Abstract

The symmedian point of a triangle is known to give rise to two circles, obtained by drawing respectively parallels and antiparallels to the sides of the triangle through the symmedian point. In this note we will explore a third circle with a similar construction - discovered by Jean-Pierre Ehrmann [3]. It is obtained by drawing circles through the symmedian point and two vertices of the triangle, and intersecting these circles with the triangle's sides. We prove the existence of this circle and identify its center and radius.

Ehrmann's third Lemoine circle

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§1. The first two Lemoine circles

Let us remind the reader about some classical triangle geometry first. Let L be the symmedian point of a triangle ABC. Then, the following two results[1] are well-known ([2], Chapter 9):

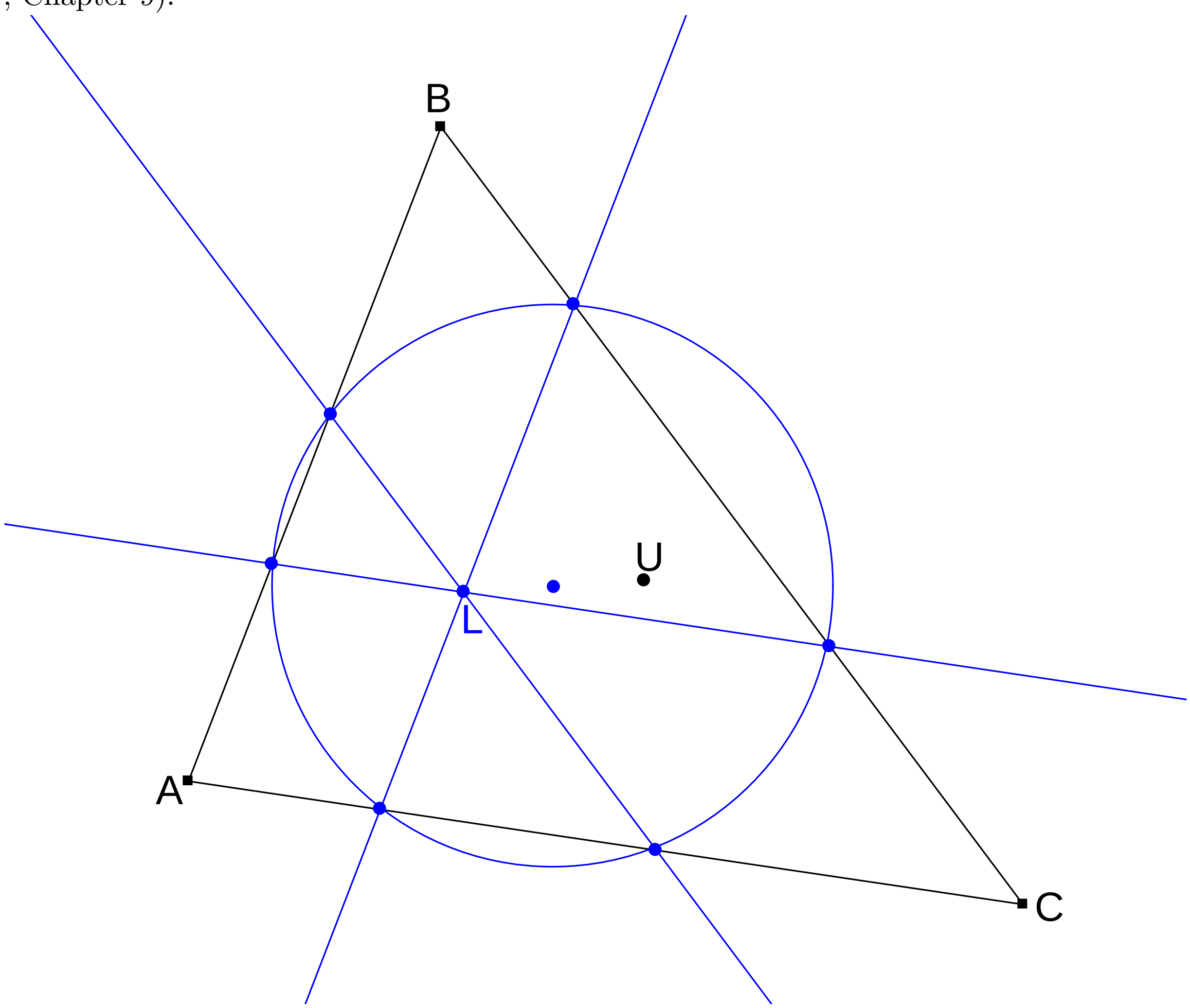

Fig. 1

Theorem 1 *Let the parallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA at six points. These six points lie on one circle,*

[1] For the definition of a *Tucker circle*, see below.

the so-called ***first Lemoine circle*** *of triangle ABC; this circle is a Tucker circle, and its center is the midpoint of the segment UL, where U is the circumcenter of triangle ABC. (See Fig. 1.)*
[The somewhat uncommon formulation "Let the parallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA at six points" means the following: Take the point where the parallel to BC through L meets CA, the point where the parallel to BC through L meets AB, the point where the parallel to CA through L meets AB, the point where the parallel to CA through L meets BC, the point where the parallel to AB through L meets BC, and the point where the parallel to AB through L meets CA.]

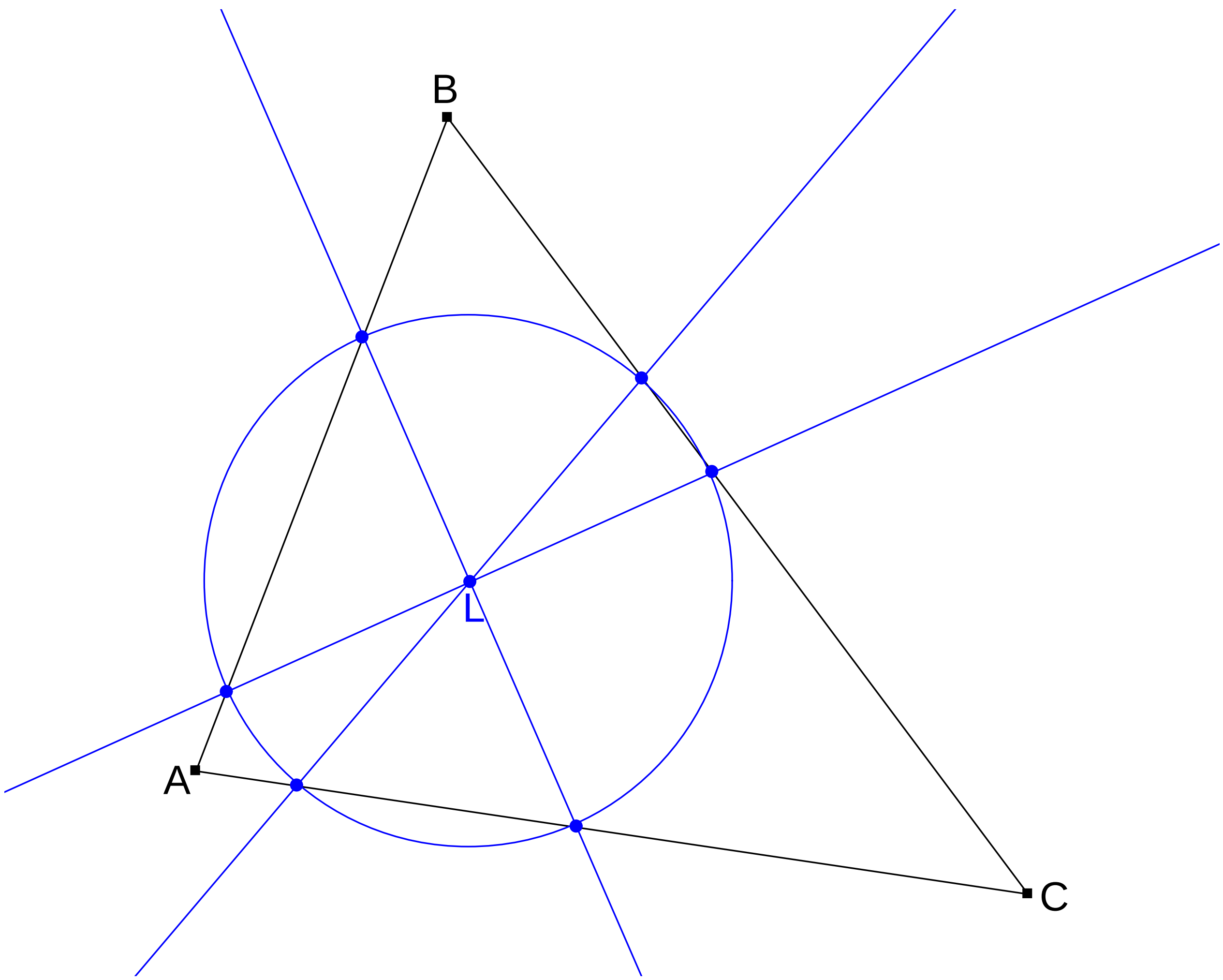

Fig. 2
Furthermore (see [2] for this as well):

