Problems from the Book – Problem 19.9

Let $n \in \mathbb{N}$. Let w_1, w_2, ..., w_n be n reals. Prove the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{ijw_iw_j}{i+j-1} \geq \left(\sum_{i=1}^{n} w_i \right)^2 .$$

Solution by Darij Grinberg

Notations.

- For any matrix A, we denote by $A \left[\begin{array}{c} j \\ i \end{array} \right]$ the entry in the j-th column and the i-th row of A. [This is usually denoted by A_{ij} or by $A_{i,j}$.]

- Let k be a field. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of k for every $(i, j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$. Then, we denote by $(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v}$ the $u \times v$ matrix A which satisfies $A \left[\begin{array}{c} j \\ i \end{array} \right] = a_{i,j}$ for every $(i, j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$.

- Let $n \in \mathbb{N}$. Let t_1, t_2, ..., t_n be n objects. Let $m \in \{1, 2, ..., n\}$. Then, we let $\left(t_1, t_2, ..., \widehat{t_m}, ..., t_n \right)$ denote the $(n-1)$-tuple $(t_1, t_2, ..., t_{m-2}, t_{m-1}, t_{m+1}, t_{m+2}, ..., t_n)$ (that is, the $(n-1)$-tuple $(s_1, s_2, ..., s_{n-1})$ defined by $s_i = \left\{ \begin{array}{c} t_i, \text{ if } i < m; \\ t_{i+1}, \text{ if } i \geq m \end{array} \right.$ for all $i \in \{1, 2, ..., n-1\}$).

- Let R be a commutative ring with unity. Let a_1, a_2, ..., a_m be m elements of R. Then, we define an element $\sigma_k (a_1, a_2, ..., a_m)$ of R by

  $$\sigma_k (a_1, a_2, ..., a_m) = \sum_{\substack{S \subseteq \{1,2,...,m\}; \\ |S|=k}} \prod_{i \in S} a(i) .$$

  This element $\sigma_k (a_1, a_2, ..., a_m)$ is simply the k-th elementary symmetric polynomial evaluated at a_1, a_2, ..., a_m.
  The Viete theorem states that

  $$\prod_{\ell \in \{1,2,...,m\}} (x - a_\ell) = \sum_{k=0}^{m} (-1)^k \sigma_k (a_1, a_2, ..., a_m) x^{m-k}$$

  for every $x \in R$. If we choose some $i \in \{1, 2, ..., m\}$ and apply this equality to the $m-1$ elements a_1, a_2, ..., $\widehat{a_i}$, ..., a_m in lieu of the m elements a_1, a_2, ..., a_m, then we obtain

  $$\prod_{\ell \in \{1,2,...,m\} \setminus \{i\}} (x - a_\ell) = \sum_{k=0}^{m-1} (-1)^k \sigma_k \left(a_1, a_2, ..., \widehat{a_i}, ..., a_m\right) x^{m-1-k}. \qquad (1)$$

Theorem 1 (Sylvester). Let $n \in \mathbb{N}$, and let $A \in \mathbb{R}^{n \times n}$ be a symmetric $n \times n$ matrix. Then, the matrix A is positive definite if and only if every $m \in \{1, 2, ..., n\}$ satisfies $\det \left(\left(A \left[\begin{array}{c} j \\ i \end{array} \right] \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) > 0$.

For a *proof of Theorem 1*, see any book on symmetric or Hermitian matrices.

