

# On cyclic quadrilaterals and the butterfly theorem

*Darij Grinberg*

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## 1. The butterfly theorem(s)

Being one of the most well-known geometric facts beyond the usual school curriculum, the butterfly theorem has received attention of various authors. In [1], 14 proofs of this theorem and a number of generalizations are presented. In this note, we are going to show two new approaches to the butterfly theorem and incidentally prove an important fact on cyclic quadrilaterals.

The butterfly theorem is known in two versions, a "strong" and a "weak" one:<sup>1</sup>

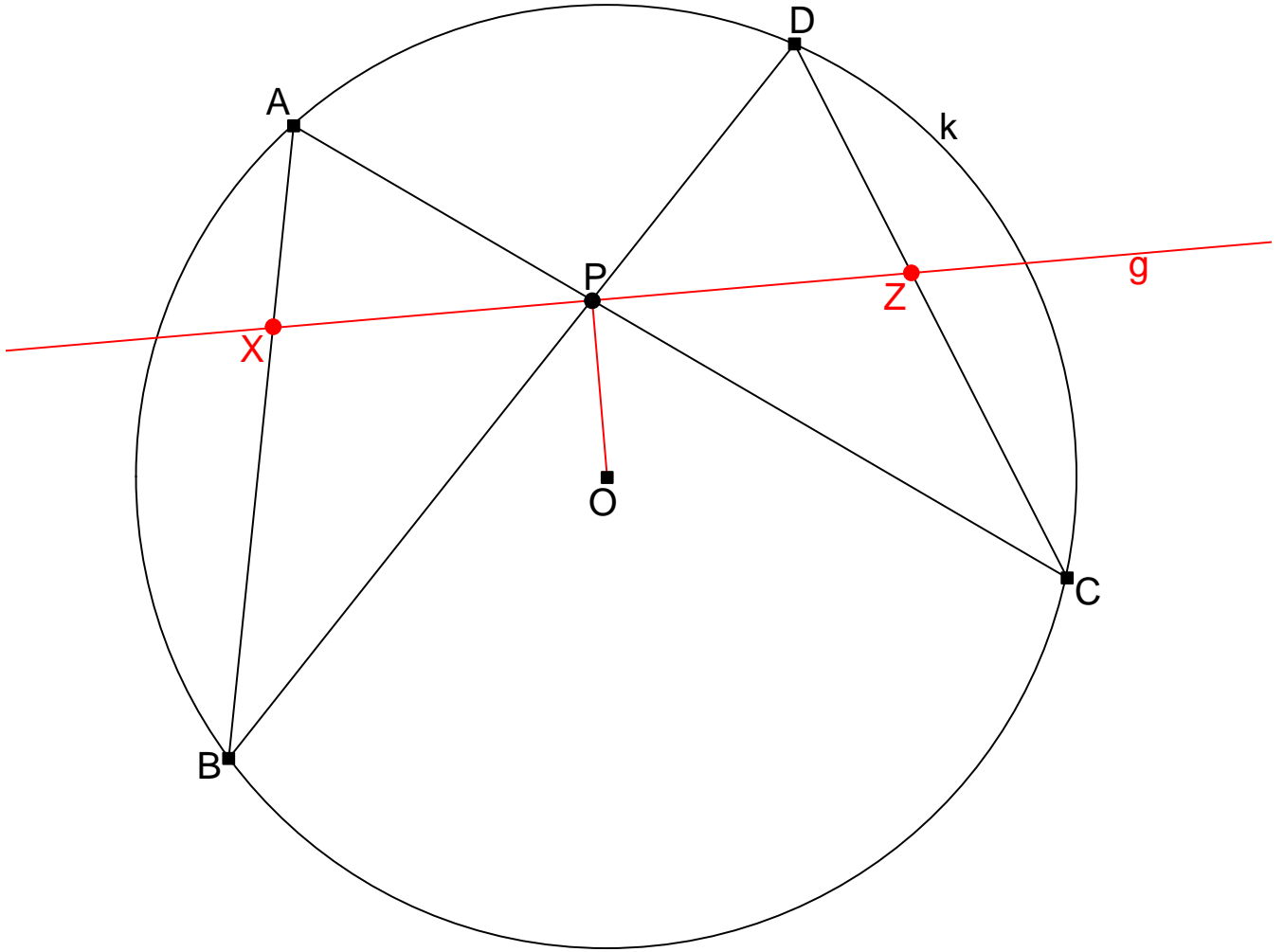


Fig. 1

**Theorem 1, the strong butterfly theorem.** Let  $k$  be a circle with center  $O$ , and let  $A, B, C, D$  be four points on this circle  $k$ . Let  $P = AC \cap BD$ . Let  $g$  be a line through the point  $P$  such that the point  $P$  is the orthogonal projection of the point  $O$  on this line  $g$ .<sup>2</sup> Let  $X = g \cap AB$  and  $Z = g \cap CD$ . Then, the point  $P$  is the midpoint

<sup>1</sup>In the following, the point of intersection of two lines  $u$  and  $v$  will be denoted by  $u \cap v$ .

<sup>2</sup>In the case when  $P \neq O$ , this simply means that the line  $g$  is the perpendicular to the line  $OP$  at the point  $P$ . In the case  $P = O$ , however,  $g$  can be any arbitrary line through the point  $P$ .

Note that, while we have thus taken care of the case  $P = O$  in the formulation of Theorem 1, we

of the segment  $XZ$ . (See Fig. 1.)

**Theorem 2, the weak butterfly theorem.** Let  $k$  be a circle with center  $O$ , and let  $T_1, T_2, A, B, C, D$  be six points on this circle  $k$ . Let  $P = AC \cap BD$ . Assume that the point  $P$  is the midpoint of the segment  $T_1T_2$ . Let  $X = T_1T_2 \cap AB$  and  $Z = T_1T_2 \cap CD$ . Then, the point  $P$  is the midpoint of the segment  $XZ$ . (See Fig. 2.)

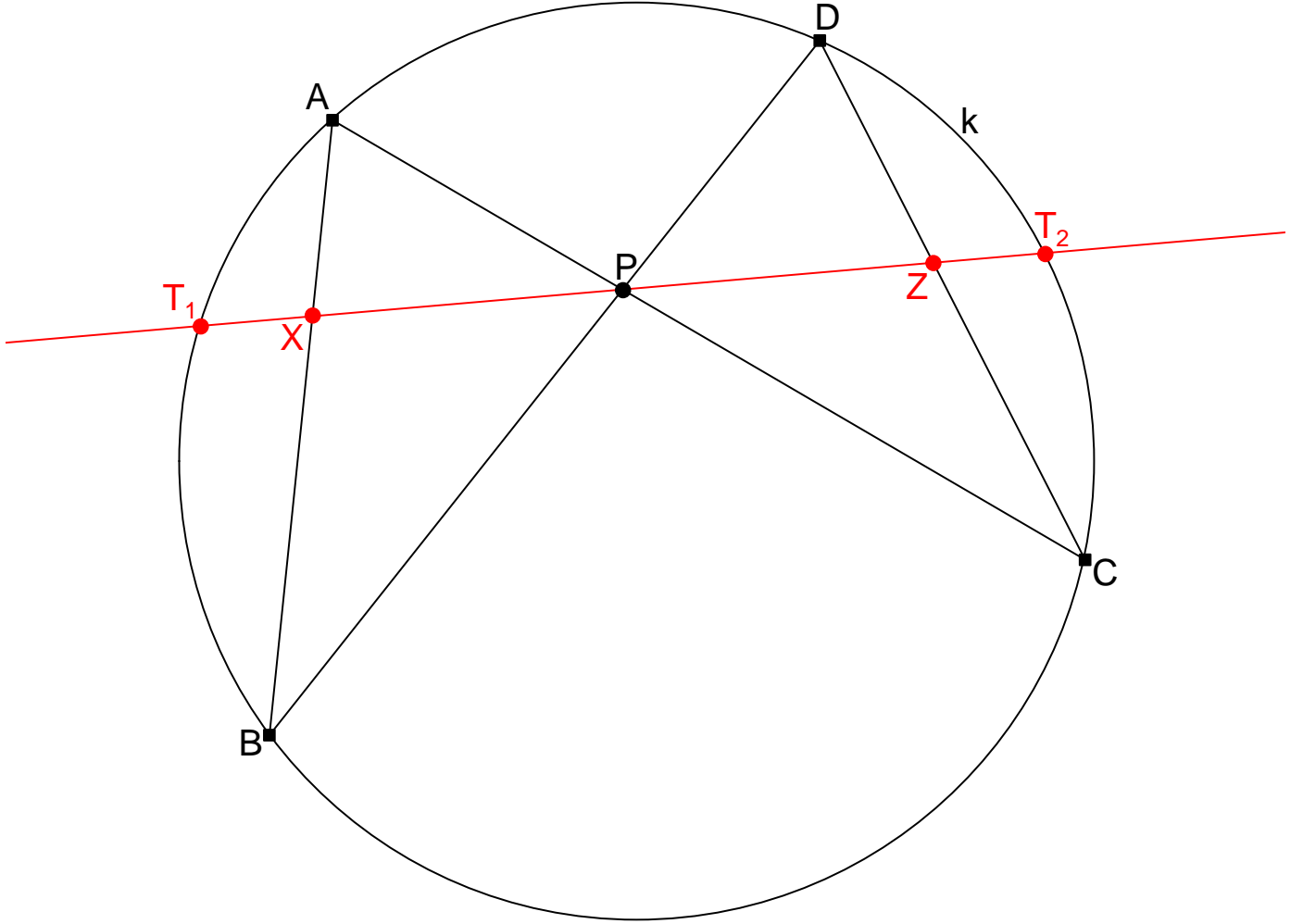


Fig. 2

First a *remark* on degenerate cases: In the formulations of both theorems, we didn't exclude the case when some of the points  $A, B, C, D$  coincide. In such cases, the following convention is to be applied: If two points  $P_1$  and  $P_2$ , which have been defined as two points on a circle  $k$ , coincide, then the line  $P_1P_2$  has to be understood as the tangent to the circle  $k$  at the point  $P_1$  (or, what is the same, at the point  $P_2$ ).

Theorems 1 and 2 are called strong and weak butterfly theorem for the reason that Theorem 2 readily follows from Theorem 1, but not conversely (it is possible to infer Theorem 1 from Theorem 2 using an algebraic argument, but this is not quite trivial).

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won't always take care of such particular cases in the proofs of our theorems.

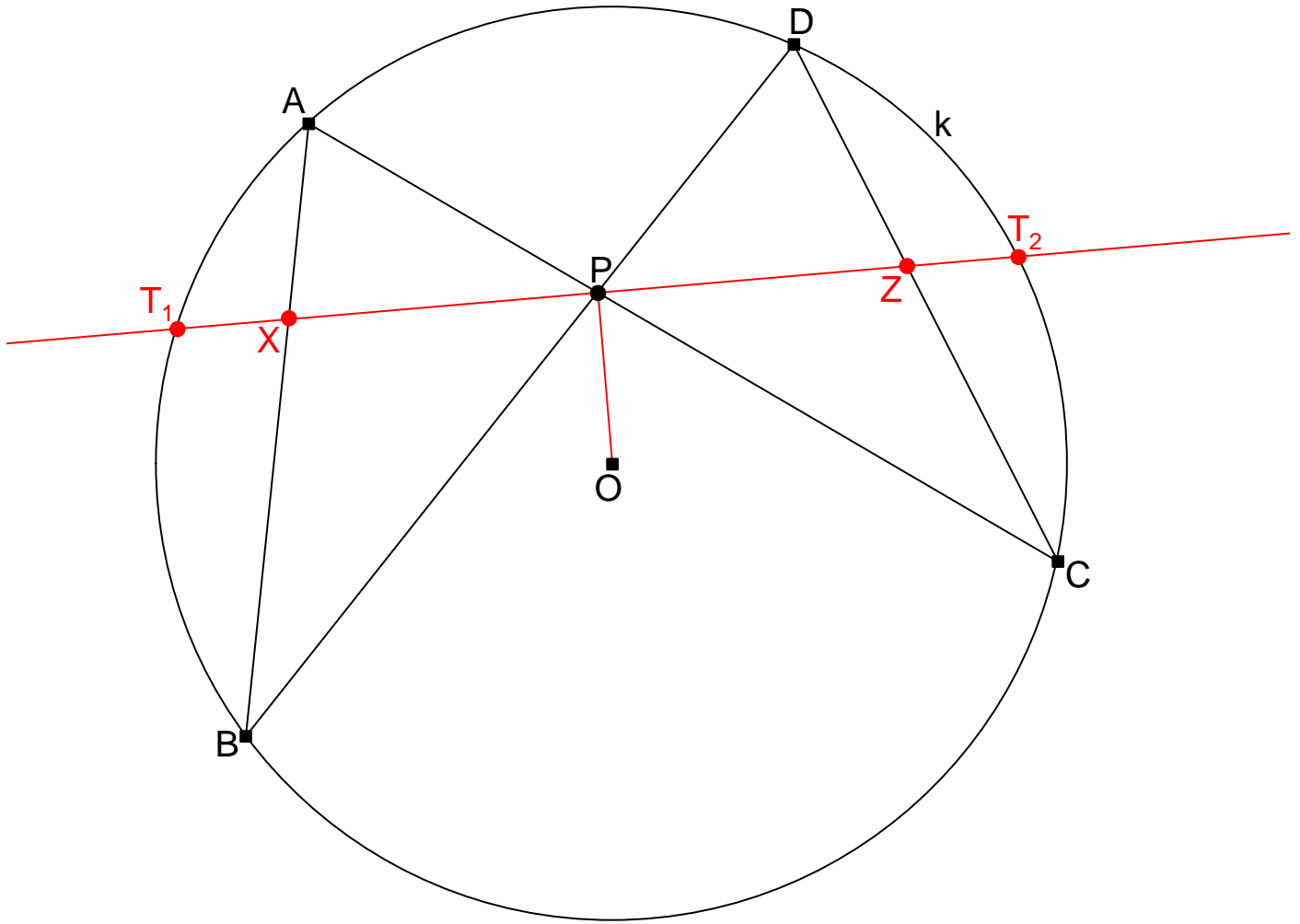


Fig. 3

The *proof of Theorem 2* using Theorem 1 works out as follows: (See Fig. 3.) Since  $T_1T_2$  is a chord of the circle  $k$ , while  $O$  is the center of this circle  $k$ , the point  $O$  lies on the perpendicular bisector of  $T_1T_2$  (since the perpendicular bisector of a chord of a circle passes through the center of the circle). Since  $P$  is the midpoint of  $T_1T_2$ , this yields that  $P$  is the orthogonal projection of the point  $O$  on the line  $T_1T_2$ . Thus, Theorem 2 follows from Theorem 1 (we just have to apply Theorem 1 to the line  $T_1T_2$  in the role of the line  $g$ ).