Theorem 2 *Let the antiparallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA at six points. These six points lie on one circle, the so-called* ***second Lemoine circle*** *(also known as the* ***cosine circle****) of triangle ABC; this circle is a Tucker circle, and its center is L. (See Fig. 2.)*

We have been using the notion of a *Tucker circle* here. This can be defined as follows:

Theorem 3 *Let ABC be a triangle. Let Q_a and R_a be two points on the line BC. Let R_b and P_b be two points on the line CA. Let P_c and Q_c be two points on the line AB. Assume that the following six conditions hold: The lines Q_bR_c, R_cP_a, P_bQ_a are parallel to the lines BC, CA, AB, respectively; the lines P_bP_c, Q_cQ_a, R_aR_b are antiparallel to the sidelines BC, CA, AB of triangle ABC, respectively. (Actually, requiring five of these conditions is enough, since any five of them imply the sixth one, as one can show.) Then, the points Q_a, R_a, R_b, P_b, P_c and Q_c lie on one circle. Such circles are called **Tucker circles** of triangle ABC. The center of each such circle lies on the line UL, where U is the circumcenter and L the symmedian point of triangle ABC. Notable Tucker circles are the circumcircle of triangle ABC, its first and second Lemoine circles (and the third one we will define below), and its Taylor circle.*

§2. The third Lemoine circle

Far less known than these two results is the existence of a third member can be added to this family of Tucker circles related to the symmedian point L. As far as I know, it has been first discovered by Jean-Pierre Ehrmann in 2002 [3]:

Theorem 4 *Let the circumcircle of triangle BLC meet the lines CA and AB at the points A_b and A_c (apart from C and B). Let the circumcircle of triangle CLA meet the lines AB and BC at the point B_c and B_a (apart from A and C). Let the circumcircle of triangle ALB meet the lines BC and CA at the points C_a and C_b (apart from B and A). Then, the six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle. This circle is a Tucker circle, and its midpoint M lies on the line UL and satisfies $LM = -\frac{1}{2} \cdot LU$ (where the segments are directed). The radius of this circle is $\frac{1}{2}\sqrt{9r_1^2 + r^2}$, where r is the circumradius and r_1 is the radius of the second Lemoine circle of triangle ABC.*

We propose to denote the circle through the points A_b, A_c, B_c, B_a, C_a, C_b as the **third Lemoine circle** of triangle ABC. (See Fig. 3.)

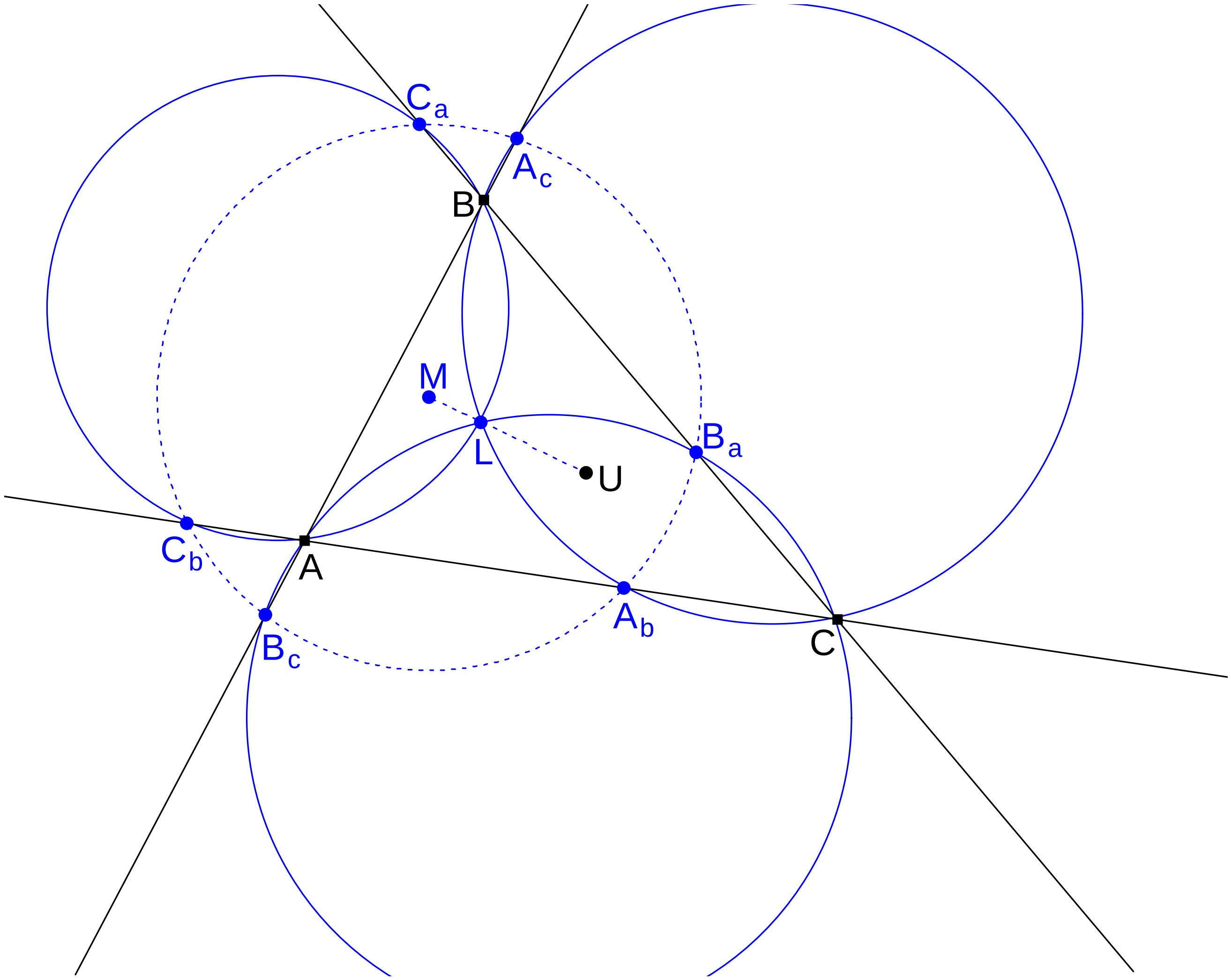

Fig. 3

The rest of this note will be about proving this theorem. First we will give a complete proof of Theorem 4 in §3-§5; this proof will use four auxiliary facts (Theorems 5, 6, 7 and 8). Then, in §6 and §7, we will give a new argument to show the part of Theorem 4 which claims that the six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle; this argument will not give us any information about the center of this circle (so that it doesn't extend to a complete second proof of Theorem 4, apparently), but it has the advantage of showing a converse to Theorem 4 (which we formulate as Theorem 10 in the final paragraph §8).

§3. A lemma

In triangle geometry, most nontrivial proofs begin by deducing further (and easier) properties of the configuration. These properties are then used as lemmas (and even if they don't turn out directly useful, they are often interesting for themselves). In the case of Theorem 4, the following result plays the role of such a lemma:

Theorem 5 *The point L is the centroid of each of the three triangles AA_bA_c, B_aBB_c, C_aC_bC. (See Fig. 4.)*

