^{n} \frac{n!}{t!} \cdot \sum_{\rho=0}^{t} \frac{t!}{\rho!}\binom{\mu+t-\rho-2}{t-\rho}\left(n+\underbrace{(X_1+X_2+...+X_{\mu-1})+(X_\mu+X_{\mu+1})}_{=X_1+X_2+...+X_{\mu+1}}\right)^\rho$$

$$= \sum_{t=0}^{n} \frac{n!}{t!} \cdot \sum_{\rho=0}^{t} \frac{t!}{\rho!}\binom{\mu+t-\rho-2}{t-\rho}(n+(X_1+X_2+...+X_{\mu+1}))^\rho$$

$$= \sum_{t=0}^{n} \sum_{\rho=0}^{t} \underbrace{\frac{n!}{t!} \cdot \frac{t!}{\rho!}}_{=\frac{n!}{\rho!}} \binom{\mu+t-\rho-2}{t-\rho}(n+(X_1+X_2+...+X_{\mu+1}))^\rho$$

$$= \underbrace{\sum_{t=0}^{n} \sum_{\rho=0}^{t}}_{=\sum_{\rho=0}^{n}\sum_{t=\rho}^{n}} \frac{n!}{\rho!}\binom{\mu+t-\rho-2}{t-\rho}(n+(X_1+X_2+...+X_{\mu+1}))^\rho$$

$$= \sum_{\rho=0}^{n} \sum_{t=\rho}^{n} \frac{n!}{\rho!}\binom{\mu+t-\rho-2}{t-\rho}(n+(X_1+X_2+...+X_{\mu+1}))^\rho$$

$$= \sum_{\rho=0}^{n} \frac{n!}{\rho!} \sum_{t=\rho}^{n} \binom{\mu+t-\rho-2}{t-\rho}(n+(X_1+X_2+...+X_{\mu+1}))^\rho . \qquad (22)$$

But for every $\rho \in \mathbb{N}$, we have

$$\sum_{t=\rho}^{n} \binom{\mu+t-\rho-2}{t-\rho} = \sum_{t=\rho}^{n} \left( \binom{(\mu+t-\rho-2)+1}{t-\rho} - \binom{\mu+t-\rho-2}{(t-\rho)-1} \right)$$

$$\left( \begin{array}{c} \text{as the recurrence of the binomial coefficients yields} \\ \binom{(\mu+t-\rho-2)+1}{t-\rho} = \binom{\mu+t-\rho-2}{(t-\rho)-1} + \binom{\mu+t-\rho-2}{t-\rho}, \\ \text{thus } \binom{\mu+t-\rho-2}{t-\rho} = \binom{(\mu+t-\rho-2)+1}{t-\rho} - \binom{\mu+t-\rho-2}{(t-\rho)-1} \end{array} \right)$$

$$= \sum_{t=\rho}^{n} \left( \binom{\mu+t-\rho-1}{t-\rho} - \underbrace{\binom{\mu+t-\rho-2}{t-\rho-1}}_{=\binom{\mu+(t-1)-\rho-1}{(t-1)-\rho}} \right)$$

$$= \sum_{t=\rho}^{n} \left( \binom{\mu+t-\rho-1}{t-\rho} - \binom{\mu+(t-1)-\rho-1}{(t-1)-\rho} \right)$$

$$= \sum_{t=\rho}^{n} \binom{\mu+t-\rho-1}{t-\rho} - \sum_{t=\rho}^{n} \binom{\mu+(t-1)-\rho-1}{(t-1)-\rho}$$

$$= \underbrace{\sum_{t=\rho}^{n} \binom{\mu+t-\rho-1}{t-\rho}}_{=\sum_{t=\rho}^{n-1} \binom{\mu+t-\rho-1}{t-\rho} + \binom{\mu+n-\rho-1}{n-\rho}} - \underbrace{\sum_{t=\rho-1}^{n-1} \binom{\mu+t-\rho-1}{t-\rho}}_{=\binom{\mu+(\rho-1)-\rho-1}{(\rho-1)-\rho} + \sum_{t=\rho}^{n-1} \binom{\mu+t-\rho-1}{t-\rho}}$$