Most of the literature considers the weak Theorem 2 as "the" butterfly theorem and disregards Theorem 1 - a pity, for Theorem 1 has numerous applications in geometry which are harder (if possible at all) to handle with Theorem 2. We are going to forget about Theorem 2 for the rest of this note and work with Theorem 1 only.

Note that proofs of Theorem 1 are also featured in [14] and [15], and [16] indicates a proof of Theorem 2 which can be easily extended to a proof of the stronger Theorem 1.

We are going to give two proofs of Theorem 1.

## 2. The first proof

(requirements: Pascal theorem)

*First proof of Theorem 1.* We show a generalization of Theorem 1:

**Theorem 3, the strong Klamkin butterfly theorem.** Let  $k$  be a circle with center  $O$ , and let  $A, B, C, D$  be four points on this circle  $k$ . Let  $g$  be an arbitrary line, and let  $P$  be the orthogonal projection of the point  $O$  on this line  $g$ . The line  $g$  intersects the lines  $AB, BC, CD, DA, AC, BD$  at the points  $X, Y, Z, W, U, V$ . Then, the following three assertions are pairwise equivalent:

*Assertion 1:* The point  $P$  is the midpoint of the segment  $XZ$ .

*Assertion 2:* The point  $P$  is the midpoint of the segment  $YW$ .

*Assertion 3:* The point  $P$  is the midpoint of the segment  $UV$ .

(See Fig. 4.)

A few words about the name of Theorem 3: Murray Klamkin found a weaker version of Theorem 3 which generalizes the weak butterfly theorem (Theorem 2) in the same way as Theorem 3 generalizes the strong butterfly theorem (Theorem 1). See [1] (Remark to Proof 5') for details on Klamkin's result. In the form given here, Theorem 3 has been proven by Virgil Nicula in [11], post #2 (P.B.2).

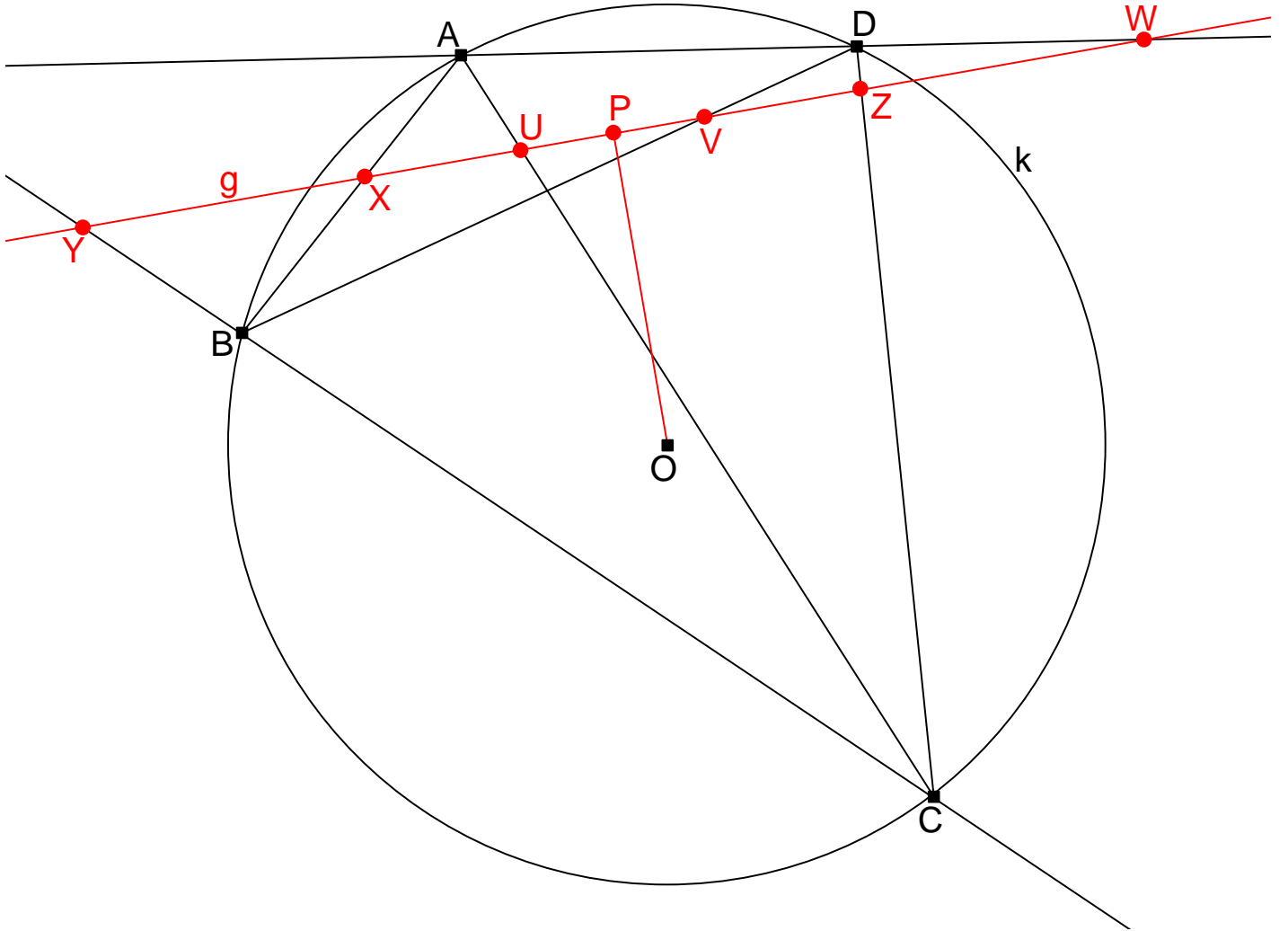


Fig. 4

The following *proof of Theorem 3* is apparently new: We start with a property of

triangles which, at the first sight, seems unrelated to our subject, but turns out to be equivalent to Theorem 3. This property was formulated by a MathLinks user with the nickname "Bismarck" in [2], post #4:

**Theorem 4.** Let  $ABC$  be a triangle, and  $g$  an arbitrary line. The line  $g$  intersects the lines  $BC$ ,  $CA$ ,  $AB$  at the points  $X$ ,  $Y$ ,  $Z$ .

Let  $O$  be the circumcenter of triangle  $ABC$ , and let  $P$  be the orthogonal projection of the point  $O$  on the line  $g$ . Let  $X'$ ,  $Y'$ ,  $Z'$  be the reflections of the points  $X$ ,  $Y$ ,  $Z$  in the point  $P$ .

Then, the lines  $AX'$ ,  $BY'$ ,  $CZ'$  concur at one point, and this point lies on the circumcircle of triangle  $ABC$ . (See Fig. 5.)

Theorem 4 is related to the Blaikie theorem, which states that the lines  $AX'$ ,  $BY'$ ,  $CZ'$  concur for *any* point  $P$  on the line  $g$  - not just for  $P$  being the orthogonal projection of the point  $O$  on the line  $g$ . The point of concurrence is called the *Blaikie point* of the line  $g$  and the point  $P$  with respect to triangle  $ABC$ . In the context of this result, Theorem 4 shows that, if the point  $P$  is the orthogonal projection of the point  $O$  on the line  $g$ , then the Blaikie point of the line  $g$  and the point  $P$  with respect to triangle  $ABC$  lies on the circumcircle of triangle  $ABC$ .

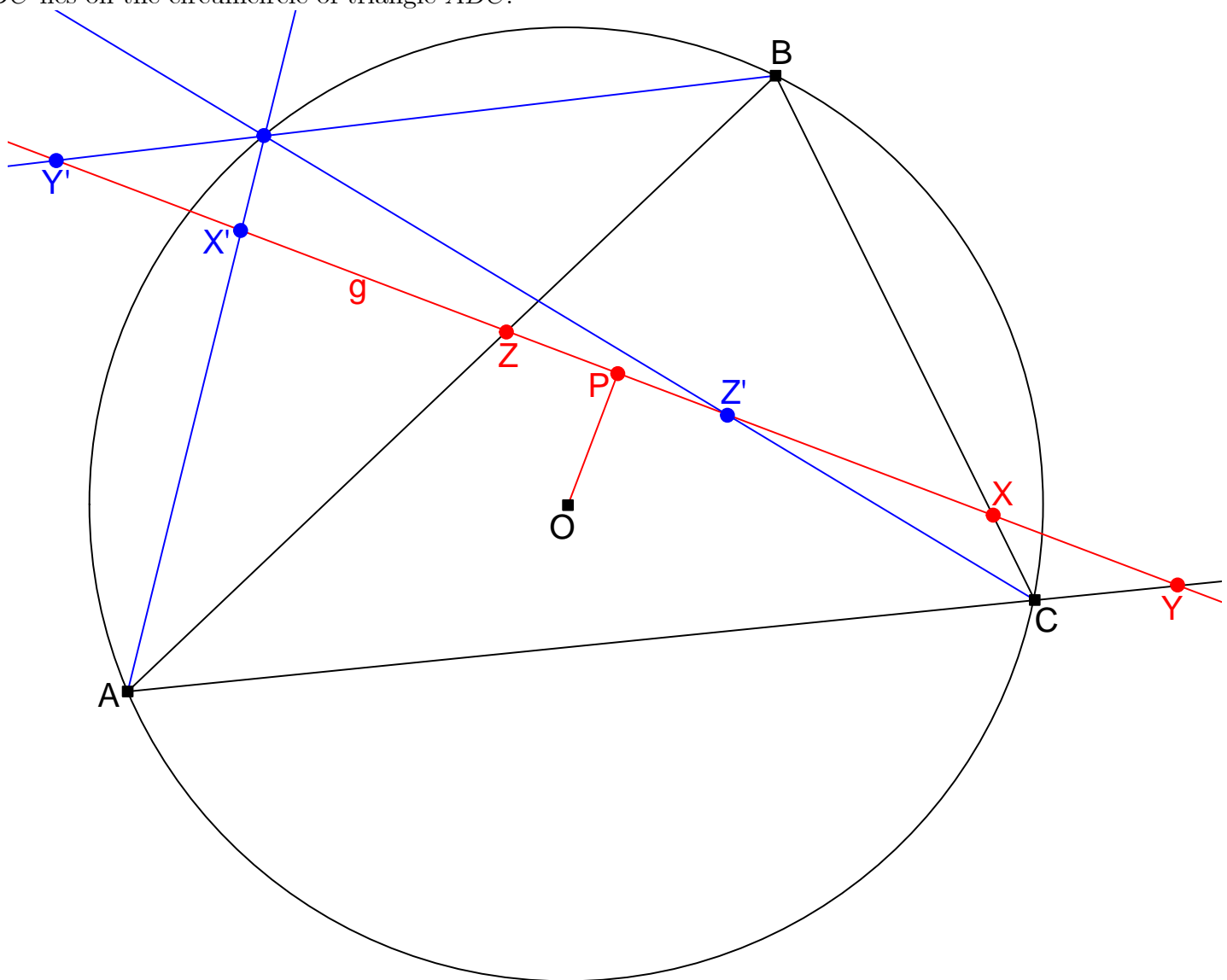


Fig. 5

*Proof of Theorem 4.* (See Fig. 6.) (In the following, we will often speak of the line  $OP$ . In the case when  $O = P$ , the line  $OP$  will mean the perpendicular to the line  $g$  at the point  $O$ .)

Since  $X'$  is the reflection of  $X$  in the point  $P$ , the point  $P$  is the midpoint of the segment  $XX'$ . Also,  $g \perp OP$ . Thus, the line  $OP$  is perpendicular to the segment  $XX'$  and passes through the midpoint  $P$  of this segment. Hence, the line  $OP$  is the perpendicular bisector of the segment  $XX'$ . Thus,  $X'$  is the reflection of  $X$  in the line  $OP$ .

Let  $A'$ ,  $B'$ ,  $C'$  be the reflections of the points  $A$ ,  $B$ ,  $C$  in the line  $OP$ . Then,  $AA' \perp OP$  and  $BB' \perp OP$ . On the other hand,  $g \perp OP$ . Thus, the lines  $AA'$ ,  $BB'$  and  $g$  are parallel to each other; therefore, they concur at one point - namely, at an infinite point. In other words: The (infinite) point of intersection  $AA' \cap BB'$  lies on the line  $g$ .

The reflection in the line  $OP$  maps the circumcircle of triangle  $ABC$  to itself (since the line  $OP$  passes through the center  $O$  of this circumcircle). Since the points  $A$ ,  $B$ ,  $C$  lie on the circumcircle of triangle  $ABC$ , it thus follows that their reflections in the line  $OP$ , i. e. the points  $A'$ ,  $B'$ ,  $C'$ , also lie on the circumcircle of triangle  $ABC$ .

Now, let  $S$  be the point of intersection of the line  $A'X$  with the circumcircle of triangle  $ABC$  different from  $A'$ . Then, the hexagon  $A'SB'BCA$  has a circumcircle (namely, the circumcircle of triangle  $ABC$ ). Therefore, by the Pascal theorem, the points  $A'S \cap BC$ ,  $SB' \cap CA$  and  $B'B \cap AA'$  lie on one line. This line must coincide with the line  $g$ , since two points on this line (namely, the point  $A'S \cap BC = X$  and the point  $B'B \cap AA' = AA' \cap BB'$ ) lie on the line  $g$ . Hence, the point  $SB' \cap CA$  lies on the line  $g$ . Thus,  $SB' \cap CA = g \cap CA$ . But  $g \cap CA = Y$ . Hence,  $SB' \cap CA = Y$ . This signifies that the point  $S$  lies on the line  $B'Y$ . Similarly, the point  $S$  lies on the line  $C'Z$ .

Altogether, the point  $S$  lies on the circumcircle of triangle  $ABC$  and on the lines  $A'X$ ,  $B'Y$ ,  $C'Z$ .

Let  $S'$  be the reflection of the point  $S$  in the line  $OP$ . Since the point  $S$  lies on the circumcircle of triangle  $ABC$ , its reflection  $S'$  in the line  $OP$  also lies on the circumcircle of triangle  $ABC$  (since the reflection in the line  $OP$  maps this circumcircle into itself).

Since  $A'$  is the reflection of  $A$  in the line  $OP$ , the point  $A$  is the reflection of  $A'$  in the line  $OP$ . On the other hand,  $X'$  is the reflection of  $X$  in the line  $OP$ . Thus, the line  $AX'$  is the reflection of the line  $A'X$  in the line  $OP$ . Similarly, the lines  $BY'$  and  $CZ'$  are the reflections of the lines  $B'Y$  and  $C'Z$  in the line  $OP$ . Since the point  $S$  lies on the lines  $A'X$ ,  $B'Y$ ,  $C'Z$ , its reflection  $S'$  in the line  $OP$  must lie on the reflections of the lines  $A'X$ ,  $B'Y$ ,  $C'Z$  in the line  $OP$ , i. e. on the lines  $AX'$ ,  $BY'$ ,  $CZ'$ .