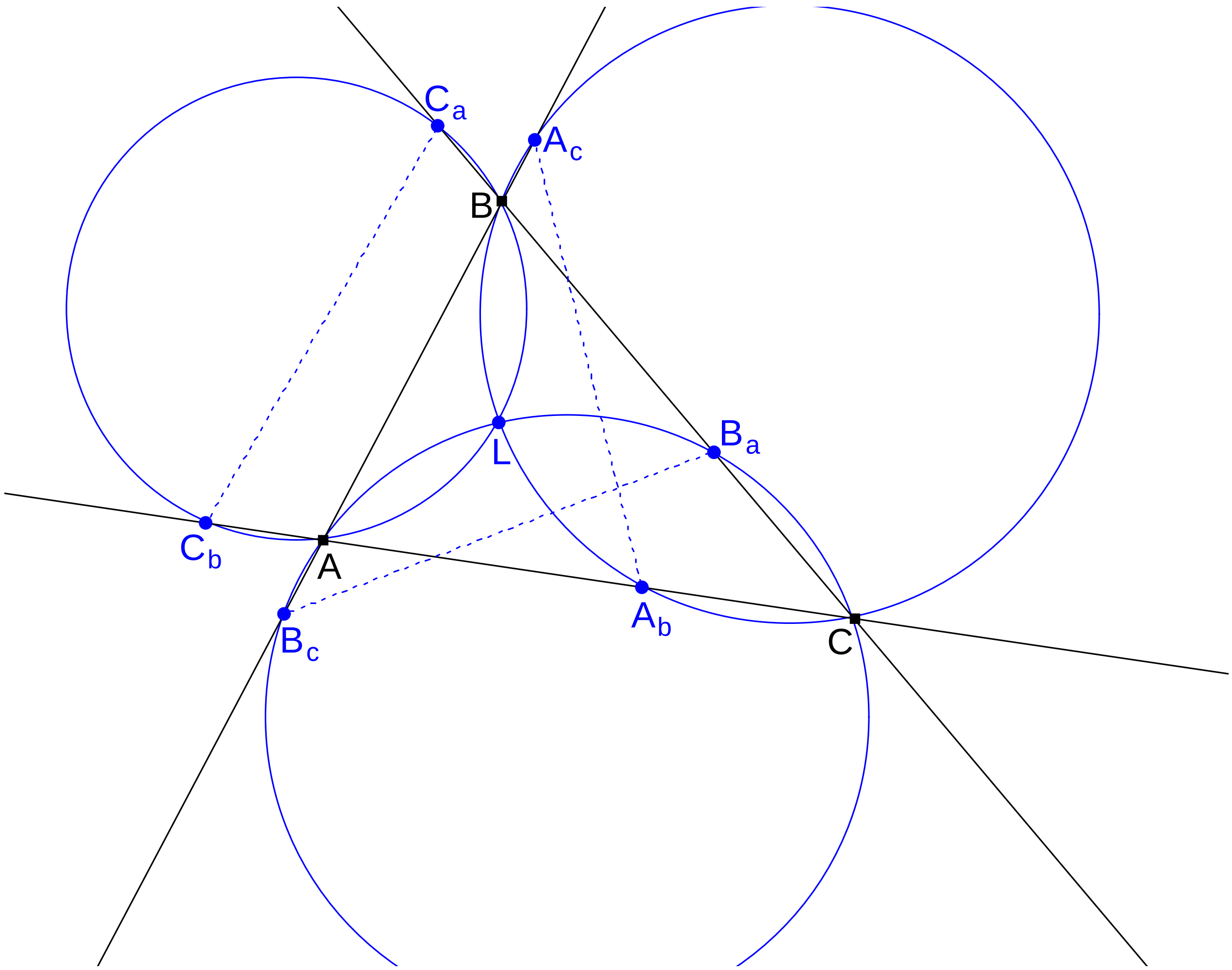

Fig. 4

Actually this result isn't as much about symmedian points and centroids, as it generalizes to arbitrary isogonal conjugates:

Theorem 6 *Let P and Q be two points isogonally conjugate to each other with respect to triangle ABC. Let the circumcircle of triangle CPA meet the lines AB and BC at the points B_c and B_a (apart from A and C). Then, the triangles B_aBB_c and ABC are oppositely similar, and the points P and Q are corresponding points in the triangles B_aBB_c and ABC. (See Fig. 5.)*

Remark. Two points P_1 and P_2 are said to be *corresponding points* in two similar triangles Δ_1 and Δ_2 if the similitude transformation that maps triangle Δ_1 to triangle Δ_2 maps the point P_1 to the point P_2.

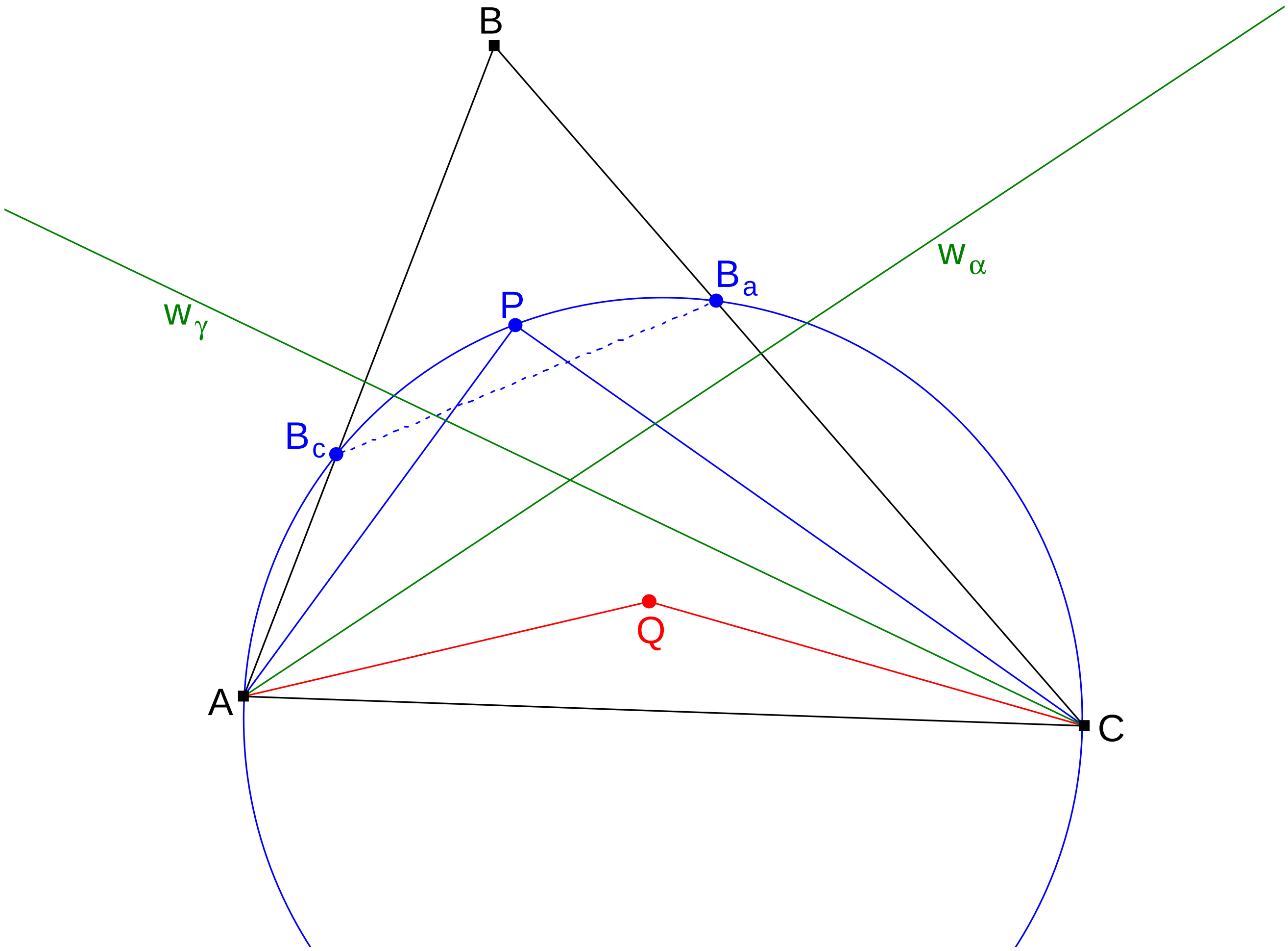

Fig. 5

Proof of Theorem 6. We will use directed angles modulo 180°. A very readable introduction into this kind of angles can be found in [1]. A list of their important properties has also been given in [4].

The point Q is the isogonal conjugate of the point P with respect to triangle ABC; thus, $\measuredangle PAB = \measuredangle CAQ$.

Since C, A, B_c, B_a are concyclic points, we have $\measuredangle CB_aB_c = \measuredangle CAB_c$, so that $\measuredangle BB_aB_c = -\measuredangle BAC$. Furthermore, $\measuredangle B_cBB_a = -\measuredangle CBA$. Thus, the triangles B_aBB_c and ABC are oppositely similar (having two pairs of oppositely equal angles).

By the chordal angle theorem, $\measuredangle PB_aB_c = \measuredangle PAB_c = \measuredangle PAB = \measuredangle CAQ = -\measuredangle QAC$. Similarly, $\measuredangle PB_cP_a = -\measuredangle QCA$. These two equations show that the triangles B_aPB_c and AQC are oppositely similar. Combining this with the opposite similarity of triangles B_aBB_c and ABC, we obtain that the quadrilaterals B_aBB_cP and $ABCQ$ are oppositely similar. Hence, P and Q are corresponding points in the triangles B_aBB_c and ABC. (See Fig. 6.) Theorem 6 is thus proven.

