Original proposal of Mathematical Reflections problem O25 / Darij Grinberg

The following problem submission made it into the periodical "Mathematical Reflections" as Problem O25 (in a shortened form). Below is my original solution of this problem. A much simpler solution was published in "Mathematical Reflections" issue 6/2006.

Problem. For any triangle ABC, prove that

$$\cos\frac{A}{2}\cot\frac{A}{2}+\cos\frac{B}{2}\cot\frac{B}{2}+\cos\frac{C}{2}\cot\frac{C}{2}\geq\frac{\sqrt{3}}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$
$$\geq\frac{9}{2}\geq 2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right).$$

Solution. The only interesting part of the inequality is

$$\cos\frac{A}{2}\cot\frac{A}{2}+\cos\frac{B}{2}\cot\frac{B}{2}+\cos\frac{C}{2}\cot\frac{C}{2}\geq\frac{\sqrt{3}}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right), \qquad (1)$$

because the other two parts of the inequality are pretty easy:

Since the angles of a triangle sum up to 180°, we have $A+B+C=180°$; since the function $f(x)=\cot x$ is convex on the interval $]0°;\ 90°[$ (the interval where the angles $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ lie), the Jensen inequality yields

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\geq 3\cot\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}=3\cot\frac{A+B+C}{6}=3\cot\frac{180°}{6}=3\cot 30°=3\sqrt{3},$$

so that

$$\frac{\sqrt{3}}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)\geq\frac{9}{2},$$

and since $\cos\varphi=2\cos^2\frac{\varphi}{2}-1$ for every angle φ, the famous triangle inequality $\cos A+\cos B+\cos C\leq\frac{3}{2}$ rewrites as

$$\left(2\cos^2\frac{A}{2}-1\right)+\left(2\cos^2\frac{B}{2}-1\right)+\left(2\cos^2\frac{C}{2}-1\right)\leq\frac{3}{2}, \qquad \text{so that}$$
$$2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)-3\leq\frac{3}{2}, \qquad \text{and thus}$$
$$2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\leq 3+\frac{3}{2}=\frac{9}{2}.$$

So it only remains to prove the inequality (1). Let $s=\frac{a+b+c}{2}$ be the semiperimeter of triangle ABC. Then, it is known that the reals $x=s-a$, $y=s-b$, $z=s-c$ are positive. Also, $x+y+z=s$ (because $x+y+z=(s-a)+(s-b)+(s-c)=3s-(a+b+c)=3s-2s=s$) and $y+z=a$ (because $y+z=(x+y+z)-x=$

$s-(s-a)=a$) and similarly $z+x=b$ and $x+y=c$. Hence, the well-known half-angle formulas $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ and $\cot\frac{A}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ (the latter is better known in the equivalent form $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$) rewrite as

$$\cos\frac{A}{2}=\sqrt{\frac{(x+y+z)x}{(z+x)(x+y)}} \qquad \text{and} \qquad \cot\frac{A}{2}=\sqrt{\frac{(x+y+z)x}{yz}}.$$

Now, using the sign $\sum$ for cyclic sums, the inequality (1) becomes

$$\sum\cos\frac{A}{2}\cot\frac{A}{2}\geq\frac{\sqrt{3}}{2}\sum\cot\frac{A}{2};$$

but

$$\sum\cos\frac{A}{2}\cot\frac{A}{2}=\sum\sqrt{\frac{(x+y+z)x}{(z+x)(x+y)}}\cdot\sqrt{\frac{(x+y+z)x}{yz}}=\sum\frac{(x+y+z)x}{\sqrt{(z+x)(x+y)yz}}$$

and

$$\sum\cot\frac{A}{2}=\sum\sqrt{\frac{(x+y+z)x}{yz}}=\sqrt{\frac{x+y+z}{xyz}}\sum x=\sqrt{\frac{x+y+z}{xyz}}(x+y+z),$$

so this inequality becomes

$$\sum\frac{(x+y+z)x}{\sqrt{(z+x)(x+y)yz}}\geq\frac{\sqrt{3}}{2}\sqrt{\frac{x+y+z}{xyz}}(x+y+z).$$