Altogether, the point  $S'$  lies on the lines  $AX'$ ,  $BY'$ ,  $CZ'$  and on the circumcircle of triangle  $ABC$ . This proves Theorem 4.

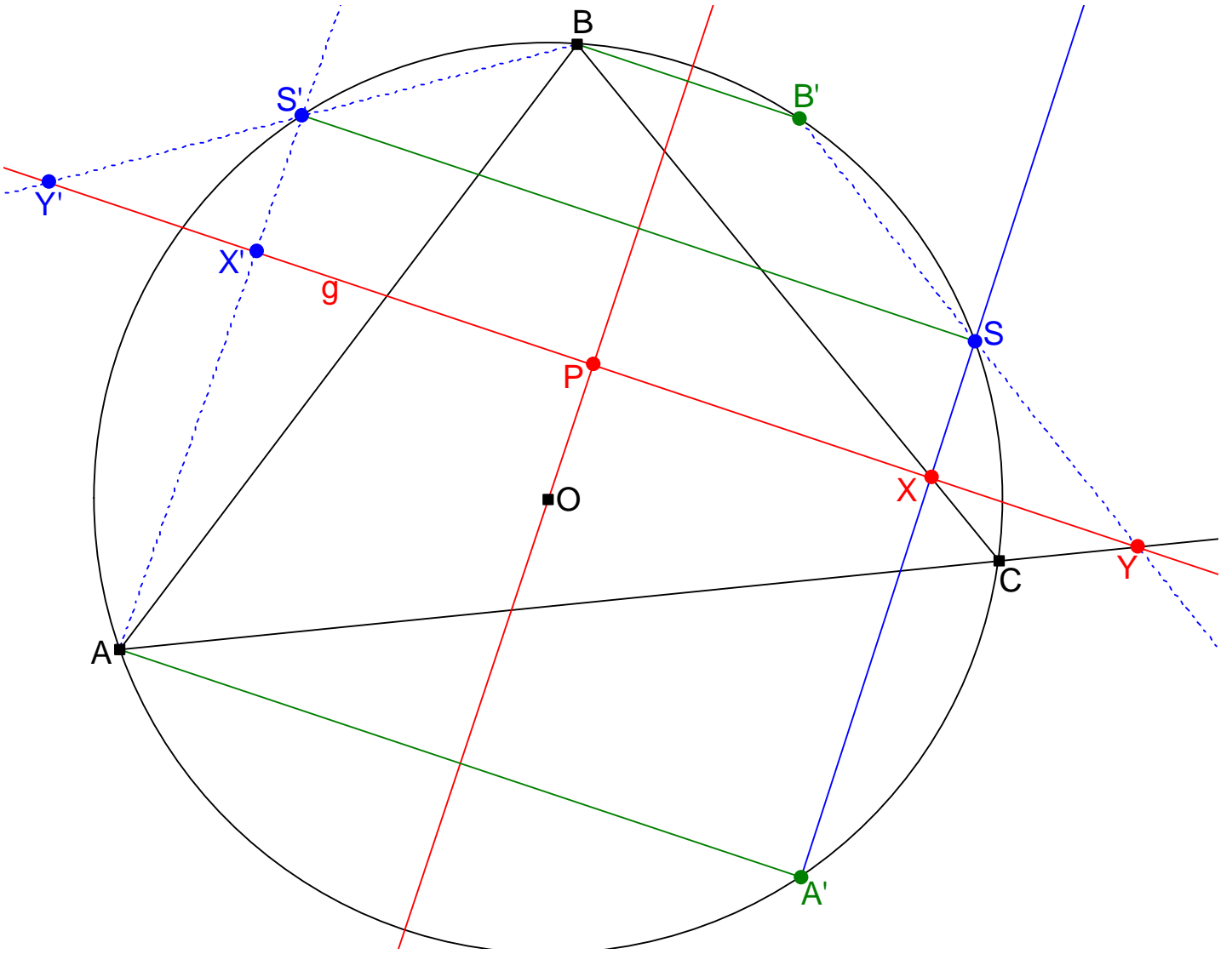


Fig. 6

Now we are going to derive Theorem 3 from Theorem 4:

(See Fig. 7.) First, we are going to show that Assertion 1 yields Assertion 2. In order to prove this, we assume Assertion 1 to hold, i. e. we assume that  $P$  is the midpoint of  $XZ$ . Then,  $Z$  is the reflection of  $X$  in the point  $P$ . Let  $V'$  and  $W'$  be the reflections of  $U$  and  $Y$  in the point  $P$ . Then, we have the following configuration:

A triangle  $ABC$  is given. The line  $g$  intersects the lines  $BC$ ,  $CA$ ,  $AB$  at the points  $Y$ ,  $U$ ,  $X$ . The point  $O$  is the circumcenter of triangle  $ABC$  (in fact, the circumcircle of triangle  $ABC$  is the circle  $k$  and thus has center  $O$ ), and the point  $P$  is the orthogonal projection of the point  $O$  on the line  $g$ . Finally,  $W'$ ,  $V'$ ,  $Z$  are the reflections of the points  $Y$ ,  $U$ ,  $X$  in the point  $P$ .

Hence, Theorem 4 yields that the lines  $AW'$ ,  $BV'$ ,  $CZ$  concur at one point, and this point lies on the circumcircle of triangle  $ABC$ . We denote this point by  $D'$ . Thus, this point  $D'$  is the point of intersection of the line  $CZ$  with the circumcircle of triangle  $ABC$  different from  $C$ . But the point of intersection of the line  $CZ$  with the circumcircle of triangle  $ABC$  different from  $C$  is the point  $D$  (in fact, the circumcircle of triangle  $ABC$  is the circle  $k$  and intersects the line  $CZ$  at the points  $C$  and  $D$ ). Hence,  $D' = D$ . As we know that  $D'$  lies on  $AW'$ , we thus conclude that  $D$  lies on  $AW'$ . In other words,

$W'$  lies on  $DA$ . On the other hand,  $W'$  lies on the line  $g$  (since  $W'$  is the reflection of  $Y$  in  $P$ , and both points  $Y$  and  $P$  lie on the line  $g$ ). Hence,  $W' = DA \cap g$ . But  $DA \cap g = W$ . Thus,  $W' = W$ . Since we have introduced the point  $W'$  as the reflection of  $Y$  in the point  $P$ , it thus follows that  $W$  is the reflection of  $Y$  in the point  $P$ . Thus,  $P$  is the midpoint of  $YW$ . Hence, Assertion 2 holds.

Thus we have shown that Assertion 1 yields Assertion 2. Similarly we can show (or conclude from the already shown result by permutation of the points  $A, B, C, D$ ) that Assertion 2 yields Assertion 3 and that Assertion 3 yields Assertion 1. Hence, the Assertions 1, 2 and 3 are pairwise equivalent. This completes the proof of Theorem 3.

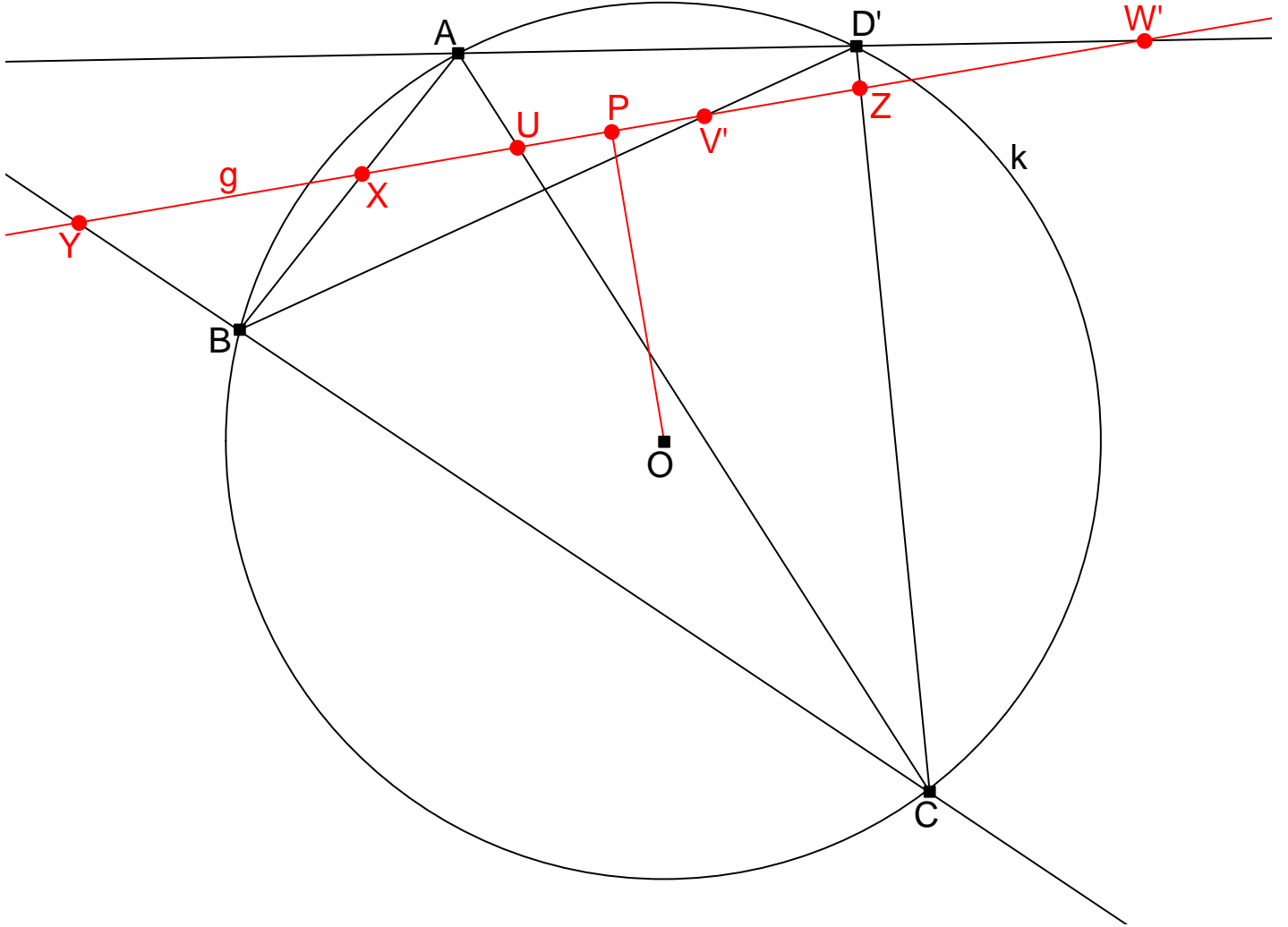


Fig. 7

Now, after Theorem 3 is verified, we can finally establish Theorem 1:

Consider the particular case of Theorem 3 when the point  $P$  happens to coincide with the point  $AC \cap BD$ . Then,  $U = g \cap AC = P$  and  $V = g \cap BD = P$ . Hence, Assertion 3 of Theorem 3 is equivalent to the point  $P$  being the midpoint of the segment  $PP$ . Of course, this assertion is trivially valid. Since, according to Theorem 3, the Assertions 1, 2 and 3 are pairwise equivalent, this entails that Assertion 1 of Theorem 3 is also



valid, i. e. the point  $P$  is the midpoint of the segment  $XZ$ . Thus we have proven the following: If the point  $P$  in Theorem 3 coincides with the point  $AC \cap BD$ , then the point  $P$  is the midpoint of the segment  $XZ$ . But this is exactly what Theorem 1 asserts. Hence, Theorem 1 is proven.

### 3. The second proof

*(requirements: Ceva AND (Desargues OR (invariance of cross-ratio AND Menelaos)) AND (polarity with respect to circles OR inversion OR radical axes))*

*Second proof of Theorem 1.* This second proof of Theorem 1 is more or less a variation of Proof 12 in [1] - the idea is exactly that of Proof 12, but the advanced concepts used will be reduced to a significantly lower amount, and a number of useful facts will be gathered on the way.

Our first lemma is an affine theorem which has been proposed independently as a problem in [3]:

**Theorem 5.** Let  $A, B, C, D$  be four points in the plane. Let  $P = AC \cap BD$ ,  $Q = AB \cap CD$  and  $R = BC \cap DA$ . The parallel to the line  $QR$  through the point  $P$  intersects the lines  $AB$  and  $CD$  at the points  $X$  and  $Z$ . Then, the point  $P$  is the midpoint of the segment  $XZ$ . (See Fig. 8.)

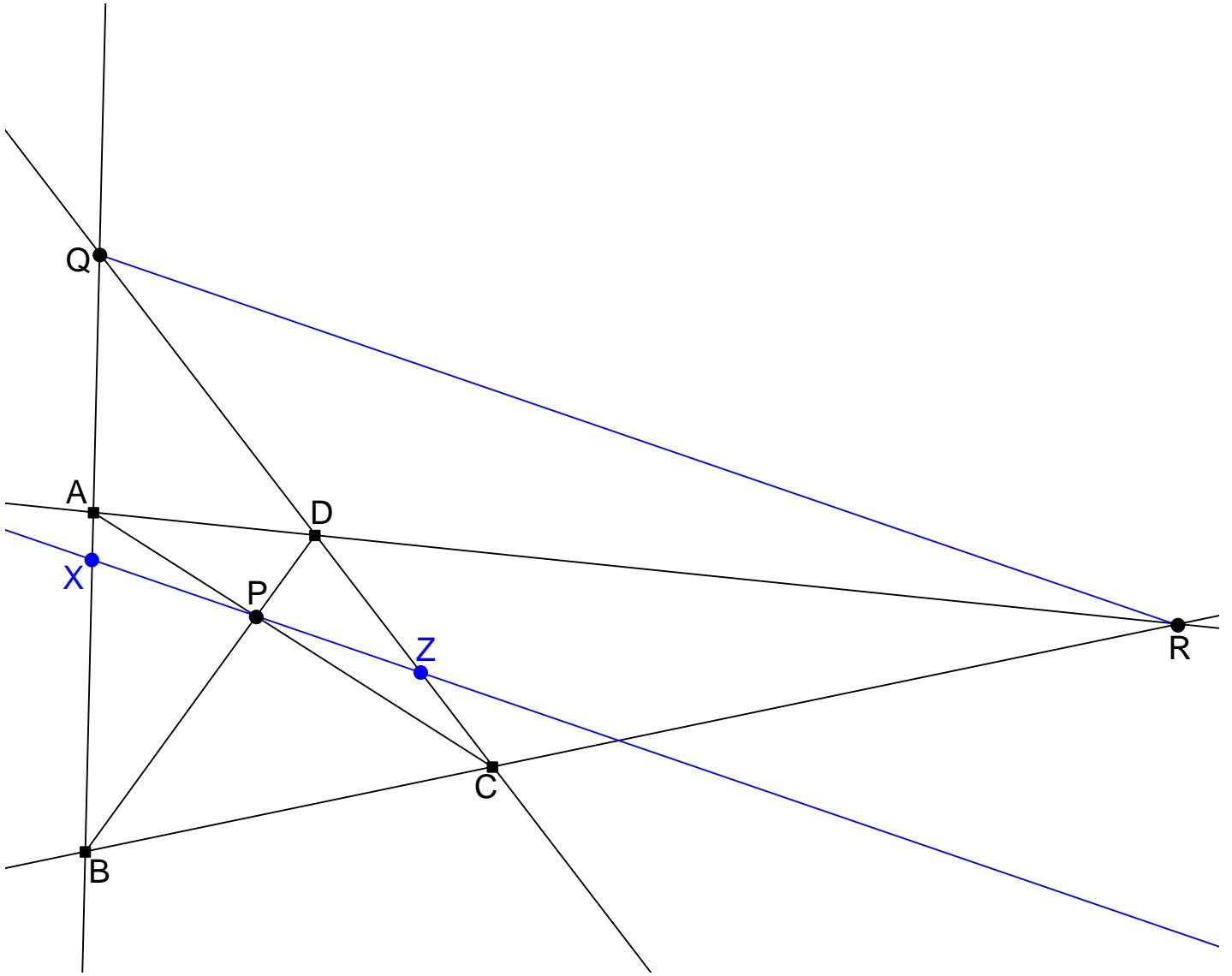


Fig. 8

We give two proofs of this fact - one using some basic projective geometry and one using just the Ceva and Desargues theorems.