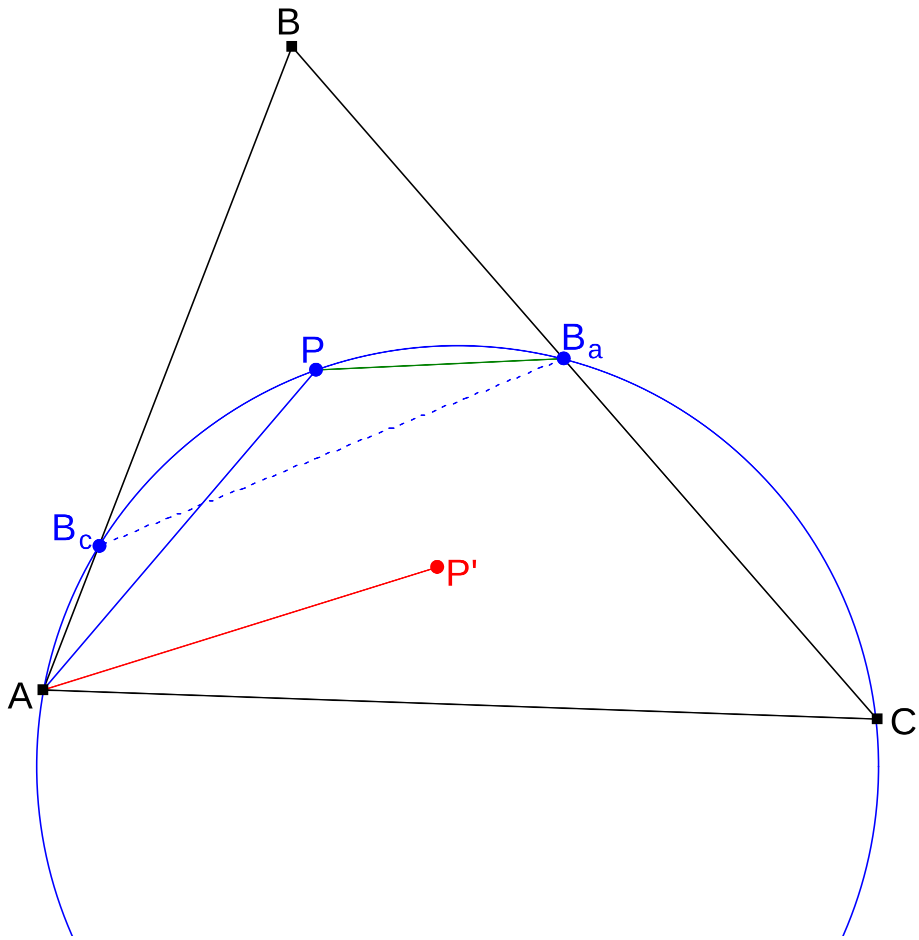

Fig. 6 ■

Proof of Theorem 5. Now return to the configuration of Theorem 4. To prove Theorem 5, we apply Theorem 6 to the case when P is the symmedian point of triangle ABC; the isogonal conjugate Q of P is, in this case, the centroid of triangle ABC. Now, Theorem 6 says that the points P and Q are corresponding points in the triangles B_aBB_c and ABC. Since Q is the centroid of triangle ABC, this means that P is the centroid of triangle B_aBB_c. But $P = L$; thus, we have shown that L is the centroid of triangle B_aBB_c. Similarly, L is the centroid of triangles AA_bA_c and C_aC_bC, and Theorem 5 follows. ■

§4. Antiparallels

Theorem 5 was the first piece of our jigsaw. Next we are going to chase some angles.

Since the points B, C, A_b, A_c are concyclic, we have $\measuredangle CA_bA_c = \measuredangle CBA_c$, so that $\measuredangle AA_bA_c = -\measuredangle ABC$. Thus, the line A_bA_c is antiparallel to BC in triangle ABC. Similarly, the lines B_cB_a and C_aC_b are antiparallel to CA and AB. We have thus shown:

Theorem 7 *In the configuration of Theorem 4, the lines A_bA_c, B_cB_a, C_aC_b are antiparallel to BC, CA, AB in triangle ABC.*

Now let X_b, X_c, Y_c, Y_a, Z_a, Z_b be the points where the antiparallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA. According to Theorem 2, these points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on one circle around

L; however, to keep this note self-contained, we do not want to depend on Theorem 2 here, but rather prove the necessary facts on our own (Fig. 7):

Partial proof of Theorem 2.

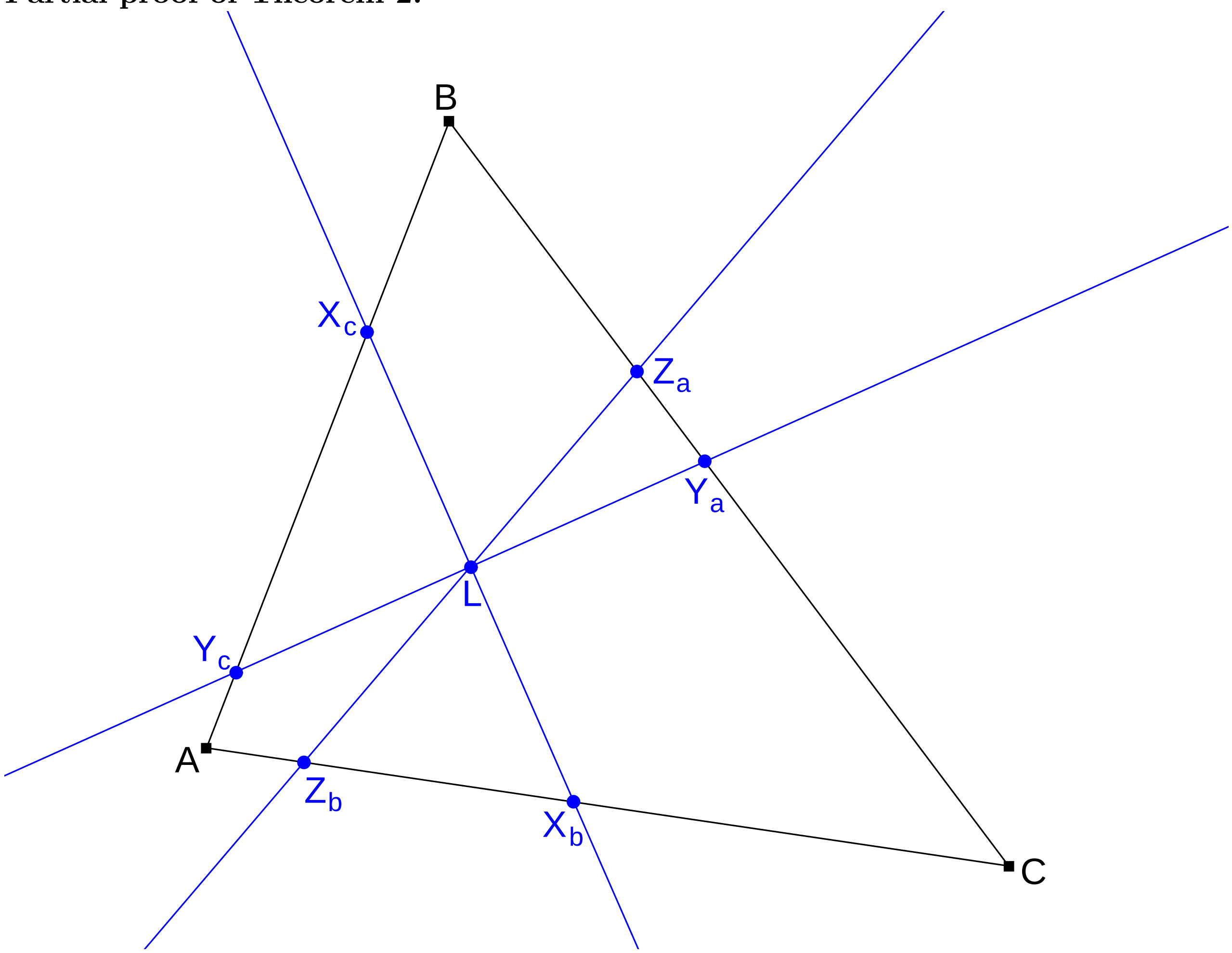

Fig. 7

Since symmedians bisect antiparallels, and since the symmedian point L of triangle ABC lies on all three symmedians, the point L must bisect the three antiparallels X_bX_c, Y_cY_a, Z_aZ_b. This means that $LX_b = LX_c$, $LY_c = LY_a$ and $LZ_a = LZ_b$. Furthermore, $\measuredangle AX_cX_b = -\measuredangle ACB$ (since X_bX_c is antiparallel to BC), thus $\measuredangle Y_cX_cL = -\measuredangle ACB$; similarly, $\measuredangle X_cY_cL = -\measuredangle BCA$, thus $\measuredangle LY_cX_c = -\measuredangle X_cY_cL = \measuredangle BCA = -\measuredangle ACB = \measuredangle Y_cX_cL$. Hence, triangle X_cLY_c is isosceles, so that $LX_c = LY_c$. Similarly, $LZ_b = LX_b$ and $LY_a = LZ_a$. Hence,

$$LX_b = LX_c = LY_c = LY_a = LZ_a = LZ_b.$$

This shows that the points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on one circle around L. This circle is the so-called second Lemoine circle of triangle ABC. Its radius is $r_1 = LX_b = LX_c = LY_c = LY_a = LZ_a = LZ_b$.