(here we substituted $t$ for $t-1$ in the second sum)

$$= \left( \sum_{t=\rho}^{n-1} \binom{\mu+t-\rho-1}{t-\rho} + \binom{\mu+n-\rho-1}{n-\rho} \right) - \left( \binom{\mu+(\rho-1)-\rho-1}{(\rho-1)-\rho} + \sum_{t=\rho}^{n-1} \binom{\mu+t-\rho-1}{t-\rho} \right)$$

$$= \binom{\mu+n-\rho-1}{n-\rho} - \underbrace{\binom{\mu+(\rho-1)-\rho-1}{(\rho-1)-\rho}}_{=0, \text{ since } (\rho-1)-\rho=-1<0} = \binom{\mu+n-\rho-1}{n-\rho} = \binom{(\mu+1)+n-\rho-2}{n-\rho}.$$

Thus, (22) becomes

$$\alpha_{\underbrace{0,0,\ldots,0}_{\mu+1 \text{ zeroes}}} = \sum_{\rho=0}^{n} \frac{n!}{\rho!} \binom{(\mu+1)+n-\rho-2}{n-\rho} (n + (X_1 + X_2 + \ldots + X_{\mu+1}))^{\rho}$$

$$= \sum_{t=0}^{n} \frac{n!}{t!} \binom{(\mu+1)+n-t-2}{n-t} (n + (X_1 + X_2 + \ldots + X_{\mu+1}))^{t}$$

(here we renamed $\rho$ as $t$ in the sum)

In other words, Theorem 5 holds for $m = \mu + 1$. This completes the induction step.

Thus, the induction proof of Theorem 5 is complete.

Theorem 4 also generalizes:

**Theorem 7.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $n \in \mathbb{N}$. Then,

$$X_1 X_2 ... X_{m-1} \cdot \alpha_{\underbrace{-1,-1,...,-1}_{m-1 \text{ times}},0} = (n + (X_1 + X_2 + ... + X_m))^n \qquad (23)$$

in the quotient field of the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_m]$.

*Proof of Theorem 7.* We will prove Theorem 7 by induction over $m$:

*Induction base:* If $m = 1$, then (23) is true (since if $m = 1$, then

$$\alpha_{\underbrace{-1,-1,...,-1}_{m-1 \text{ times}},0} = \alpha_0 \qquad (\text{since } m = 1 \text{ yields } m-1 = 0)$$

$$= \sum_{\substack{(k_1,k_2,...,k_m) \in \mathbb{N}^m; \\ k_1+k_2+...+k_m = n}} \frac{n!}{k_1! k_2! ... k_m!} \prod_{j=1}^{m} (X_j + k_j)^{k_j + 0}$$

$$= \sum_{\substack{(k_1) \in \mathbb{N}^1; \\ k_1 = n}} \underbrace{\frac{n!}{k_1!}}_{\substack{=1, \text{ since} \\ k_1 = n \text{ yields } k_1! = n!}} \underbrace{\prod_{j=1}^{1} (X_j + k_j)^{k_j + 0}}_{\substack{=(X_1+k_1)^{k_1+0} \\ =(X_1+k_1)^{k_1} \\ =(X_1+n)^n, \\ \text{since } k_1 = n}} = \sum_{\substack{(k_1) \in \mathbb{N}^1; \\ k_1 = n}} (X_1 + n)^n = (X_1 + n)^n$$

and $X_1 X_2 ... X_{m-1} = 1$ (since $m = 1$ yields $m - 1 = 0$) yield

$$X_1 X_2 ... X_{m-1} \cdot \alpha_{\underbrace{-1,-1,...,-1}_{m-1 \text{ times}},0} = 1 \cdot (X_1 + n)^n = (X_1 + n)^n$$

$$= \left( n + \underbrace{X_1}_{=X_1+X_2+...+X_m} \right)^n = (n + (X_1 + X_2 + ... + X_m))^n,$$

what yields (23)). Hence, if $m = 1$, then Theorem 7 holds. This completes the induction base.