*First proof of Theorem 5.* The following proof of Theorem 5 uses some projective geometry - namely the basic properties of infinite points and the invariance of the cross-ratio.

We work with directed segments. (See Fig. 9.) Let  $Q' = QP \cap BC$ . Since the lines  $QQ'$ ,  $BD$ ,  $CA$  concur (in fact, they concur at the point  $P$ ), the Ceva theorem (applied to triangle  $QBC$  and the points  $Q'$ ,  $D$ ,  $A$  on its sides  $BC$ ,  $CQ$ ,  $QB$ ) yields

$$\frac{BQ'}{Q'C} \cdot \frac{CD}{DQ} \cdot \frac{QA}{AB} = 1.$$

Since the points  $R$ ,  $D$ ,  $A$  are collinear, the Menelaos theorem (applied to triangle  $QBC$  and the points  $R$ ,  $D$ ,  $A$  on its sides  $BC$ ,  $CQ$ ,  $QB$ ) yields

$$\frac{BR}{RC} \cdot \frac{CD}{DQ} \cdot \frac{QA}{AB} = -1.$$

Comparison of these two equations yields  $\frac{BR}{RC} = -\frac{BQ'}{Q'C}$ , so that  $\frac{BR}{RC} : \frac{BQ'}{Q'C} = -1$ .

Since  $XZ \parallel QR$ , the lines  $XZ$  and  $QR$  intersect at an infinite point. Denote this point by  $T$ . Then, the four points  $B, C, R, Q'$  lie on the line  $BC$ , and the points  $X, Z, T, P$  are the projections of these four points from the point  $Q$  onto the line  $XZ$ . Thus, by the invariance of the cross-ratio under central projection, we have  $\frac{XT}{TZ} : \frac{XP}{PZ} = \frac{BR}{RC} : \frac{BQ'}{Q'C}$ . Since  $\frac{BR}{RC} : \frac{BQ'}{Q'C} = -1$ , this becomes  $\frac{XT}{TZ} : \frac{XP}{PZ} = -1$ , so that  $\frac{XT}{TZ} = -\frac{XP}{PZ}$ . But since  $T$  is the infinite point of the line  $XZ$ , we have  $\frac{XT}{TZ} = -1$ . Thus,  $-1 = -\frac{XP}{PZ}$ , so that  $1 = \frac{XP}{PZ}$ . Hence,  $XP = PZ$ , so that the point  $P$  is the midpoint of  $XZ$ . This proves Theorem 5.

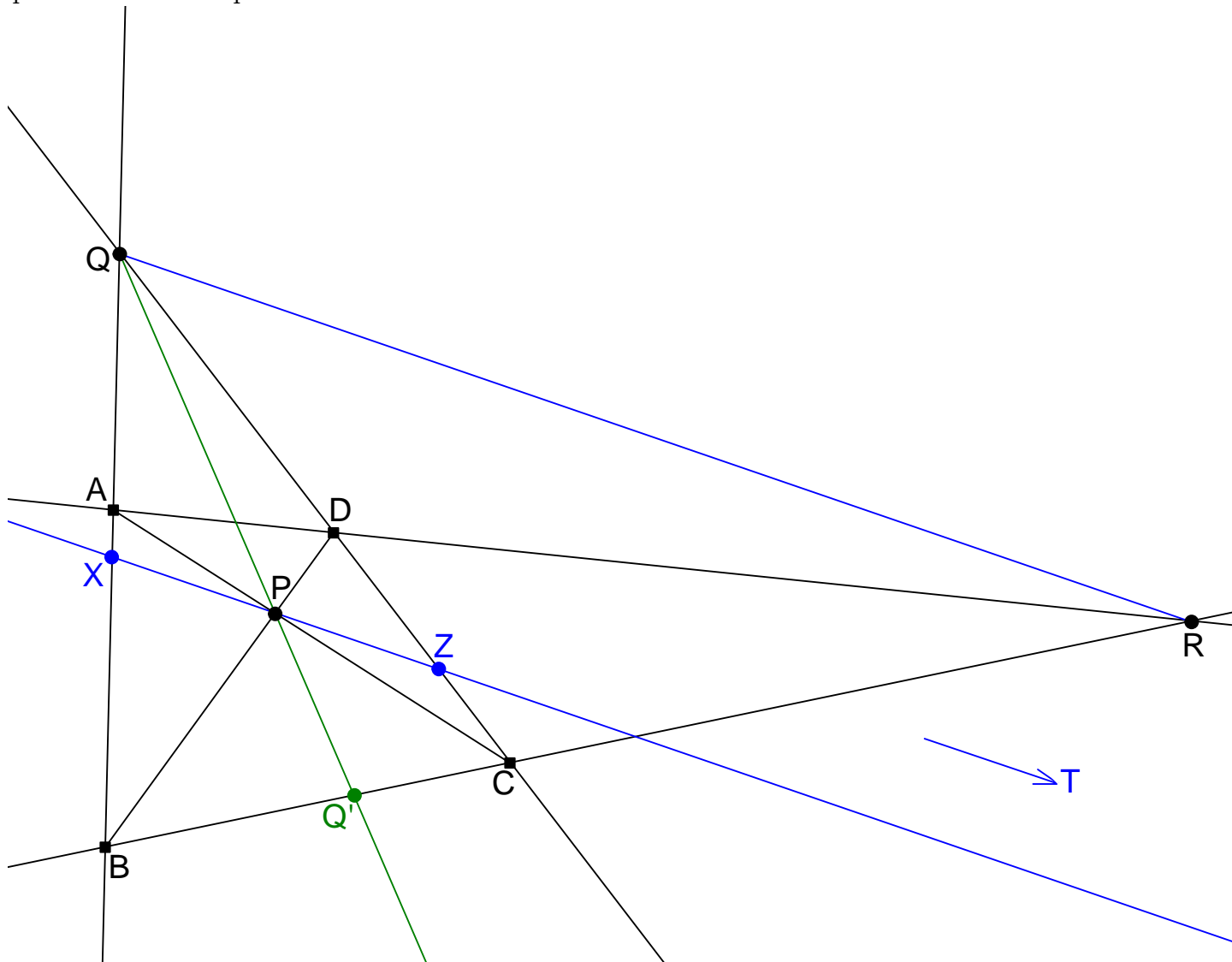


Fig. 9

*Second proof of Theorem 5.* We will use directed segments. Hereby, the parallel lines  $XZ$  and  $QR$  are assumed to be directed equally. (See Fig. 10.) Let  $B' = BD \cap QR$  and  $C' = AC \cap QR$ . Since  $XZ \parallel QR$ , Thales yields  $\frac{XP}{QB'} = \frac{BP}{BB'}$  and  $\frac{C'Q}{PZ} = \frac{CC'}{CP}$ , and

therefore

$$\begin{aligned}\frac{XP}{PZ} &= \frac{XP}{QB'} \cdot \frac{QB'}{C'Q} \cdot \frac{C'Q}{PZ} = \frac{BP}{BB'} \cdot \frac{QB'}{C'Q} \cdot \frac{CC'}{CP} = \left(-\frac{PB}{BB'}\right) \cdot \frac{B'Q}{QC'} \cdot \left(-\frac{C'C}{CP}\right) \\ &= \frac{PB}{BB'} \cdot \frac{B'Q}{QC'} \cdot \frac{C'C}{CP}.\end{aligned}$$

Now, the points  $B'C' \cap CB = R$ ,  $C'P \cap BQ = A$  and  $PB' \cap QC = D$  are collinear. According to the Desargues theorem (applied to the triangles  $PB'C'$  and  $QCB$ ), this yields that the lines  $PQ$ ,  $B'C$  and  $C'B$  concur. Hence, according to the Ceva theorem (applied to the triangle  $PB'C'$  and the points  $Q, C, B$  on its sidelines  $B'C', C'P, PB'$ ), we have  $\frac{PB}{BB'} \cdot \frac{B'Q}{QC'} \cdot \frac{C'C}{CP} = 1$ . Thus,  $\frac{XP}{PZ} = \frac{PB}{BB'} \cdot \frac{B'Q}{QC'} \cdot \frac{C'C}{CP} = 1$ , so that  $XP = PZ$ . Consequently,  $P$  is the midpoint of  $XZ$ . This proves Theorem 5.

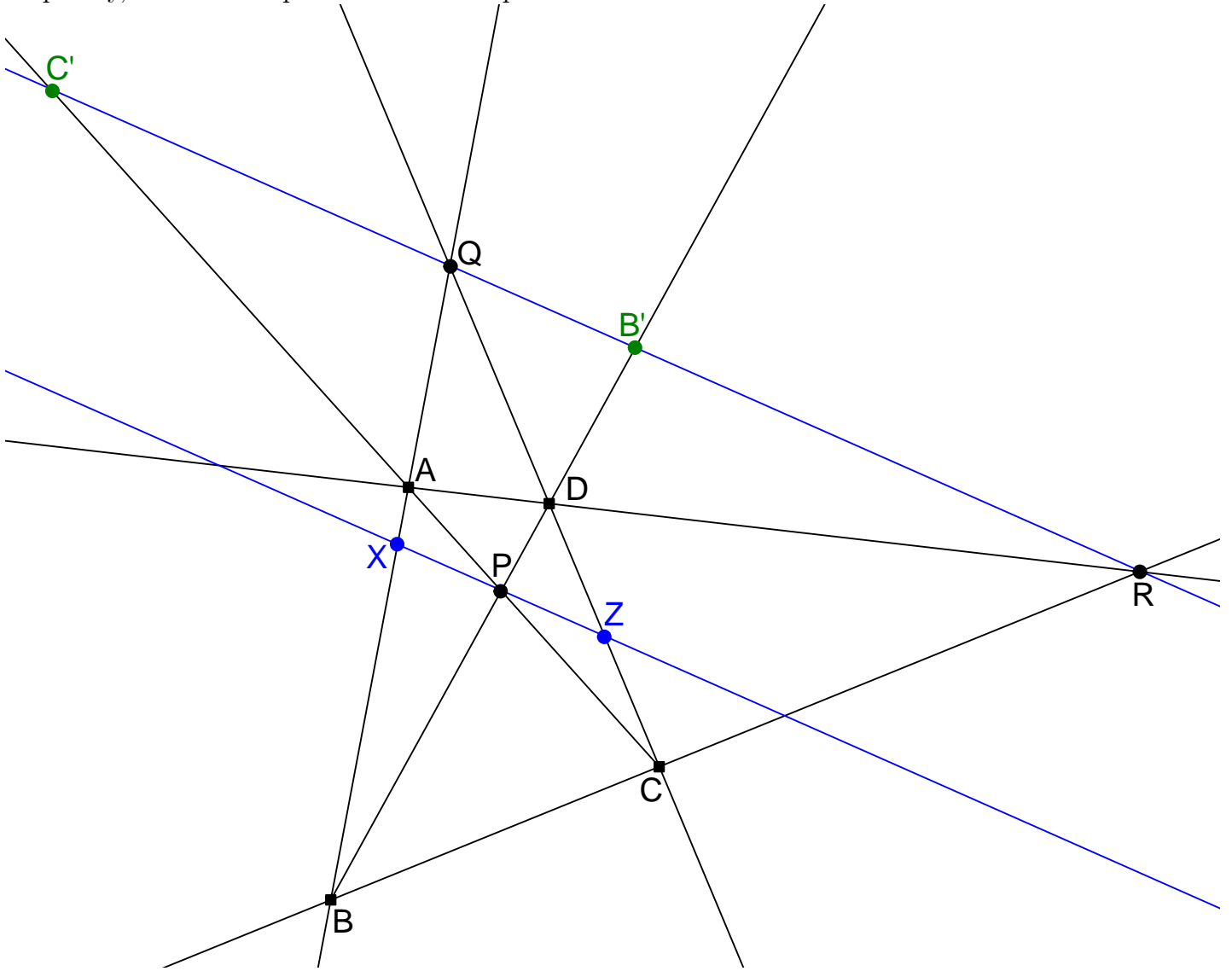


Fig. 10

Our second lemma is an important and known fact from the geometry of cyclic quadrilaterals. In an equivalent version, it has been discussed, e. g., at [4]. We will use it in the following form:

**Theorem 6.** Let  $k$  be a circle with center  $O$ , and let  $A, B, C, D$  be four points on this circle  $k$ . Let  $P = AC \cap BD$ ,  $Q = AB \cap CD$  and  $R = BC \cap DA$ . Then, the point  $O$  is the orthocenter of triangle  $PQR$ . (See Fig. 11.)