Upon multiplication by $\frac{\sqrt{xyz}}{x+y+z}$, this rewrites as

$$\sum\frac{x\sqrt{x}}{\sqrt{(z+x)(x+y)}}\geq\frac{\sqrt{3}}{2}\sqrt{x+y+z}, \qquad \text{or, equivalently,}$$

$$\sum\frac{x^2}{\sqrt{x(z+x)(x+y)}}\geq\frac{\sqrt{3}}{2}\sqrt{x+y+z}.$$

Now, by the Cauchy-Schwarz inequality in Engel form,

$$\sum\frac{x^2}{\sqrt{x(z+x)(x+y)}}\geq\frac{(x+y+z)^2}{\sum\sqrt{x(z+x)(x+y)}},$$

so it remains to prove that

$$\frac{(x+y+z)^2}{\sum\sqrt{x(z+x)(x+y)}}\geq\frac{\sqrt{3}}{2}\sqrt{x+y+z}.$$

This simplifies to

$$\sqrt{(x+y+z)^3} \geq \frac{\sqrt{3}}{2} \sum \sqrt{x(z+x)(x+y)}.$$

Squaring this yields

$$(x+y+z)^3 \geq \frac{3}{4} \left(\sum \sqrt{x(z+x)(x+y)} \right)^2, \qquad \text{i. e.}$$
$$4(x+y+z)^3 \geq 3 \left(\sum \sqrt{x(z+x)(x+y)} \right)^2.$$

This rewrites as

$$4(x+y+z)^3 \geq 3 \left(\sum x(z+x)(x+y) + 2 \sum \sqrt{y(x+y)(y+z)} \cdot \sqrt{z(y+z)(z+x)} \right), \qquad \text{i. e.}$$
$$4(x+y+z)^3 \geq 3 \left(\sum x(z+x)(x+y) + 2 \sum (y+z) \sqrt{yz(z+x)(x+y)} \right), \qquad \text{i. e.}$$
$$4(x+y+z)^3 \geq 3 \sum x(z+x)(x+y) + 6 \sum (y+z) \sqrt{yz(z+x)(x+y)}, \qquad \text{i. e.}$$
$$4(x+y+z)^3 - 3 \sum x(z+x)(x+y) \geq 6 \sum (y+z) \sqrt{yz(z+x)(x+y)}.$$

Now,

$$\begin{aligned}
&4(x+y+z)^3 - 3 \sum x(z+x)(x+y) \\
&= 4(x+y+z)^3 - 3 \sum ((x+y+z)(z+x)(x+y) - (y+z)(z+x)(x+y)) \\
&= 4(x+y+z)^3 - 3(x+y+z) \sum (z+x)(x+y) + 9(y+z)(z+x)(x+y) \\
&= (x+y+z) \cdot \left(4(x+y+z)^2 - 3 \sum (z+x)(x+y) \right) + 9(y+z)(z+x)(x+y) \\
&= (x+y+z) \cdot \left(\underbrace{x^2+y^2+z^2-yz-zx-xy}_{\geq 0, \text{ as you know}} \right) + 9(y+z)(z+x)(x+y) \\
&\geq 9(y+z)(z+x)(x+y) = 3 \sum (y+z)(z+x)(x+y) \\
&= 3 \left(\sum y(z+x)(x+y) + \sum z(z+x)(x+y) \right) \\
&= 3 \left(\sum z(x+y)(y+z) + \sum y(y+z)(z+x) \right) \\
&= 3 \sum (z(x+y)(y+z) + y(y+z)(z+x)) = 3 \sum (y+z)(y(z+x) + z(x+y)),
\end{aligned}$$

so that, in order to prove the above inequality, it will be enough to show that

$$3 \sum (y+z)(y(z+x) + z(x+y)) \geq 6 \sum (y+z) \sqrt{yz(z+x)(x+y)}, \qquad \text{or, equivalently,}$$
$$\sum (y+z)(y(z+x) + z(x+y)) \geq 2 \sum (y+z) \sqrt{yz(z+x)(x+y)}.$$

But this is obvious, since AM-GM yields $y(z+x)+z(x+y) \geq 2\sqrt{y(z+x) \cdot z(x+y)} = 2\sqrt{yz(z+x)(x+y)}$. Thus, the proof of inequality (1) is complete, and the problem is solved.