*Induction step:* Let $\mu \in \mathbb{N} \setminus \{0\}$. Assume that Theorem 7 holds for $m = \mu$. In order to complete the induction step, we must prove that Theorem 7 also holds for $m = \mu + 1$.

We have assumed that Theorem 7 holds for $m = \mu$. In other words, we have

$$X_1 X_2 ... X_{\mu-1} \cdot \alpha_{\underbrace{-1,-1,...,-1}_{\mu-1 \text{ times}},0} = (n + (X_1 + X_2 + ... + X_\mu))^n \qquad (24)$$

in the quotient field of the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu]$ for every $n \in \mathbb{N}$.

Now, let $n \in \mathbb{N}$. In the quotient field of the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$, we have

$$\alpha_{\underbrace{-1,-1,...,-1}_{\mu \text{ times}},0}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu; \\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j + k_j)^{k_j+(-1)} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu + \rho)^{\rho+(-1)} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho+0}$$

$$\left(\text{by Lemma 6, applied to } (a_1, a_2, ..., a_{\mu+1}) = \left(\underbrace{-1,-1,...,-1}_{\mu \text{ times}}, 0\right)\right)$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu; \\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j + k_j)^{k_j-1} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu + \rho)^{\rho-1} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho}. \tag{25}$$

Fix some $k_\mu \in \mathbb{N}$. In the quotient field of the polynomial ring $\mathbb{Z}[X, Y]$, we have

$$\sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} X (X+\rho)^{\rho-1} (Y-\rho)^{k_\mu-\rho} = \sum_{k=0}^{k_\mu} \binom{k_\mu}{k} X (X+k)^{k-1} (Y-k)^{k_\mu-k}$$

$$(\text{here we have renamed } \rho \text{ as } k \text{ in the sum})$$

$$= (X+Y)^{k_\mu} \qquad (\text{by (11), applied to } k_\mu \text{ instead of } n). \tag{26}$$

This is actually an identity in the polynomial ring $\mathbb{Z}[X, Y]$ (and not just in its quotient field), since $\binom{k_\mu}{\rho} X (X+\rho)^{\rho-1} (Y-\rho)^{k_\mu-\rho} \in \mathbb{Z}[X,Y]$ for every $\rho \in \{0, 1, ..., k_\mu\}$ (because $\binom{k_\mu}{\rho} \in \mathbb{Z}[X,Y]$, $X (X+\rho)^{\rho-1} \in \mathbb{Z}[X,Y]$ [5] and $(Y-\rho)^{k_\mu-\rho} \in \mathbb{Z}[X,Y]$) and $(X+Y)^{k_\mu} \in \mathbb{Z}[X,Y]$.

By the universal property of the polynomial ring, there exists a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[X,Y] \to \mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$ which maps $X$ to $X_\mu$ and $Y$ to $X_{\mu+1} + k_\mu$. Applying this homomorphism to both sides of the equation (26), we obtain

$$\sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} X_\mu (X_\mu+\rho)^{\rho-1} (X_{\mu+1} + k_\mu - \rho)^{k_\mu-\rho} = (X_\mu + X_{\mu+1} + k_\mu)^{k_\mu} \tag{27}$$

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[5] *Proof of the relation* $X (X+\rho)^{\rho-1} \in \mathbb{Z}[X,Y]$.