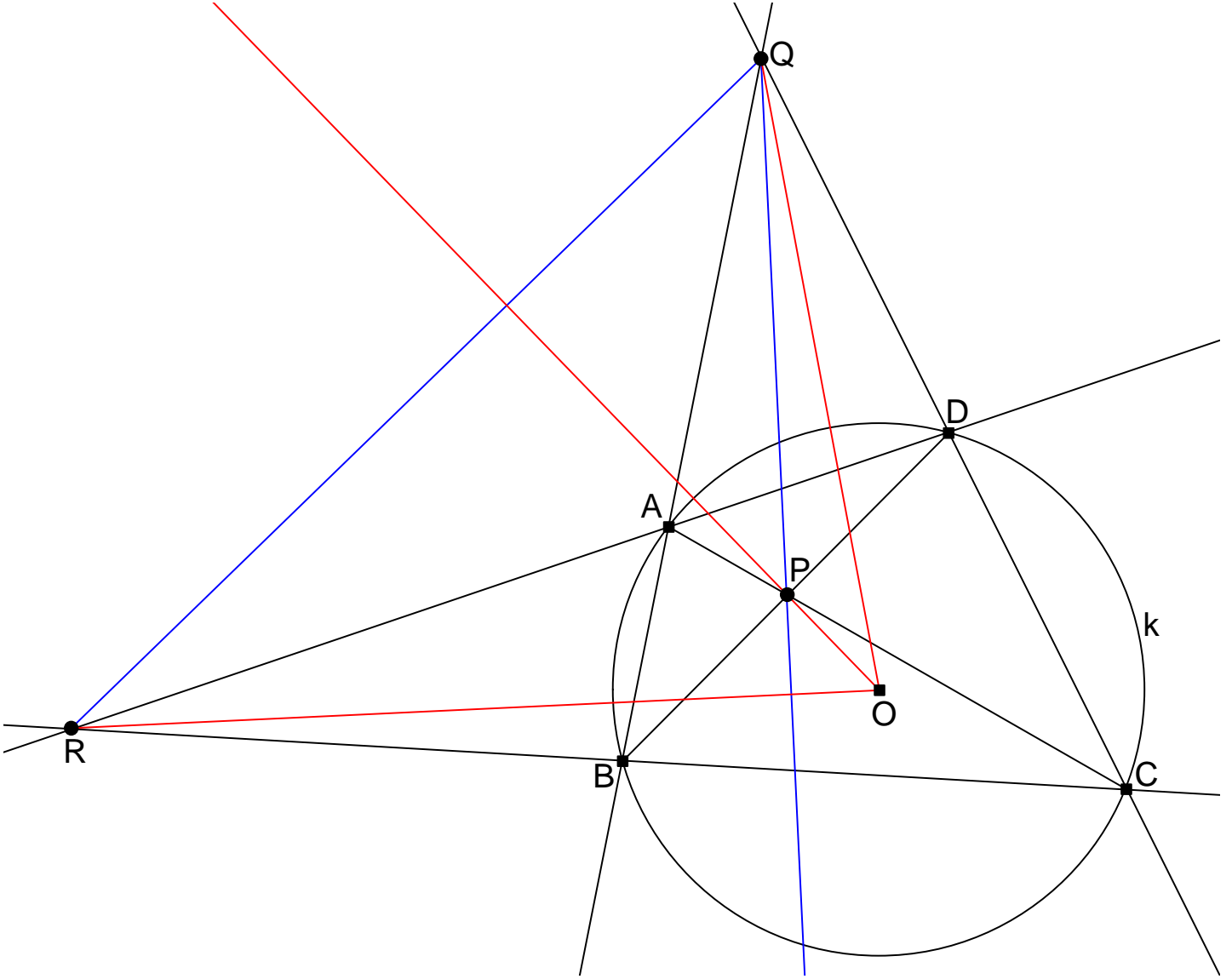


Fig. 11

We will give two proofs of Theorem 6. The first one deduces it from a result in the theory of poles and polars with respect to a circle, while the second one uses no theory beyond radical axes *or* inversion - either of these is enough! - and directed angles modulo  $180^\circ$ .

*First proof of Theorem 6.* This proof relies on a fact from the theory of poles and polars with respect to a circle:

**Theorem 7.** Let  $X, Y, Z, W$  be four points on a circle  $k_1$ . Then, the point  $XY \cap ZW$  lies on the polar of the point  $XZ \cap YW$  with respect to the circle  $k_1$ .

This fact appears as Theorem 1 in [5], where it is proven using the Pascal theorem. Below we will give a different proof of this fact based on our second proof of Theorem 6.

Assuming Theorem 7 as given, Theorem 6 is easy to verify:

Applying Theorem 7 to the points  $X = A, Y = B, Z = C, W = D$  on the circle

$k_1 = k$ , we see that the point  $AB \cap CD$  lies on the polar of the point  $AC \cap BD$  with respect to the circle  $k$ . Since  $AC \cap BD = P$  and  $AB \cap CD = Q$ , this means that  $Q$  lies on the polar of  $P$  with respect to  $k$ . Applying Theorem 7 to the points  $X = B$ ,  $Y = C$ ,  $Z = D$ ,  $W = A$  on the circle  $k_1 = k$ , we obtain that the point  $BC \cap DA$  lies on the polar of the point  $BD \cap CA$  with respect to the circle  $k$ . Since  $BC \cap DA = R$  and  $BD \cap CA = P$ , this means that  $R$  lies on the polar of  $P$  with respect to  $k$ .

Since the two points  $Q$  and  $R$  both lie on the polar of  $P$  with respect to  $k$ , this yields that the line  $QR$  is the polar of  $P$  with respect to  $k$ . Now, since  $O$  is the center of  $k$ , and since the polar of a point with respect to a circle is always perpendicular to the line joining this point with center of the circle, we thus obtain  $OP \perp QR$ . Similarly,  $OQ \perp RP$  and  $OR \perp PQ$ . Thus, the lines  $OP$ ,  $OQ$ ,  $OR$  are the altitudes of triangle  $PQR$ . Hence, the orthocenter of triangle  $PQR$  is the point of intersection of these lines  $OP$ ,  $OQ$ ,  $OR$ , so it must be the point  $O$ . This proves Theorem 6.

*Second proof of Theorem 6.* In the following proof, we are going to use directed angles modulo  $180^\circ$ .

We commence with a classical result about four lines in a plane:

**Theorem 8, the Miquel fourline theorem.** Let  $a$ ,  $b$ ,  $c$ ,  $d$  be four lines in the plane. Denote  $A = b \cap c$ ,  $B = c \cap a$ ,  $C = a \cap b$ ,  $D = a \cap d$ ,  $E = b \cap d$ ,  $F = c \cap d$ . Then, the circumcircles of triangles  $EAF$ ,  $BDF$ ,  $EDC$ ,  $BAC$  have a common point.

This point is called the *Miquel point* of the four lines  $a$ ,  $b$ ,  $c$ ,  $d$ . (See Fig. 12, where the Miquel point is marked red.)

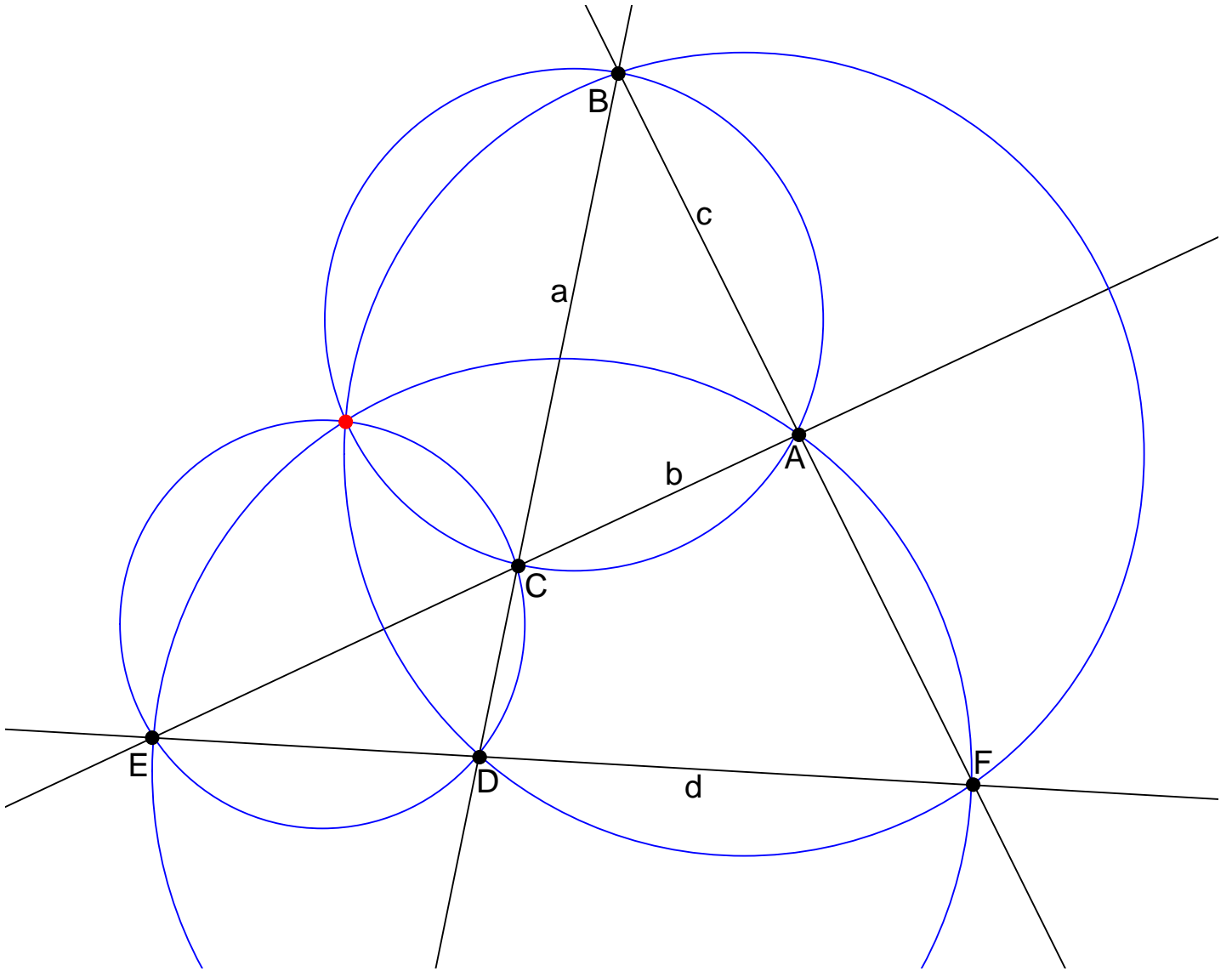


Fig. 12

*Proof of Theorem 8.* (See Fig. 13.) Let  $M$  be the point of intersection of the circumcircles of triangles  $EAF$  and  $EDC$  distinct from  $E$ . Then,  $\angle MCD = \angle MED$  (since  $M$  lies on the circumcircle of triangle  $EDC$ ) and  $\angle MEF = \angle MAF$  (since  $M$  lies on the circumcircle of triangle  $EAF$ ). Hence,  $\angle MCB = \angle MCD = \angle MED = \angle MEF = \angle MAF = \angle MAB$ . Thus, the points  $M, B, C, A$  lie on one circle. Equivalently,  $M$  lies on the circumcircle of triangle  $BAC$ . Similarly,  $M$  lies on the circumcircle of triangle  $BDF$ . Thus, the circumcircles of triangles  $EAF, BDF, EDC, BAC$  have a common point - namely, the point  $M$ . This proves Theorem 8.

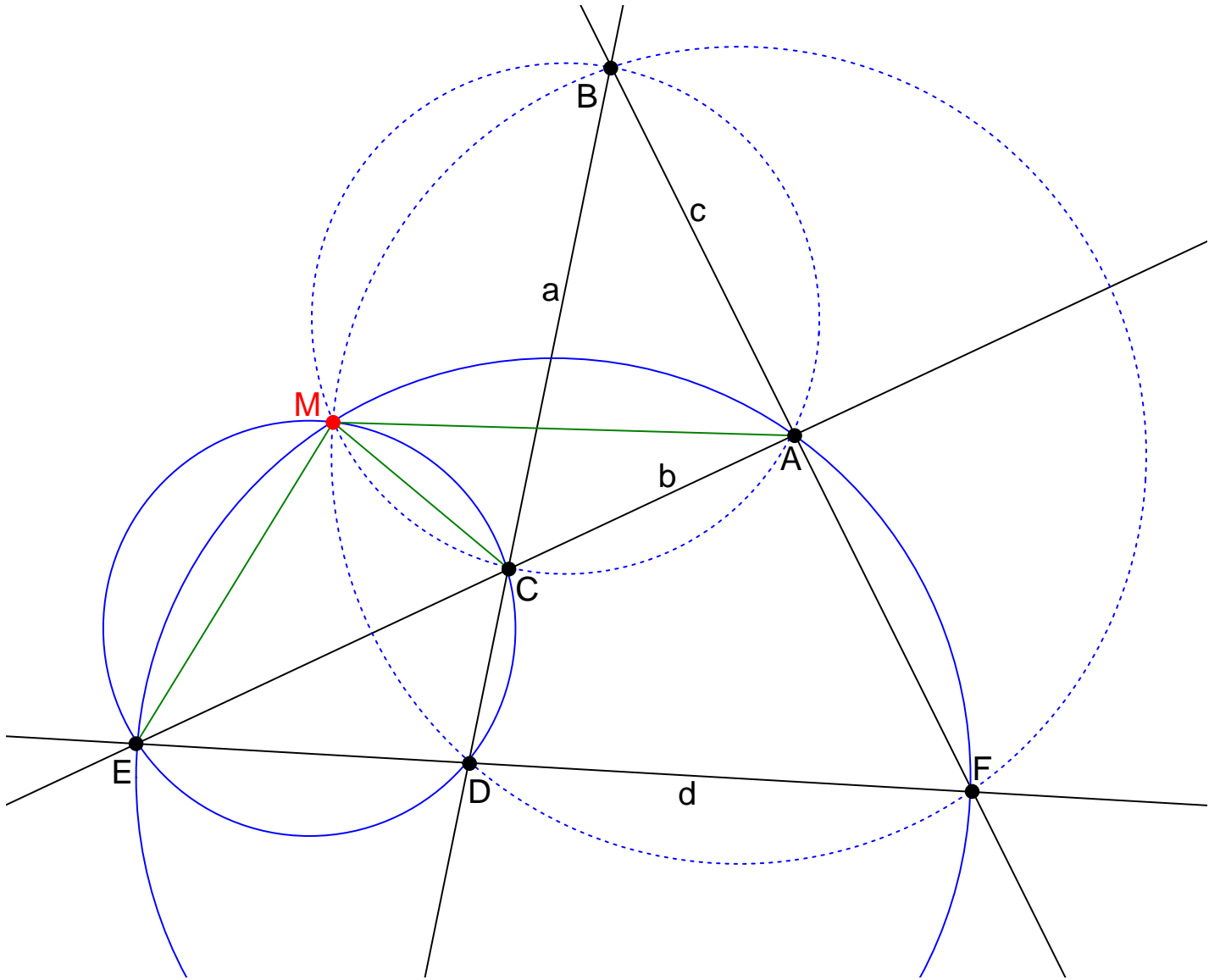
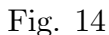


Fig. 13

Throughout mathematics it can be observed that proving more is usually easier than proving less. According to this, and also for the sake of completeness, with the next theorem we are going to prove a whole catalogue of properties of a configuration, despite the fact that the only one that we will need in our proof of Theorem 6 is Theorem 9 **h**). [Thus, the reader not familiar with inversion can skip Theorem 9 **f**), and the reader not familiar with polarity can skip **i**.] Note that the other properties are of interest, too: as an exercise, the reader can kill three olympiad problems - [6], [7], [8] - using Theorem 9. Theorem 9 **d**) has also been discussed in [9], while Theorem 9 **b**), **d**) and **f**) yield the result of [10].