We thus have incidentally proven most of Theorem 2 (to complete the proof of Theorem 2, we would only have to show that this circle is a Tucker circle, which is easy); but we have also made headway to the proof of Theorem 4. ■

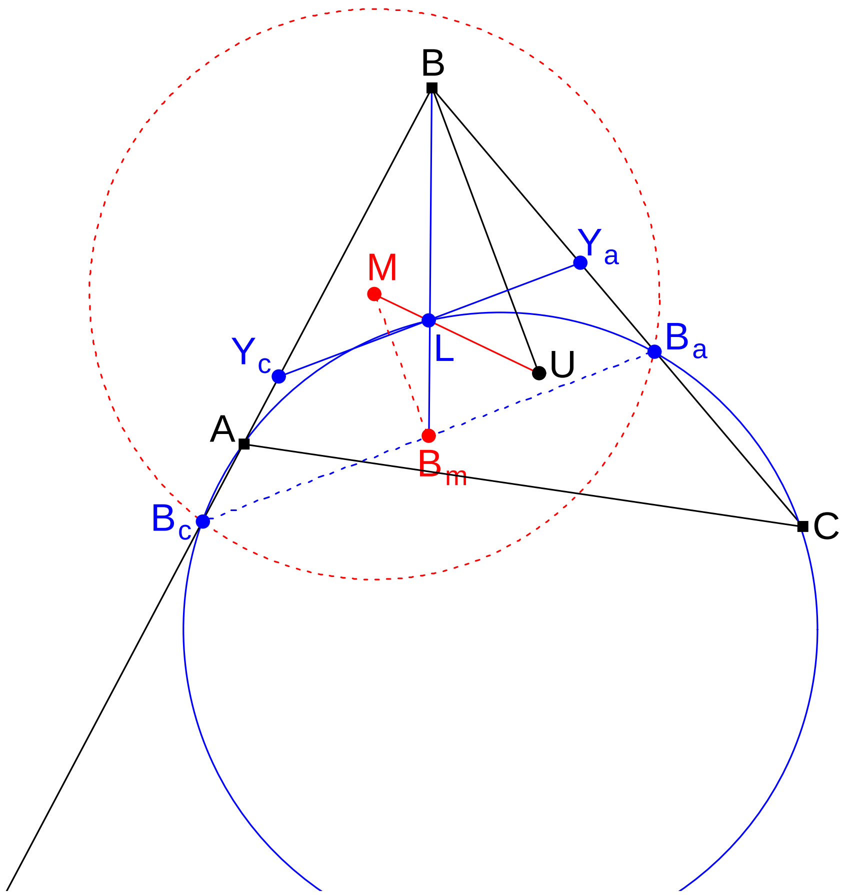

Fig. 8

Proof of Theorem 9, part 1. Now consider Fig. 8. Let B_m be the midpoint of the segment B_cB_a. According to Theorem 5, the point L is the centroid of triangle B_aBB_c; thus, it lies on the median BB_m and divides it in the ratio $2:1$. Hence, $BL : LB_m = 2$ (with directed segments).

Let M be the point on the line UL such that $LM = -\frac{1}{2} \cdot LU$ (with directed segments); then, $LU = -2 \cdot LM$, so that $UL = -LU = 2 \cdot LM$, and thus $UL : LM = 2 = BL : LB_m$. By the converse of Thales' theorem, this yields $B_mM \parallel BU$.

The lines Y_cY_a and B_cB_a are both antiparallel to CA, and thus parallel to each other. Hence, Thales' theorem yields $B_mB_c : LY_c = BB_m : BL$. But since $BL : LB_m = 2$, we have $BL = 2 \cdot LB_m$, so that $LB_m = \frac{1}{2} \cdot BL$ and therefore $BB_m = BL + LB_m = BL + \frac{1}{2} \cdot BL = \frac{3}{2} \cdot BL$ and $BB_m : BL = \frac{3}{2}$. Consequently, $B_mB_c : LY_c = \frac{3}{2}$ and $B_mB_c = \frac{3}{2} \cdot LY_c = \frac{3}{2}r_1$.

It is a known fact that every line antiparallel to the side CA of triangle ABC is perpendicular to the line BU (where, as we remind, U is the circumcenter of triangle ABC). Thus, the line B_cB_a (being antiparallel to CA) is perpendicular to the line BU. Since $B_mM \parallel BU$, this yields $B_cB_a \perp B_mM$. Furthermore, $B_mM \parallel BU$ yields $BU : B_mM = BL : LB_m$ (by Thales), so that $BU : B_mM = 2$ and $BU = 2 \cdot B_mM$, and thus $B_mM = \frac{1}{2} \cdot BU = \frac{1}{2}r$, where r is the circumradius of triangle ABC.

Now, Pythagoras' theorem in the right-angled triangle MB_mB_c yields

$$MB_c = \sqrt{B_mB_c{}^2 + B_mM^2} = \sqrt{\left(\frac{3}{2}r_1\right)^2 + \left(\frac{1}{2}r\right)^2} = \sqrt{\frac{9}{4}r_1^2 + \frac{1}{4}r^2} = \frac{1}{2}\sqrt{9r_1^2 + r^2}.$$

Similarly, we obtain the same value $\frac{1}{2}\sqrt{9r_1^2 + r^2}$ for each of the lengths MB_a, MC_a,

MC_b, MA_b and MA_c. Hence, the points A_b, A_c, B_c, B_a, C_a, C_b all lie on the circle with center M and radius $\frac{1}{2}\sqrt{9r_1^2 + r^2}$. The point M, in turn, lies on the line UL and satisfies $LM = -\frac{1}{2} \cdot LU$.

This already proves most of Theorem 4. The only part that has yet to be shown is that this circle is a Tucker circle. We will do this next. ■