We have either $\rho = 0$ or $\rho \geq 1$ (since $\rho \in \{0, 1, ..., k_\mu\}$). In both of these cases, we have $X (X+\rho)^{\rho-1} \in \mathbb{Z}[X,Y]$, because:

- if $\rho = 0$, then $X (X+\rho)^{\rho-1} = X (X+0)^{0-1} = XX^{-1} = 1 \in \mathbb{Z}[X,Y]$;
- if $\rho \geq 1$, then $\underbrace{X}_{\in\mathbb{Z}[X,Y]} \underbrace{(X+\rho)^{\rho-1}}_{\in\mathbb{Z}[X,Y]} \in \mathbb{Z}[X,Y]$.

Thus, $X (X+\rho)^{\rho-1} \in \mathbb{Z}[X,Y]$ holds for every $\rho \in \{0, 1, ..., k_\mu\}$, qed.

in the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$. Hence, in the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$, we have

$$X_1X_2...X_\mu \cdot \alpha_{\underbrace{-1,-1,...,-1}_{\mu \text{ times}},0}$$

$$= \underbrace{X_1X_2...X_\mu}_{=\prod_{j=1}^{\mu} X_j = \prod_{j=1}^{\mu-1} X_j \cdot X_\mu} \cdot \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j-1} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho-1} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho}$$

(by (25))

$$= \prod_{j=1}^{\mu-1} X_j \cdot X_\mu \cdot \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j-1} \cdot \sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} (X_\mu+\rho)^{\rho-1} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \underbrace{\prod_{j=1}^{\mu-1} X_j \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j-1}}_{=\prod_{j=1}^{\mu-1}\left(X_j(X_j+k_j)^{k_j-1}\right)} \cdot \underbrace{\sum_{\rho=0}^{k_\mu} \binom{k_\mu}{\rho} X_\mu (X_\mu+\rho)^{\rho-1} (X_{\mu+1}+k_\mu-\rho)^{k_\mu-\rho}}_{=(X_\mu+X_{\mu+1}+k_\mu)^{k_\mu} \text{ by (27)}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} \left(X_j (X_j+k_j)^{k_j-1}\right) \cdot (X_\mu+X_{\mu+1}+k_\mu)^{k_\mu}. \qquad (28)$$

On the other hand, in the quotient field of the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu]$, we have

$$\alpha_{\underbrace{-1,-1,...,-1}_{\mu-1 \text{ times}},0}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \underbrace{\prod_{j=1}^{\mu} (X_j+k_j)^{k_j+\begin{cases} 0, & \text{if } j=\mu; \\ -1, & \text{if } j<\mu \end{cases}}}_{=\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j+\begin{cases} 0, & \text{if } j=\mu; \\ -1, & \text{if } j<\mu \end{cases}} \cdot (X_\mu+k_\mu)^{k_\mu+\begin{cases} 0, & \text{if } \mu=\mu; \\ -1, & \text{if } \mu<\mu \end{cases}}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} \underbrace{(X_j+k_j)^{k_j+\begin{cases} 0, & \text{if } j=\mu; \\ -1, & \text{if } j<\mu \end{cases}}}_{\substack{=(X_j+k_j)^{k_j+(-1)},\\ \text{since } j<\mu}} \cdot \underbrace{(X_\mu+k_\mu)^{k_\mu+\begin{cases} 0, & \text{if } \mu=\mu; \\ -1, & \text{if } \mu<\mu \end{cases}}}_{\substack{=(X_\mu+k_\mu)^{k_\mu+0},\\ \text{since } \mu=\mu}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} \underbrace{(X_j+k_j)^{k_j+(-1)}}_{=(X_j+k_j)^{k_j-1}} \cdot \underbrace{(X_\mu+k_\mu)^{k_\mu+0}}_{=(X_\mu+k_\mu)^{k_\mu}}$$