**a)** The point  $M$  lies on the circumcircles of triangles  $BCD$ ,  $QAD$ ,  $RAB$ ,  $QCB$ .

**b)** The point  $M$  lies on the line  $QR$ . (See Fig. 14.)

d) The point  $M$  lies on the line  $OP$ .

e) The line  $OP$  bisects the angle  $AMC$  and bisects the angle  $BMD$ .

f) The point  $M$  is the image of the point  $P$  under the inversion with respect to the circle  $k$ .

**г)** The point  $M$  is the orthogonal projection of the point  $P$  on the line  $QR$ . (See Fig. 16.)

**h)** We have  $OP \perp QR$ .

i) The line  $QR$  is the polar of the point  $P$  with respect to the circle  $k$ .

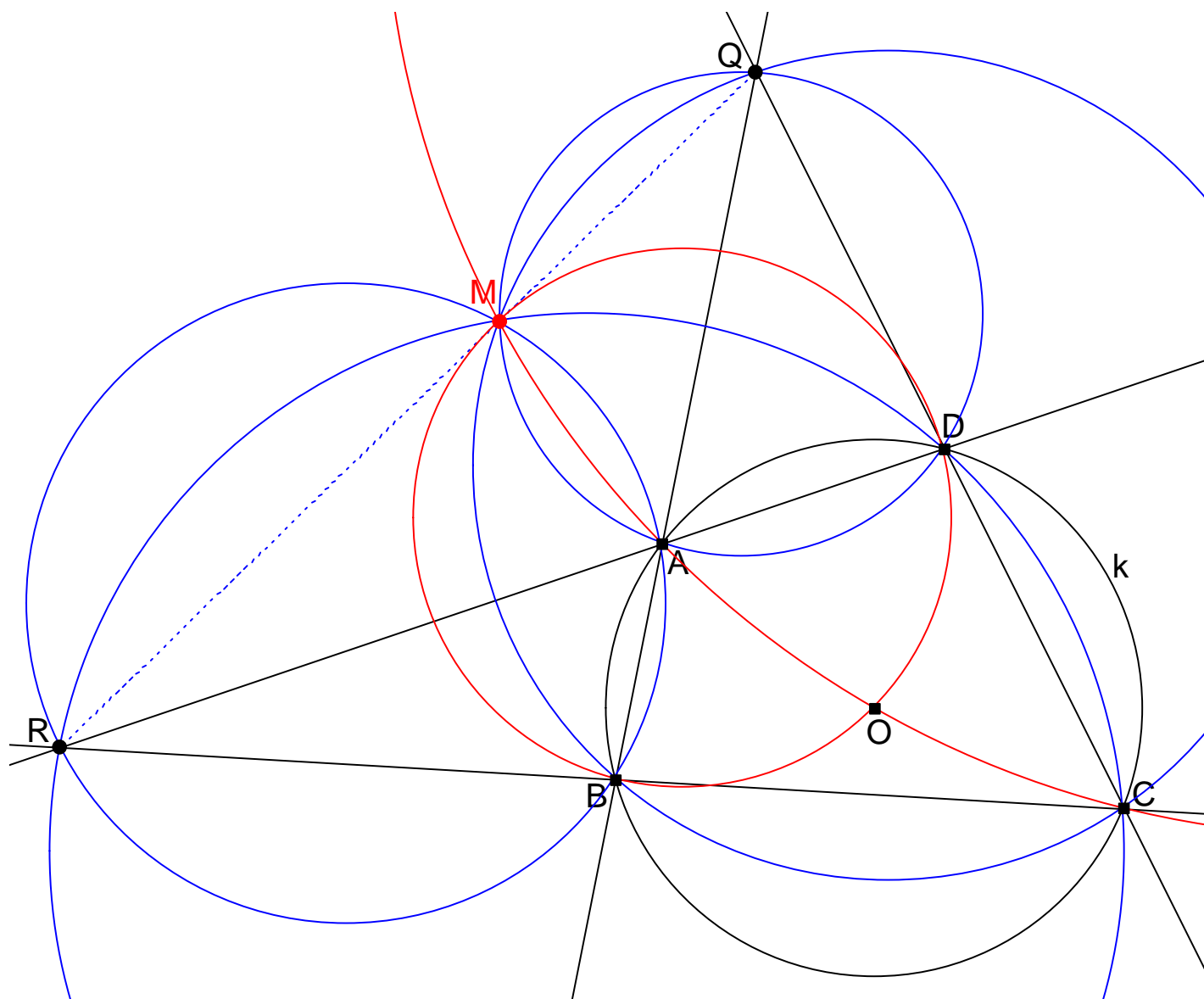


Fig. 15

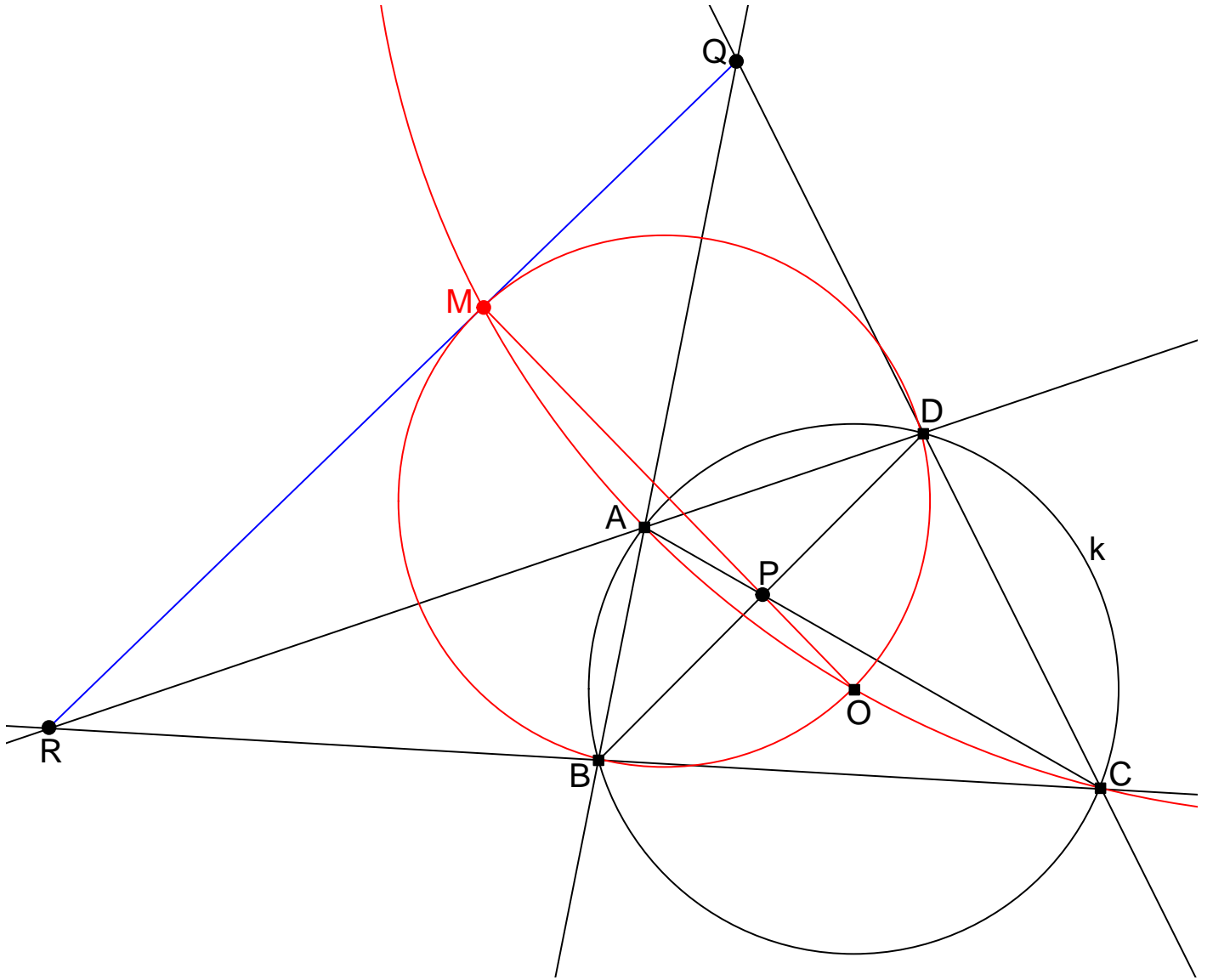


Fig. 16

*Proof of Theorem 9.* We have  $BC \cap CD = C$ ,  $CD \cap AB = Q$ ,  $AB \cap BC = B$ ,  $AB \cap DA = A$ ,  $BC \cap DA = R$ ,  $CD \cap DA = D$ . Thus, due to its definition, the Miquel point  $M$  of the four lines  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  is the common point of the circumcircles of triangles  $RCD$ ,  $QAD$ ,  $RAB$ ,  $QCB$ . This proves Theorem 9 a).

(See Fig. 17.) Since  $M$  lies on the circumcircle of triangle  $RAB$ , we have  $\angle RMB = \angle RAB$ . Since the points  $A$ ,  $B$ ,  $C$ ,  $D$  lie on one circle (namely, on the circle  $k$ ), we have  $\angle DAB = \angle DCB$ . Since  $M$  lies on the circumcircle of triangle  $QCB$ , we have  $\angle QCB = \angle QMB$ . Hence,  $\angle RMB = \angle RAB = \angle DAB = \angle DCB = \angle QCB = \angle QMB$ . Thus, the points  $M$ ,  $Q$ ,  $R$  are collinear, i. e. the point  $M$  lies on the line  $QR$ . This proves Theorem 9 b).

Since  $O$  is the center of the circle  $k$ , while the points  $B$ ,  $C$ ,  $D$  lie on this circle, the central angle theorem yields  $\angle BOD = 2 \cdot \angle BCD$ . Now we know that  $\angle RMB = \angle RAB$ , and similarly we can get  $\angle RMD = \angle RCD$ . With the aid of the relation

$\angle DAB = \angle DCB$  shown above, we thus get

$$\begin{aligned}\angle BMD &= \angle RMD - \angle RMB = \angle RCD - \angle RAB = \angle BCD - \angle DAB = \angle BCD - \angle DCB \\ &= \angle BCD - (-\angle BCD) = 2 \cdot \angle BCD = \angle BOD.\end{aligned}$$

Thus, the points  $B, D, M, O$  lie on one circle, i. e. the point  $M$  lies on the circumcircle of triangle  $BOD$ . Similarly,  $M$  also lies on the circumcircle of triangle  $AOC$ . This proves Theorem 9 c).

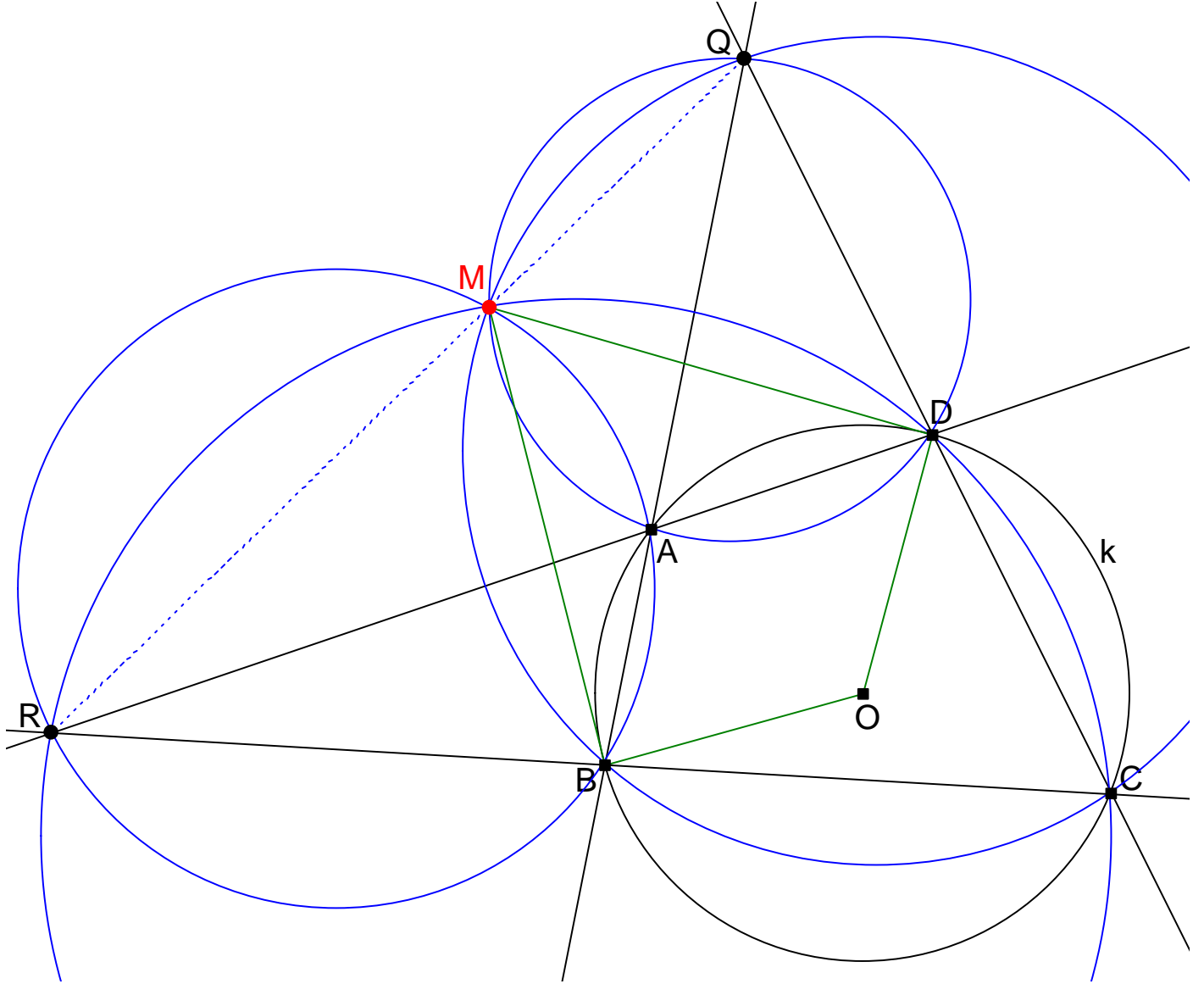


Fig. 17

Now we are going to prove Theorem 9 d) and f) in two different ways - a first one using inversion, and a second one using radical axes:

*First proof of Theorem 9 d) and f).* (See Fig. 16.) The inversion with respect to the circle  $k$  maps the points  $A$  and  $C$  to themselves (since these points lie on  $k$ ). On the other hand, the inversion with respect to the circle  $k$  maps circles through the point  $O$  to lines (since  $O$  is the center of the circle  $k$ , hence the center of our inversion, and any inversion maps circles through the center of inversion to lines). Hence, the image of the circumcircle of triangle  $AOC$  under this inversion is a line (since the circumcircle of

triangle  $AOC$  is a circle through the point  $O$ ), and this line passes through the points  $A$  and  $C$  (in fact, this line is the image of the circumcircle of triangle  $AOC$  under our inversion, and thus must pass through the images of the points  $A$  and  $C$  under this inversion - but these images are these points  $A$  and  $C$  themselves). Thus, the image of the circumcircle of triangle  $AOC$  under our inversion is the line  $AC$ . Similarly, the image of the circumcircle of triangle  $BOD$  under our inversion is the line  $BD$ .