§5. Parallels

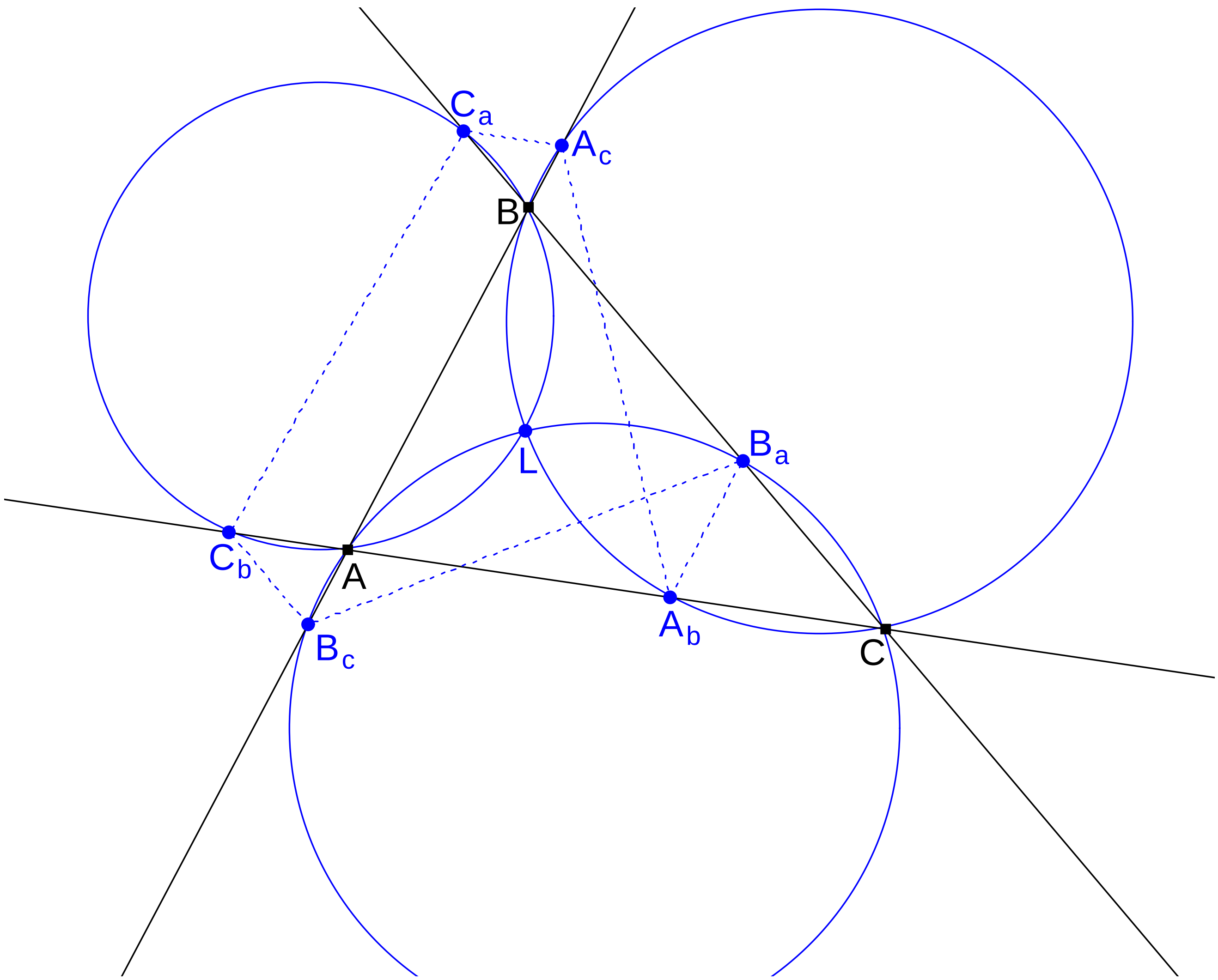

Fig. 9

Since the points A_b, B_a, C_a, C_b are concyclic, we have $\measuredangle C_aB_aA_b = \measuredangle C_aC_bA_b$, thus $\measuredangle CB_aA_b = \measuredangle C_aC_bC$. But since C_aC_b is antiparallel to AB, we have $\measuredangle CC_bC_a = -\measuredangle CBA$, so that $\measuredangle C_aC_bC = -\measuredangle CC_bC_a = \measuredangle CBA$, thus $\measuredangle CB_aA_b = \measuredangle CBA$; consequently, $A_bB_a \parallel AB$. Similarly, $B_cC_b \parallel BC$ and $C_aA_c \parallel CA$. Altogether, we have thus seen:

Theorem 8 *The lines B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB. (See Fig. 9.)*

Proof of Theorem 4, part 2. If we now combine Theorem 7 and Theorem 8, we conclude that the sides of the hexagon $A_bA_cC_aC_bB_cB_a$ are alternately antiparallel and parallel to the sides of triangle ABC. Thus, $A_bA_cC_aC_bB_cB_a$ is a Tucker hexagon, and the circle passing through its vertices A_b, A_c, B_c, B_a, C_a, C_b is a Tucker circle. This concludes the proof of Theorem 4. ■

One remark: It is known that the radius r_1 of the second Lemoine circle ΔABC is $r \tan \omega$, where ω is the Brocard angle of triangle ABC. [2] Thus, the radius of the third Lemoine circle is

$$\frac{1}{2}\sqrt{9r_1^2 + r^2} = \frac{1}{2}\sqrt{9\left(r\tan\omega\right)^2 + r^2} = \frac{1}{2}\sqrt{9r^2\tan^2\omega + r^2} = \frac{r}{2}\sqrt{9\tan^2\omega + 1}.$$

§6. A different approach: a lemma about four points

At this point, we are done with our job: Theorem 4 is proven. However, our proof depended on the construction of a number of auxiliary points (not only B_m, but also the six points X_b, X_c, Y_c, Y_a, Z_a, Z_b). One might wonder whether there isn't also a (possibly more complicated, but) more straightforward approach to proving the concyclicity of the points A_b, A_c, B_c, B_a, C_a, C_b without auxiliary constructions. We will show such an approach now. It will not yield the complete Theorem 4, but on the upside, it helps proving a kind of converse.

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[2] For a quick proof of this fact, let X, Y, Z denote the feet of the perpendiculars from the point L to the sides BC, CA, AB of triangle ABC. Then, the radius r_1 of the second Lemoine circle is $r_1 = LY_a$. Working *without* directed angles now, we see that $LY_a = \dfrac{LX}{\sin A}$ (from the right-angled triangle LXY_a), so that $r_1 = LY_a = \dfrac{LX}{\sin A}$. Since $\sin A = \dfrac{a}{2r}$ by the extended law of sines, this rewrites as $r_1 = \dfrac{LX}{\left(\dfrac{a}{2r}\right)} = \dfrac{LX \cdot 2r}{a}$. This rewrites as $\dfrac{a^2}{2r}r_1 = a \cdot LX$. Similarly, $\dfrac{b^2}{2r}r_1 = b \cdot LY$ and $\dfrac{c^2}{2r}r_1 = c \cdot LZ$. Adding these three equations together, we obtain

$$\frac{a^2}{2r}r_1 + \frac{b^2}{2r}r_1 + \frac{c^2}{2r}r_1 = a \cdot LX + b \cdot LY + c \cdot LZ.$$

On the other hand, let S denote the area of triangle ABC. But the area of triangle BLC is $\frac{1}{2}a \cdot LX$ (since a is a sidelength of triangle BLC, and LX is the corresponding altitude), and similarly the areas of triangles CLA and ALB are $\frac{1}{2}b \cdot LY$ and $\frac{1}{2}c \cdot LZ$. Thus,

$$\begin{aligned}\frac{1}{2}a \cdot LX + \frac{1}{2}b \cdot LY + \frac{1}{2}c \cdot LZ &= (\text{area of triangle } BLC) + (\text{area of triangle } CLA) + (\text{area of triangle } ALB) \\ &= (\text{area of triangle } ABC) = S,\end{aligned}$$

so that $a \cdot LX + b \cdot LY + c \cdot LZ = 2S$. The equation $\dfrac{a^2}{2r}r_1 + \dfrac{b^2}{2r}r_1 + \dfrac{c^2}{2r}r_1 = a \cdot LX + b \cdot LY + c \cdot LZ$ thus becomes $\dfrac{a^2}{2r}r_1 + \dfrac{b^2}{2r}r_1 + \dfrac{c^2}{2r}r_1 = 2S$, so that

$$r_1 = \frac{2S}{\dfrac{a^2}{2r}r_1 + \dfrac{b^2}{2r}r_1 + \dfrac{c^2}{2r}r_1} = \frac{4rS}{a^2 + b^2 + c^2} = r \Big/ \underbrace{\frac{a^2 + b^2 + c^2}{4S}}_{= \cot \omega} = r \Big/ \cot \omega = r \tan \omega,$$

qed.

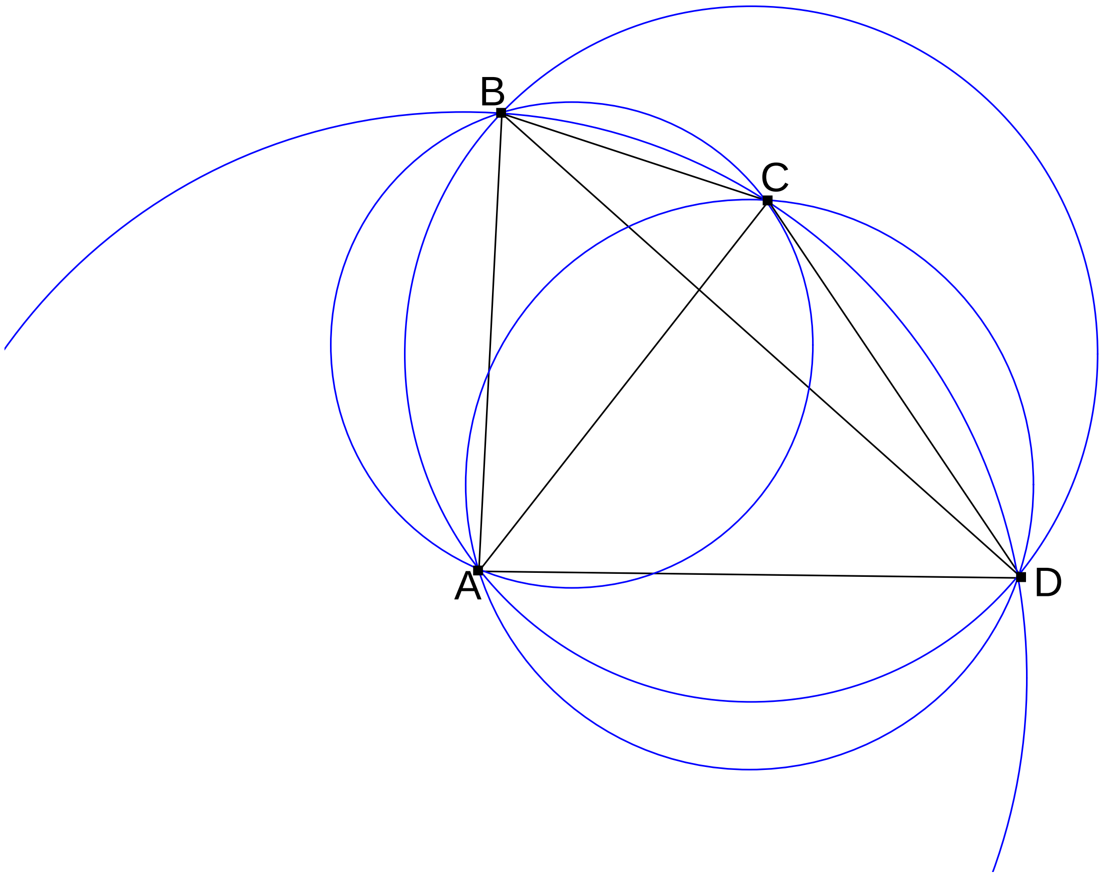

Fig. 10

Let us use directed areas and powers of points with respect to circles. Our main vehicle is the following fact:

Theorem 9 *Let A, B, C, D be four points. Let p_A, p_B, p_C, p_D denote the powers of the points A, B, C, D with respect to the circumcircles of triangles BCD, CDA, DAB, ABC. Furthermore, we denote by $[P_1P_2P_3]$ the directed area of any triangle $P_1P_2P_3$. Then,*

$$-p_A \cdot [BCD] = p_B \cdot [CDA] = -p_C \cdot [DAB] = p_D \cdot [ABC] \,. \tag{1}$$

(See Fig. 10.)

Proof of Theorem 9.. In the following, all angles are directed angles modulo 180°, and all segments and areas are directed. Let the circumcircle of triangle BCD meet the line AC at a point A' (apart from C). Let the circumcircle of triangle CDA meet the line BD at a point B' (apart from D). Let the lines AC and BD intersect at P. Since the points C, D, A, B' are concyclic, we have $\measuredangle DB'A = \measuredangle DCA$, thus $\measuredangle PB'A = \measuredangle DCA$. But since the points C, D, B, A' are concyclic, we have $\measuredangle DBA' = \measuredangle DCA'$, so that $\measuredangle PBA' = \measuredangle DCA$. Therefore, $\measuredangle PB'A = \measuredangle PBA'$, which leads to $B'A \parallel BA'$. Hence, by the Thales theorem,

$$\frac{A'A}{BB'} = \frac{PA'}{PB}.$$

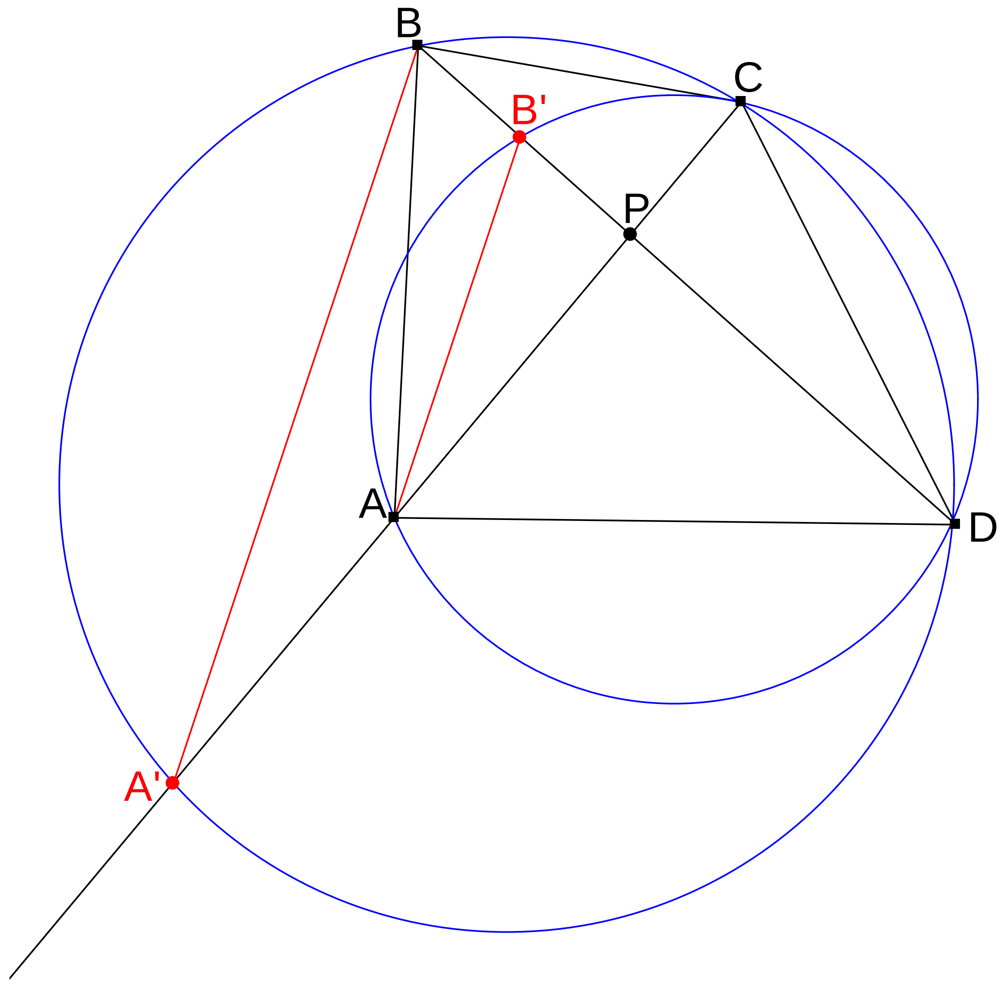

Fig. 11

The power p_A of the point A with respect to the circumcircle of triangle BCD is $AA' \cdot AC$; similarly, $p_B = BB' \cdot BD$. Thus,

$$\frac{-p_A \cdot [BCD]}{p_B \cdot [CDA]} = \frac{-AA' \cdot AC \cdot [BCD]}{BB' \cdot BD \cdot [CDA]} = \frac{A'A \cdot AC \cdot [BCD]}{BB' \cdot BD \cdot [CDA]} = \frac{A'A}{BB'} \cdot \frac{AC}{BD} \cdot \frac{[BCD]}{[CDA]}$$
$$= \frac{PA'}{PB} \cdot \frac{AC}{BD} \cdot \frac{[BCD]}{[CDA]}.$$

But it is a known fact that whenever $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are three collinear points and $\mathcal{P}$ is a point not collinear with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, then we have $\dfrac{\mathcal{AB}}{\mathcal{AC}} = \dfrac{[\mathcal{APB}]}{[\mathcal{APC}]}$. This fact yields

$$\frac{CP}{CA} = \frac{[CDP]}{[CDA]} \qquad \text{and} \qquad \frac{DP}{DB} = \frac{[DCP]}{[DCB]},$$

so that

$$\frac{CP}{CA} : \frac{DP}{DB} = \frac{[CDP]}{[CDA]} : \frac{[DCP]}{[DCB]} = \frac{[DCB] \cdot [CDP]}{[DCP] \cdot [CDA]} = \frac{-[BCD] \cdot [CDP]}{-[CDP] \cdot [CDA]} = \frac{[BCD]}{[CDA]},$$

and thus

$$\frac{-p_A\cdot[BCD]}{p_B\cdot[CDA]}=\frac{PA'}{PB}\cdot\frac{AC}{BD}\cdot\frac{[BCD]}{[CDA]}=\frac{PA'}{PB}\cdot\frac{AC}{BD}\cdot\left(\frac{CP}{CA}:\frac{DP}{DB}\right)$$
$$=\frac{PA'}{PB}\cdot\frac{AC}{BD}\cdot\left(\frac{PC}{AC}:\frac{PD}{BD}\right)=\frac{PA'}{PB}\cdot\frac{AC}{BD}\cdot\frac{PC}{AC}\cdot\frac{BD}{PD}=\frac{PA'\cdot PC}{PB\cdot PD}.$$

But the intersecting chord theorem yields $PA'\cdot PC=PB\cdot PD$, so that

$$\frac{-p_A\cdot[BCD]}{p_B\cdot[CDA]}=1;$$

in other words, $-p_A\cdot[BCD]=p_B\cdot[CDA]$. Similarly, $-p_B\cdot[CDA]=p_C\cdot[DAB]$, so that $p_B\cdot[CDA]=-p_C\cdot[DAB]$, and $-p_C\cdot[DAB]=p_D\cdot[ABC]$. Combining these equalities, we get (1). This proves Theorem 9. ■