$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} (X_j+k_j)^{k_j-1} \cdot (X_\mu+k_\mu)^{k_\mu}, \qquad (29)$$

and thus

$$X_1X_2...X_{\mu-1}\cdot\alpha_{\underbrace{-1,-1,...,-1}_{\mu-1\text{ times}},0}$$

$$=\underbrace{X_1X_2...X_{\mu-1}}_{=\prod_{j=1}^{\mu-1}X_j}\cdot\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}}\frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j-1}\cdot(X_\mu+k_\mu)^{k_\mu}$$

$$=\prod_{j=1}^{\mu-1}X_j\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}}\frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j-1}\cdot(X_\mu+k_\mu)^{k_\mu}$$

$$=\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}}\frac{n!}{k_1!k_2!...k_\mu!}\underbrace{\prod_{j=1}^{\mu-1}X_j\prod_{j=1}^{\mu-1}(X_j+k_j)^{k_j-1}}_{=\prod_{j=1}^{\mu-1}\left(X_j(X_j+k_j)^{k_j-1}\right)}\cdot(X_\mu+k_\mu)^{k_\mu}$$

$$=\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}}\frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}\left(X_j(X_j+k_j)^{k_j-1}\right)\cdot(X_\mu+k_\mu)^{k_\mu}.$$

Hence, (24) becomes

$$\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}}\frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}\left(X_j(X_j+k_j)^{k_j-1}\right)\cdot(X_\mu+k_\mu)^{k_\mu}$$
$$=\left(n+\underbrace{(X_1+X_2+...+X_\mu)}_{=(X_1+X_2+...+X_{\mu-1})+X_\mu}\right)^n$$
$$=(n+(X_1+X_2+...+X_{\mu-1})+X_\mu)^n. \tag{30}$$

This is actually an identity in the polynomial ring $\mathbb{Z}[X_1,X_2,...,X_\mu]$ (and not just in its quotient field), since $\frac{n!}{k_1!k_2!...k_\mu!}\prod_{j=1}^{\mu-1}\left(X_j(X_j+k_j)^{k_j-1}\right)\cdot(X_\mu+k_\mu)^{k_\mu}\in\mathbb{Z}[X_1,X_2,...,X_\mu]$ for every $(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu$ such that $k_1+k_2+...+k_\mu=n$ (because $\frac{n!}{k_1!k_2!...k_\mu!}\in\mathbb{Z}[X_1,X_2,...,X_\mu]$ [6], $X_j(X_j+k_j)^{k_j-1}\in\mathbb{Z}[X_1,X_2,...,X_\mu]$ for every $j\in\{1,2,...,\mu-1\}$ [7] and $(X_\mu+k_\mu)^{k_\mu}\in\mathbb{Z}[X_1,X_2,...,X_\mu]$) and $(n+(X_1+X_2+...+X_{\mu-1})+X_\mu)^n\in\mathbb{Z}[X_1,X_2,...,X_\mu]$.

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[6] since $\frac{n!}{k_1!k_2!...k_\mu!}\in\mathbb{Z}$

[7] *Proof of the relation* $X_j(X_j+k_j)^{k_j-1}\in\mathbb{Z}[X_1,X_2,...,X_\mu]$ for every $j\in\{1,2,...,\mu-1\}$.

Fix some $j\in\{1,2,...,\mu-1\}$.

We have either $k_j=0$ or $k_j\geq 1$ (since $k_j\in\mathbb{N}$). In both of these cases, we have $X_j(X_j+k_j)^{k_j-1}\in\mathbb{Z}[X_1,X_2,...,X_\mu]$, because:

- if $k_j=0$, then $X_j(X_j+k_j)^{k_j-1}=X_j(X_j+0)^{0-1}=X_jX_j^{-1}=1\in\mathbb{Z}[X_1,X_2,...,X_\mu]$;

By the universal property of the polynomial ring, there exists a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[X_1, X_2, ..., X_\mu] \to \mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$ which maps $X_i$ to $X_i$ (for every $i \in \{1, 2, ..., \mu-1\}$) and $X_\mu$ to $X_\mu + X_{\mu+1}$. Applying this homomorphism to both sides of the equation (30), we obtain

$$\sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} \left(X_j (X_j+k_j)^{k_j-1}\right) \cdot (X_\mu + X_{\mu+1} + k_\mu)^{k_\mu}$$
$$= (n + (X_1 + X_2 + ... + X_{\mu-1}) + (X_\mu + X_{\mu+1}))^n . \qquad (31)$$