Now, since the point  $M$  lies on the circumcircles of triangles  $AOC$  and  $BOD$ , the image of this point  $M$  under the inversion with respect to the circle  $k$  must lie on the images of these circumcircles, thus on the lines  $AC$  and  $BD$ . Hence, the image of the point  $M$  under the inversion with respect to  $k$  is the point of intersection of the lines  $AC$  and  $BD$ . But the point of intersection of the lines  $AC$  and  $BD$  is the point  $P$ . Hence, the image of the point  $M$  under the inversion with respect to  $k$  is the point  $P$ . Thus, in turn, the point  $M$  is the image of the point  $P$  under the inversion with respect to  $k$ . Theorem 9 f) is hence proven.<sup>3</sup>

The image of a point under the inversion with respect to a circle always lies on the line through the center of the circle and the point. Since  $M$  is the image of the point  $P$  under the inversion with respect to the circle  $k$ , we can thus conclude that the point  $M$  lies on the line  $OP$  (since  $O$  is the center of the circle  $k$ ). This proves Theorem 9 d).

*Second proof of Theorem 9 d) and f).* If two circles intersect, then the line joining the two points of intersection is the radical axis of the two circles. This yields that:

- The line  $OM$  is the radical axis of the circumcircles of triangles  $AOC$  and  $BOD$  (since the two points of intersection of these circles are  $O$  and  $M$ ).
- The line  $AC$  is the radical axis of the circle  $k$  and the circumcircle of triangle  $AOC$  (since the two points of intersection of these circles are  $A$  and  $C$ ).
- The line  $BD$  is the radical axis of the circle  $k$  and the circumcircle of triangle  $BOD$  (since the two points of intersection of these circles are  $B$  and  $D$ ).

Now, the pairwise radical axes of three circles always concur. Applied to the circle  $k$  and the circumcircles of triangles  $AOC$  and  $BOD$ , this yields that the lines  $OM$ ,  $AC$  and  $BD$  concur. In other words, the line  $OM$  passes through the point  $AC \cap BD$ . Since  $AC \cap BD = P$ , this is equivalent to saying that the line  $OM$  passes through  $P$ . In other words,  $M$  lies on the line  $OP$ . Hence, Theorem 9 d) is proven.

(See Fig. 18.) Since  $O$  is the center of the circle  $k$ , while the points  $B$  and  $D$  lie on this circle, we have  $OB = OD$ . Thus, triangle  $BOD$  is isosceles, so that  $\angle OBD = \angle BDO$ . Since  $M$  lies on the circumcircle of triangle  $BOD$ , we have  $\angle OBD = \angle OMD$  and  $\angle BDO = \angle BMO$ . Hence,  $\angle OBD = \angle BDO$  becomes  $\angle OMD = \angle BMO$ . This equation shows that the line  $OP$  bisects the angle  $BMD$ . Similarly, the line  $OP$  bisects the angle  $AMC$ . Thus we have shown Theorem 9 e).

From  $\angle OBD = \angle BDO$  and  $\angle BDO = \angle BMO$  we can conclude that  $\angle OBD = \angle BMO$ , so that  $\angle OBP = -\angle OMB$ . Further, obviously  $\angle BOP = -\angle MOB$ . Thus,

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<sup>3</sup>This proof was not particularly watertight. In fact, as we are working in the inversive plane, the lines  $AC$  and  $BD$  have not just one, but two points of intersection: the usual point of intersection  $P$  and the infinite point of the inversive plane. But it is readily seen that the image of the point  $M$  under the inversion with respect to  $k$  is the "right" point of intersection, i. e. the point  $P$ .

A geometric diagram illustrating a circle  $k$  with center  $O$ . Points  $A, B, C, D$  are on the circumference of  $k$ . A red circle passes through points  $M, B, O, D$ . A green quadrilateral  $MBOD$  is inscribed in the red circle. Lines connect  $R$  to  $M, B, D$  and  $Q$  to  $D$ . The diagram shows the intersection of two circles and the construction of a cyclic quadrilateral within one of them.

Now, we have shown Theorem 9 **a)**, **b)** and **c)**, then proved Theorem 9 **d)** and **f)** two ways, additionally showing Theorem 9 **e)** in the second proof of Theorem 9 **d)** and **f)**. What remains now is to verify Theorem 9 **g)**, **h)** and **i)**:

$\angle RMO = \angle RMB + \angle BMO = \angle RAB + \angle BDO = \angle DAB + (90^\circ - \angle DAB) = 90^\circ$ ,  
hence  $OM \perp QR$ . Since the point  $M$  lies on  $OP$ , this becomes  $OP \perp QR$ . Thus,

Theorem 9 **h)** is proven. The relation  $OP \perp QR$ , together with the fact that the point  $M$  lies on the lines  $OP$  and  $QR$ , yields that the point  $M$  is the orthogonal projection of the point  $P$  on the line  $QR$ . Hence, Theorem 9 **g)** is proven as well.

The polar of the point  $P$  with respect to the circle  $k$  is defined as the perpendicular to the line  $OP$  through the image of the point  $P$  under the inversion with respect to the circle  $k$  (since  $O$  is the center of  $k$ ). Now, the image of the point  $P$  under the inversion with respect to the circle  $k$  is the point  $M$ . Hence, the polar of the point  $P$  with respect to the circle  $k$  is the perpendicular to the line  $OP$  through  $M$ . This perpendicular is obviously the line  $QR$  (since the line  $QR$  passes through  $M$  and is perpendicular to  $OP$ ). Hence, the polar of the point  $P$  with respect to the circle  $k$  is the line  $QR$ . Hence, Theorem 9 **i)** is proven, what concludes our proof of Theorem 9.

Now, Theorem 9 swiftly implies Theorem 6: Applying Theorem 9 **h)** directly to the points  $A, B, C, D$  on the circle  $k$ , we get  $OP \perp QR$ . But applying Theorem 9 **h)** to the points  $C, B, D, A$  (in this order) on the circle  $k$ , we get  $OQ \perp RP$ , and applying Theorem 9 **h)** to the points  $B, D, C, A$  (in this order) on the circle  $k$ , we obtain  $OR \perp PQ$ .

Since  $OP \perp QR$ ,  $OQ \perp RP$  and  $OR \perp PQ$ , the lines  $OP, OQ, OR$  must be the altitudes of the triangle  $PQR$ . Thus, the point  $O$ , being the point of intersection of these lines  $OP, OQ, OR$ , must be the point of intersection of the altitudes of triangle  $PQR$ , i. e. the orthocenter of triangle  $PQR$ . Thus, Theorem 6 is proven.

As we promised, we can also immediately conclude Theorem 7 from Theorem 9: In the configuration of Theorem 9, according to Theorem 9 **i)**, the line  $QR$  is the polar of the point  $P$  with respect to the circle  $k$ . Now, the point  $Q$  lies on the line  $QR$ ; hence, the point  $Q$  lies on the polar of the point  $P$  with respect to the circle  $k$ . In other words: The point  $AB \cap CD$  lies on the polar of the point  $AC \cap BD$  with respect to the circle  $k$ . Renaming the points  $A, B, C, D$  into  $X, Y, Z, W$  and the circle  $k$  into  $k_1$  in this assertion, we get Theorem 7.

After Theorem 6 has been proved in two ways, we finally deduce Theorem 1 from Theorems 5 and 6:

(See Fig. 19.) Consider the configuration of Theorem 1. Assume that  $O \neq P$  (in fact, in the case  $O = P$ , the lines  $AC$  and  $BD$  are diameters of the circle  $k$ , so that the quadrilateral  $ABCD$  is symmetric with respect to  $O$ , and thus Theorem 1 becomes trivial from symmetry). Then, we can speak of the line  $OP$ . Define two points  $Q = AB \cap CD$  and  $R = BC \cap DA$ . Then, according to Theorem 6, the point  $O$  is the orthocenter of triangle  $PQR$ . Hence,  $OP \perp QR$ . Together with  $OP \perp g$ , this yields  $g \parallel QR$ . Thus, the line  $g$  is the parallel to the line  $QR$  through the point  $P$ . Since this line  $g$  intersects the lines  $AB$  and  $CD$  at the points  $X$  and  $Z$ , Theorem 5 now yields that the point  $P$  is the midpoint of the segment  $XZ$ . Thus, Theorem 1 is proven.

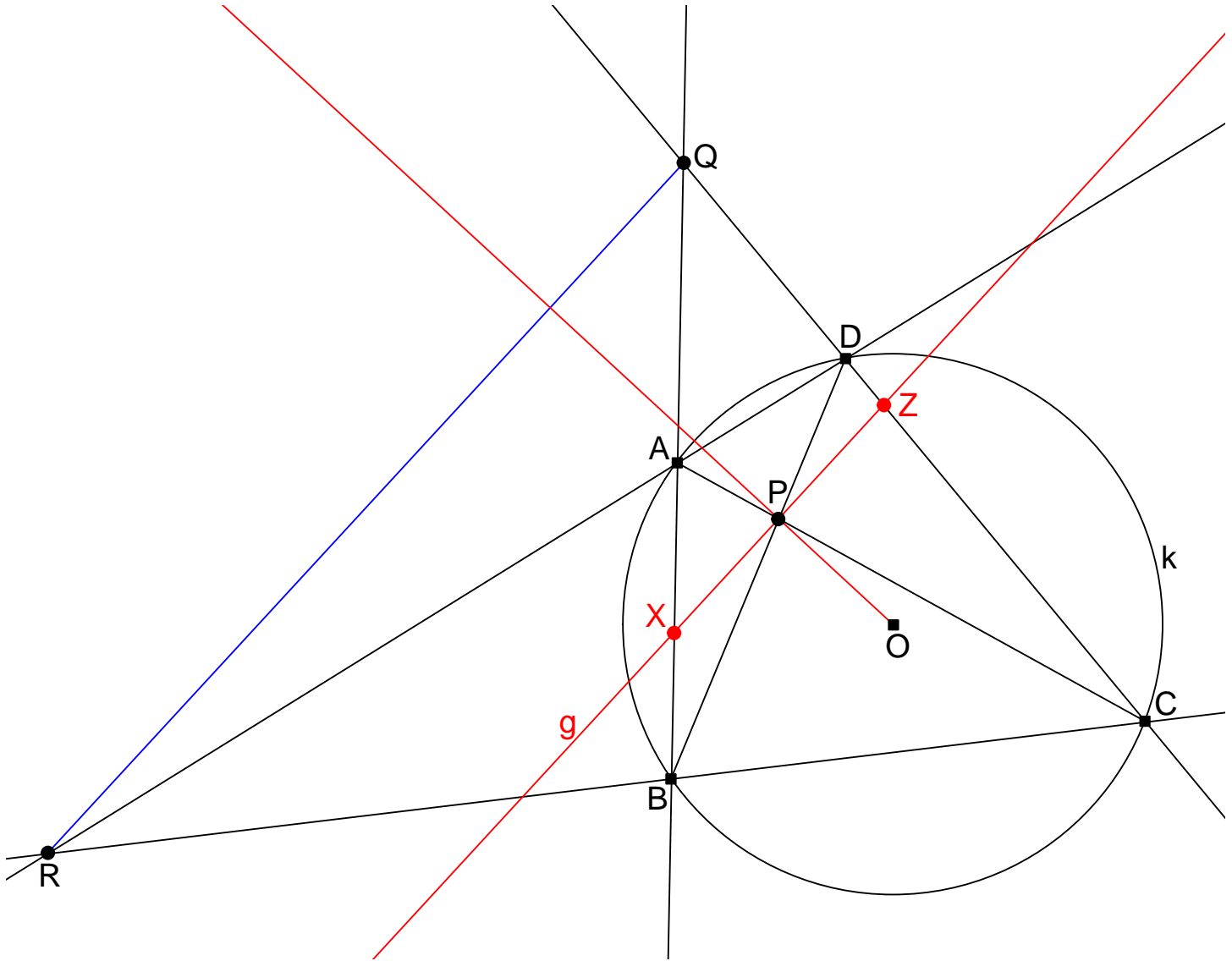


Fig. 19

#### 4. An application of the first proof

We have proven Theorem 1 in two different ways now. We conclude this note with an application of Theorem 3 noticed by Virgil Nicula in [12].

We start with a trivial particular case of Theorem 3:

**Theorem 10.** Let  $ABC$  be a triangle with the circumcenter  $O$ . Let  $g$  be an arbitrary line, and let  $P$  be the orthogonal projection of the point  $O$  on this line  $g$ . The line  $g$  intersects the lines  $BC$ ,  $CA$ ,  $AB$  at some points  $X$ ,  $Y$ ,  $Z$ . Let  $W$  be the point of intersection of the line  $g$  with the tangent to the circumcircle of triangle  $ABC$  at the point  $A$ . Then, the point  $P$  is the midpoint of the segment  $YZ$  if and only if the point  $P$  is the midpoint of the segment  $XW$ . (See Fig. 20.)