§7. Application to the triangle

Now the promised alternative proof of the fact that the points A_b, A_c, B_c, B_a, C_a, C_b are concyclic:

Proof. The identity (1) can be rewritten as

$$p_A\cdot[BDC]=p_B\cdot[CDA]=p_C\cdot[ADB]=p_D\cdot[ABC]$$

(since $[BCD]=-[BDC]$ and $[DAB]=-[ADB]$). Applying this equation to the case when D is the symmedian point L of triangle ABC, we get

$$p_A\cdot[BLC]=p_B\cdot[CLA]=p_C\cdot[ALB]$$

(we have dropped the third equality sign since we don't need it), where p_A, p_B, p_C are the powers of the points A, B, C with respect to the circumcircles of triangles BLC, CLA, ALB. But we have $p_A=AC\cdot AA_b$ and $p_B=BC\cdot BB_a$ (Fig. 4); thus, $p_A\cdot[BLC]=p_B\cdot[CLA]$ becomes

$$AC\cdot AA_b\cdot[BLC]=BC\cdot BB_a\cdot[CLA],$$

so that

$$\frac{AA_b}{BB_a}=\frac{BC}{AC}\cdot\frac{[CLA]}{[BLC]}. \tag{2}$$

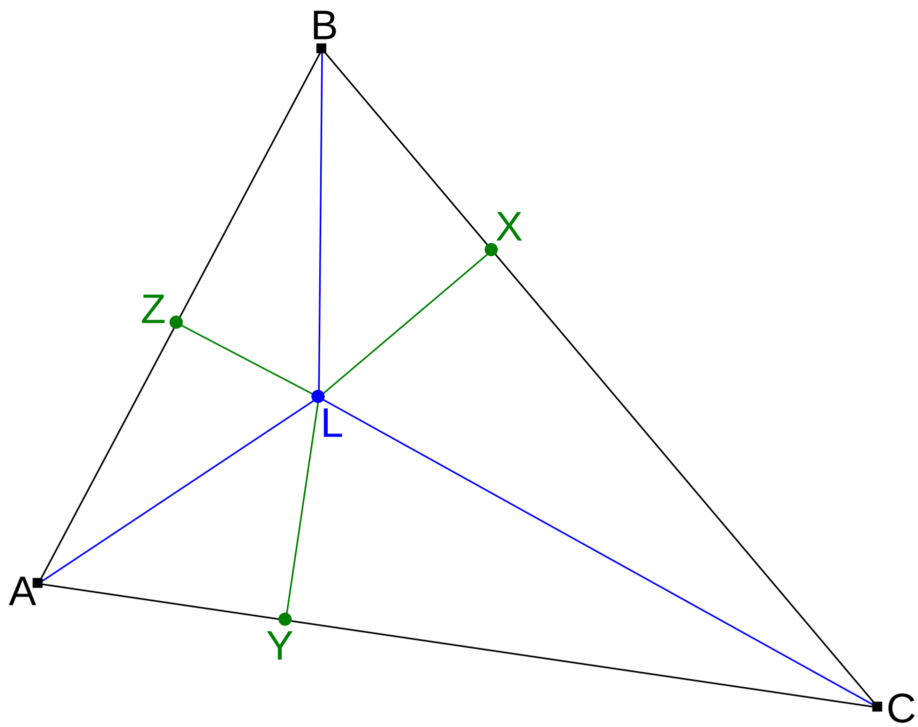

Fig. 12

Now let X, Y and Z be the feet of the perpendiculars from the symmedian point L onto the lines BC, CA, AB. It is a known fact that the distances from the symmedian point of a triangle to its sides are proportional to these sides; thus, $LX : LY : LZ = BC : CA : AB$. [3]. But since the area of a triangle equals $\frac{1}{2}\cdot$ (one of its sidelengths) $\cdot$ (corresponding altitude), we have $[BLC] = \frac{1}{2}\cdot BC \cdot LX$, $[CLA] = \frac{1}{2}\cdot CA\cdot LY$ and $[ALB] = \frac{1}{2}\cdot AB\cdot LZ$. Thus, (2) becomes

$$\frac{AA_b}{BB_a} = \frac{BC}{AC}\cdot\frac{\frac{1}{2}\cdot CA\cdot LY}{\frac{1}{2}\cdot BC\cdot LX} = \frac{BC}{AC}\cdot\frac{CA}{BC}\cdot\frac{LY}{LX} = \frac{BC}{AC}\cdot\frac{CA}{BC}\cdot\frac{CA}{BC}$$
$$= \frac{BC}{AC}\cdot\frac{CA^2}{BC^2} = \frac{BC}{AC}\cdot\frac{AC^2}{BC^2} = \frac{AC}{BC}.$$

By the converse of Thales' theorem, this yields $A_bB_a \parallel AB$. Similarly, $B_cC_b \parallel BC$ and $C_aA_c \parallel CA$; this proves Theorem 8. The proof of Theorem 7 can be done in the same way as §4 (it was too trivial to have any reasonable alternative). Combined, this yields that the sides of the hexagon $A_bA_cC_aC_bB_cB_aA_b$ are alternately antiparallel and parallel to the sides of triangle ABC. Consequently, this hexagon is a Tucker hexagon, and since every Tucker hexagon is known to be cyclic, we thus conclude that the points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle. This way we have reproven a part of Theorem 4. ■

Here is an alternative way to show that the points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle, without using the theory of Tucker hexagons:

Proof. (See Fig. 9.) Since C_aC_b is antiparallel to AB, we have $\measuredangle CC_bC_a =$

[3]Actually, there is no need to refer to this known fact here, because we have almost completely proven it above. Indeed, while proving that $r_1 = r\tan\omega$, we showed that $r_1 = \frac{LX\cdot 2r}{a}$, so that $LX = \frac{r_1}{2r}a = \frac{r_1}{2r}BC$. Similarly, $LY = \frac{r_1}{2r}CA$ and $LZ = \frac{r_1}{2r}AB$, so that $LX : LY : LZ = BC : CA : AB$.

$-\measuredangle CBA$, and thus

$$\begin{aligned}\measuredangle C_bC_aA_c &= \measuredangle\left(C_aC_b;\ C_aA_c\right) = \measuredangle\left(C_aC_b;\ CA\right) \qquad \text{(since } C_aA_c \parallel CA\text{)} \\ &= \measuredangle C_aC_bC = -\measuredangle CC_bC_a = \measuredangle CBA.\end{aligned}$$

Since A_bA_c is antiparallel to BC, we have $\measuredangle AA_bA_c = -\measuredangle ABC$, so that

$$\measuredangle C_bA_bA_c = \measuredangle AA_bA_c = -\measuredangle ABC = \measuredangle CBA.$$

Therefore, $\measuredangle C_bA_bA_c = \measuredangle C_bC_aA_c$, so that the points C_a, C_b, A_b, A_c lie on one circle. The point B_a also lies on this circle, since

$$\begin{aligned}\measuredangle C_aB_aA_b &= \measuredangle\left(BC;\ A_bB_a\right) = \measuredangle\left(BC;\ AB\right) \qquad \text{(since } A_bB_a \parallel AB\text{)} \\ &= \measuredangle CBA = -\measuredangle CC_bC_a \qquad \text{(since } \measuredangle CC_bC_a = -\measuredangle CBA \text{ was shown above)} \\ &= \measuredangle C_aC_bA_b.\end{aligned}$$

Similarly, the point B_c lies on this circle as well. This shows that all six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle. ■

This argument did never use anything but the facts that the lines A_bA_c, B_cB_a, C_aC_b are antiparallel to BC, CA, AB and that the lines B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB. It can therefore be used as a general argument why Tucker hexagons are cyclic.

§8. A converse

The alternative proof in §7 allows us to show a converse of Theorem 4:

Theorem 10 *Let P be a point in the plane of a triangle ABC but not on its circumcircle. Let the circumcircle of triangle BPC meet the lines CA and AB at the points A_b and A_c (apart from C and B). Let the circumcircle of triangle CPA meet the lines AB and BC at the point B_c and B_a (apart from A and C). Let the circumcircle of triangle APB meet the lines BC and CA at the points C_a and C_b (apart from B and A). If the six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle, then P is the symmedian point of triangle ABC.*

We will not give a complete proof of this theorem here, but we only sketch its path: First, it is easy to see that the lines A_bA_c, B_cB_a, C_aC_b are antiparallel to BC, CA, AB. Now, we can reverse the argument from §7 to show that the lines B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB, and use this to conclude that the distances from P to the sidelines BC, CA, AB of triangle ABC are proportional to the lengths of BC, CA, AB. But this implies that P is the symmedian point of triangle ABC.

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