Thus, in the polynomial ring $\mathbb{Z}[X_1, X_2, ..., X_\mu, X_{\mu+1}]$, we have

$$X_1X_2...X_\mu \cdot \alpha_{\underbrace{-1,-1,...,-1}_{\mu \text{ times}},0}$$
$$= \sum_{\substack{(k_1,k_2,...,k_\mu)\in\mathbb{N}^\mu;\\ k_1+k_2+...+k_\mu=n}} \frac{n!}{k_1!k_2!...k_\mu!} \prod_{j=1}^{\mu-1} \left(X_j (X_j+k_j)^{k_j-1}\right) \cdot (X_\mu + X_{\mu+1} + k_\mu)^{k_\mu} \qquad (\text{by } (28))$$
$$= \left(n + \underbrace{(X_1 + X_2 + ... + X_{\mu-1}) + (X_\mu + X_{\mu+1})}_{=X_1+X_2+...+X_{\mu+1}}\right)^n \qquad (\text{by } (31))$$
$$= (n + (X_1 + X_2 + ... + X_{\mu+1}))^n .$$

In other words, Theorem 7 holds for $m = \mu + 1$. This completes the induction step.

Thus, the induction proof of Theorem 7 is complete.

Finally, notice how Theorem 1 yields an important fact:

**Theorem 8.** Let $R$ be a commutative ring with unity. Let $N \in \mathbb{N}$. Let $P \in R[X]$ be a polynomial such that $\deg P < N$. Then,

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} P(k) = 0.$$

*Proof of Theorem 8.* Since $P \in R[X]$ is a polynomial such that $\deg P < N$, there exist elements $a_0, a_1, ..., a_{N-1}$ of $R$ such that $P(X) = \sum\limits_{\ell=0}^{N-1} a_\ell X^\ell$. Thus, $P(k) = \sum\limits_{\ell=0}^{N-1} a_\ell k^\ell$ for every $k \in \mathbb{Z}$. Hence,

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} P(k) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \sum_{\ell=0}^{N-1} a_\ell k^\ell = \sum_{\ell=0}^{N-1} a_\ell \underbrace{\sum_{k=0}^{N} (-1)^k \binom{N}{k} k^\ell}_{\substack{=0 \text{ by (2), since}\\ \ell\in\{0,1,...,N-1\}}} = \sum_{\ell=0}^{N-1} a_\ell 0 = 0,$$

- if $k_j \geq 1$, then $\underbrace{X_j}_{\in\mathbb{Z}[X_1,X_2,...,X_\mu]} \underbrace{(X_j+k_j)^{k_j-1}}_{\in\mathbb{Z}[X_1,X_2,...,X_\mu]} \in \mathbb{Z}[X_1, X_2, ..., X_\mu]$.

Thus, $X_j (X_j + k_j)^{k_j-1} \in \mathbb{Z}[X_1, X_2, ..., X_\mu]$ holds for every $j \in \{1, 2, ..., \mu-1\}$, qed.

and Theorem 8 is proven.

**Remarks on authorship:**

The identity (1) is known as the *Cauchy identity* and appears in [1], section 1.5.

Theorem 4 is the so-called *Abel's generalized binomial formula* and appears in [1], section 1.5, and in [2] as Theorem 5 in section 3.1.

Theorems 5 and 7 are the so-called *first and second Hurwitz identities* and appear in [1], section 1.6.

## References


[1] John Riordan, *Combinatorial Identities*, John Wiley & Sons, 1968.

[2] Louis Comtet, *Advanced Combinatorics*, Revised and enlarged edition, D. Reidel, 1974.