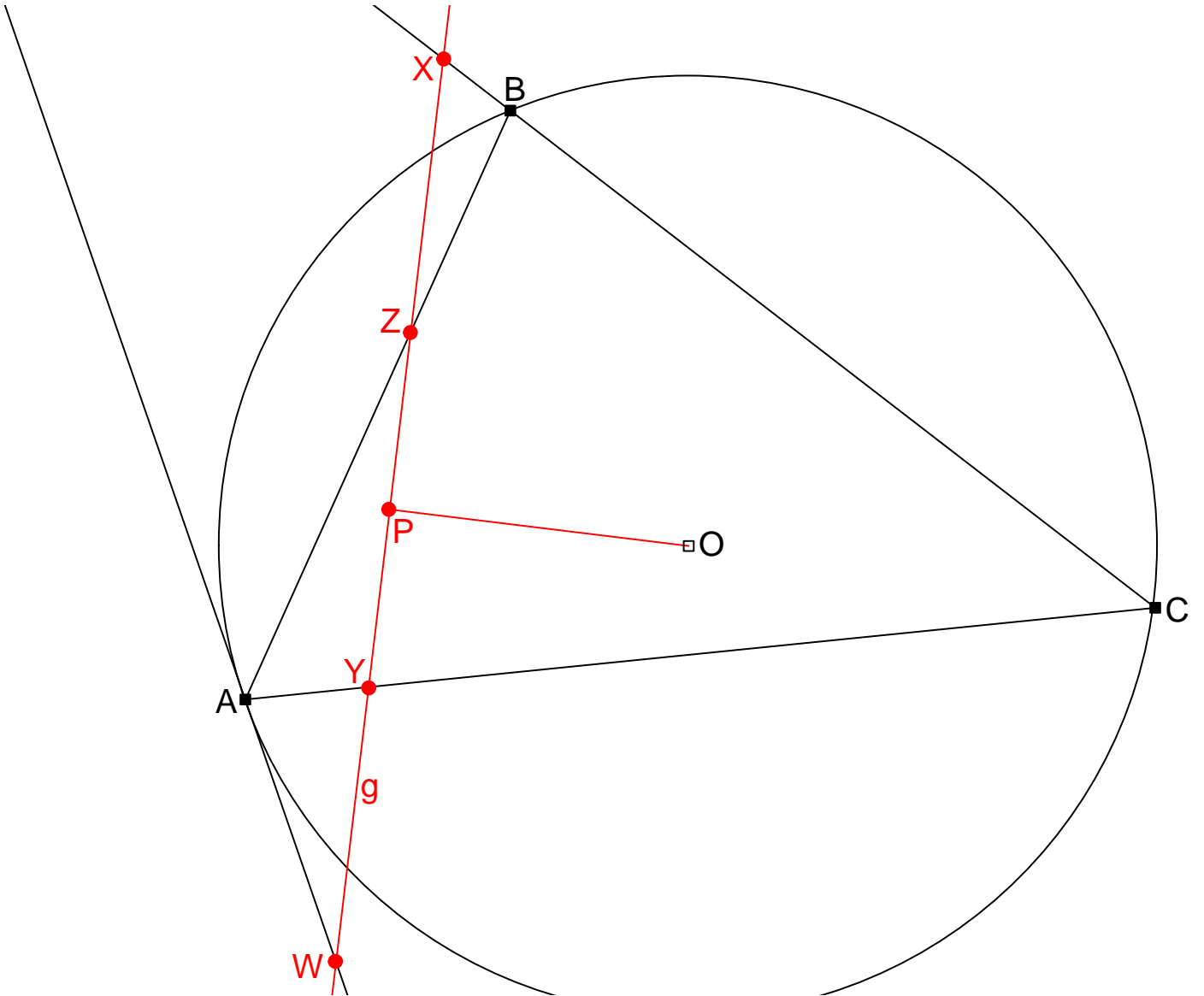


Fig. 20

*Proof of Theorem 10.* We consider our configuration from a slightly different viewpoint:

The circumcircle of triangle  $ABC$  has the center  $O$ , and  $A, C, B, A$  are four points on this circumcircle. The point  $P$  is the orthogonal projection of the point  $O$  on the line  $g$ . The line  $g$  intersects the lines  $AC, CB, BA, AA, AB, CA$  at the points  $Y, X, Z, W, Z, Y$  (hereby, the line  $AA$  is considered to mean the tangent to the circumcircle of triangle  $ABC$  at the point  $A$ ).

Hence we can apply Theorem 3 to the circumcircle of triangle  $ABC$  (in the role of the circle  $k$ ), the four points  $A, C, B, A$  on this circumcircle (in the role of the points  $A, B, C, D$ ) and the line  $g$  (in the role of the line  $g$ ), and we conclude that the following three assertions are pairwise equivalent:

*Assertion 1:* The point  $P$  is the midpoint of the segment  $YZ$ .

*Assertion 2:* The point  $P$  is the midpoint of the segment  $XW$ .

*Assertion 3:* The point  $P$  is the midpoint of the segment  $ZY$ .

The equivalence of Assertions 1 and 2 is exactly the statement of Theorem 10. Thus, Theorem 10 is proven.

An application of Theorem 10 is the following fact from triangle geometry:

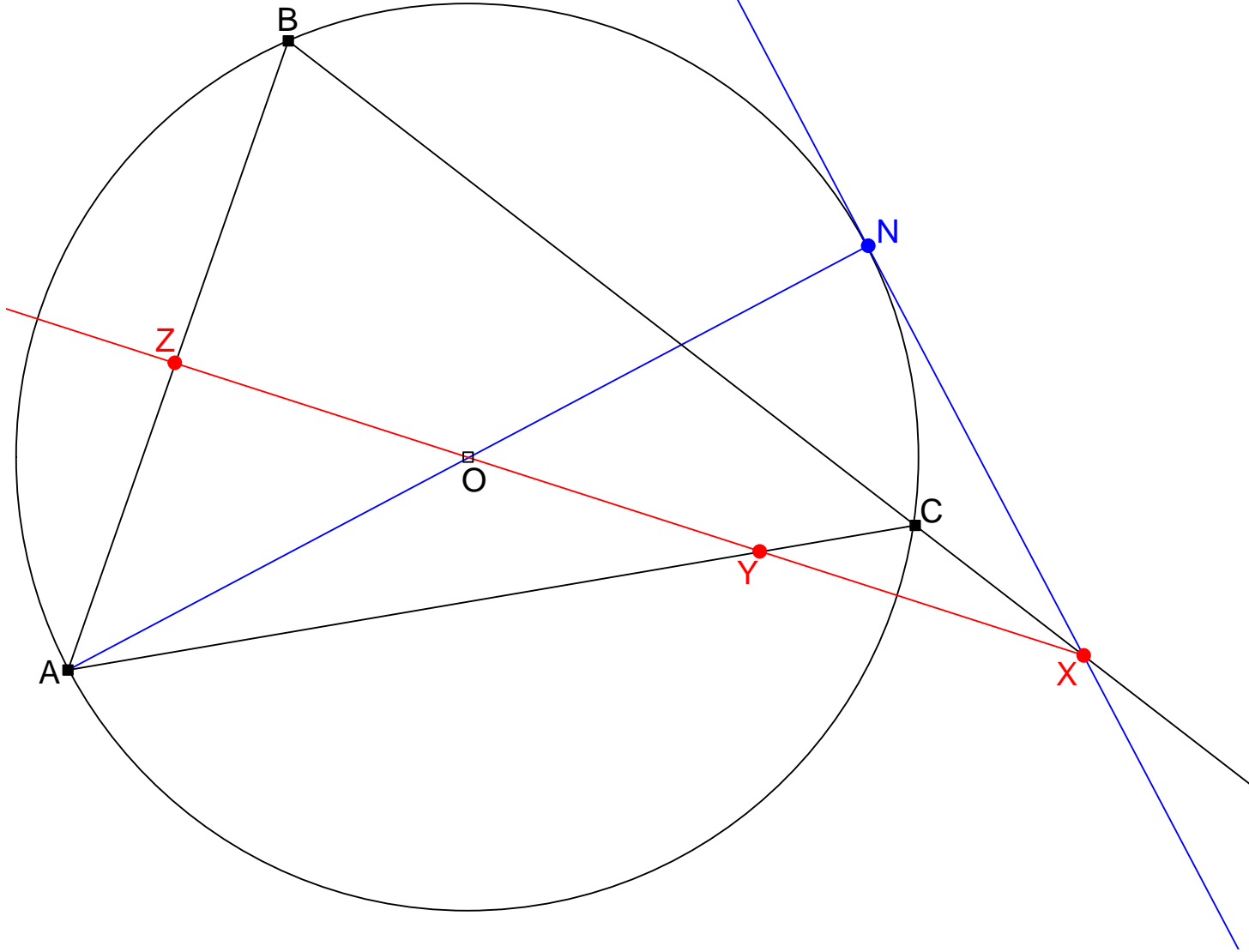


Fig. 21

**Theorem 11.** Let  $O$  be the circumcenter of a triangle  $ABC$ . Let  $N$  be the reflection of the point  $A$  in the point  $O$ , or, equivalently, the point diametrically opposite to the point  $A$  on the circumcircle of triangle  $ABC$ . The tangent to the circumcircle of triangle  $ABC$  at the point  $N$  intersects the line  $BC$  at a point  $X$ . The line  $OX$  intersects the lines  $CA$  and  $AB$  at the points  $Y$  and  $Z$ . Then, the point  $O$  is the midpoint of the segment  $YZ$ . (See Fig. 21.)

This theorem has been discussed in [12] and [13] and allows for different approaches. It has been given in a slightly more complicated form as problem 6 in the selection round of the St. Petersburg Mathematical Olympiad 2002 (SPbMO) for the 9th grade. Here we show two proofs of Theorem 11 - one by Virgil Nicula using Theorem 3 and one being a slight variation of the proposed solution of the SPbMO problem.



$AW \perp AO$ , this yields  $NX \parallel AW$ . Hence, after the Thales theorem,  $\frac{XO}{OW} = \frac{NO}{OA}$ . Now,  $NO = OA$  (since  $N$  is the reflection of  $A$  in the point  $O$ ), and thus  $\frac{XO}{OW} = \frac{NO}{OA} = \frac{OA}{OA} = 1$ , so that  $XO = OW$ . Thus, the point  $O$  is the midpoint of the segment  $XW$ . As we have said, this proves Theorem 11.

*Second proof of Theorem 11.* As a contrast, here comes a completely elementary proof of Theorem 11 - actually, more or less a restatement of the proposed solution of the SPbMO problem. We will use directed angles modulo  $180^\circ$ , but we will use non-directed segments. (See Fig. 23.) Let  $M$  be the midpoint of the segment  $BC$ . Being the circumcenter of triangle  $ABC$ , the point  $O$  must lie on the perpendicular bisector of its side  $BC$ . Thus,  $OM \perp BC$ . In other words,  $\angle OMX = 90^\circ$ . On the other hand,  $NX$  is the tangent to the circumcircle of triangle  $ABC$  at  $N$ , and thus  $NX \perp NO$  (since  $O$  is the center of this circumcircle). This yields  $\angle ONX = 90^\circ$ .

Since  $\angle OMX = 90^\circ$  and  $\angle ONX = 90^\circ$ , the points  $M$  and  $N$  lie on the circle with diameter  $OX$ . Thus,  $\angle XMN = \angle XON$ . On the other hand,  $\angle BCN = \angle BAN$  since the point  $N$  lies on the circumcircle of triangle  $ABC$ .

Thus,  $\angle CMN = \angle XMN = \angle XON = -\angle AOZ$  and  $\angle MCN = \angle BCN = \angle BAN = -\angle OAZ$ . Hence, the triangles  $CMN$  and  $AOZ$  are oppositely similar. This entails  $\frac{OZ}{AO} = \frac{MN}{CM}$ . Similarly,  $\frac{OY}{AO} = \frac{MN}{BM}$ . Since  $CM = BM$  (what is because the point  $M$  is the midpoint of  $BC$ ), this leads to  $\frac{OZ}{AO} = \frac{MN}{CM} = \frac{MN}{BM} = \frac{OY}{AO}$ , and thus  $OZ = OY$ . Hence,  $O$  is the midpoint of  $YZ$ , and Theorem 11 is proven once again.

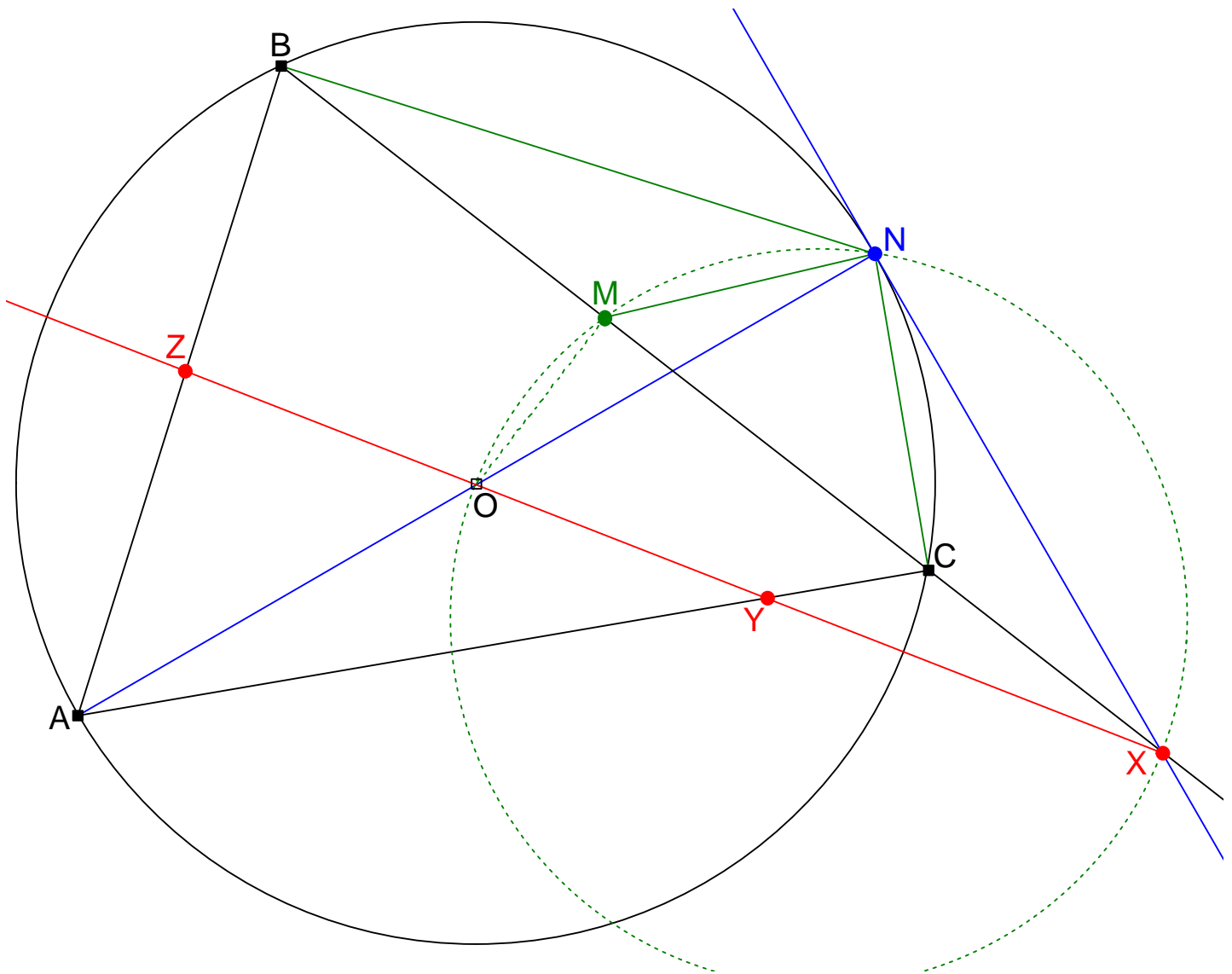


Fig. 23

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