Theorem 2 (Cauchy determinant). Let k be a field. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of k. Let b_1, b_2, ..., b_m be m elements of k. Assume that $a_j \neq b_i$ for every $(i, j) \in \{1, 2, ..., m\}^2$. Then,

$$\det \left(\left(\frac{1}{a_j - b_i} \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \frac{\prod\limits_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} \left((a_i - a_j)(b_j - b_i) \right)}{\prod\limits_{(i,j) \in \{1,2,...,m\}^2} (a_j - b_i)}.$$

In the following, I attempt to give the most conceptual proof of Theorem 2. First we recall a known fact we are not going to prove:

Theorem 3 (Vandermonde determinant). Let S be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of S. Then,

$$\det \left(\left(a_i^{j-1} \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Besides, a trivial fact:

Lemma 4. Let S be a commutative ring with unity. Let $a \in S$. In the ring $S[X]$ (the polynomial ring over S in one indeterminate X), the element $X - a$ is not a zero divisor.

And a consequence of this fact:

Lemma 5. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. In the ring $R[X_1, X_2, ..., X_m]$ (the polynomial ring over R in m indeterminates X_1, X_2, ..., X_m), the element $\prod\limits_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (X_i - X_j)$ is not a zero divisor.

Proof of Lemma 5. We will first show that:

For any $(i, j) \in \{1, 2, ..., m\}^2$ satisfying $i > j$, the element $X_i - X_j$ of the ring $R[X_1, X_2, ..., X_m]$ is not a zero divisor. (2)

Proof of (2). Let $R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ denote the sub-R-algebra of $R[X_1, X_2, ..., X_m]$ generated by the elements X_1, X_2, ..., X_{i-2}, X_{i-1}, X_{i+1}, X_{i+2}, ..., X_m (that is, the m elements X_1, X_2, ..., X_m except of X_i). Consider the ring $\left(R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]\right)[X]$

(this is the polynomial ring over the ring $R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ in one indeterminate X). It is known that there exists an R-algebra isomorphism $\phi : \left(R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]\right)[X] \to R[X_1, X_2, ..., X_m]$ such that $\phi(X) = X_i$ and $\phi(X_k) = X_k$ for every $k \in \{1, 2, ..., m\} \setminus \{i\}$. Hence, $\phi(X - X_j) = \underbrace{\phi(X)}_{=X_i} - \underbrace{\phi(X_j)}_{\substack{=X_j\text{, as}\\ j\in\{1,2,...,m\}\setminus\{i\}}} = X_i - X_j$. Since $X - X_j$ is not a zero divisor in $\left(R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]\right)[X]$ (by Lemma 4, applied to $S = R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ and $a = X_j$), it thus follows that $\phi(X - X_j) = X_i - X_j$ is not a zero divisor in $R[X_1, X_2, ..., X_m]$ (since ϕ is an R-algebra isomorphism). This proves (2).

It is known that if we choose some elements of a ring such that each of these elements is not a zero divisor, then the product of these elements is not a zero divisor. Hence, (2) yields that the element $\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$ of the ring $R[X_1, X_2, ..., X_m]$ is not a zero divisor. This proves Lemma 5.

Now comes a rather useful fact:

> **Theorem 6.** Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Consider the ring $R[X_1, X_2, ..., X_m]$ (the polynomial ring over R in m indeterminates X_1, X_2, ..., X_m). Then,
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> $$\det\left(\left((-1)^{m-j}\sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j).$$

Proof of Theorem 6. Let $V = \left(X_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $V\begin{bmatrix} j \\ i \end{bmatrix} = X_i^{j-1}$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, and

$$\det V = \det\left(\left(X_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j) \tag{3}$$

(by Theorem 3, applied to $S = R[X_1, X_2, ..., X_m]$ and $a_i = X_i$).

Let $W = \left((-1)^{m-j}\sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $W\begin{bmatrix} j \\ i \end{bmatrix} = (-1)^{m-j}\sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

For every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we can apply (1) to $x = X_j$ and $a_k = X_k$, and obtain

$$\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_j - X_\ell) = \sum_{k=0}^{m-1} (-1)^k \sigma_k\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right) X_j^{m-1-k}. \tag{4}$$

Now, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$\left(WV^T\right)\begin{bmatrix} j \\ i \end{bmatrix} = \sum_{k=1}^{m} \underbrace{W\begin{bmatrix} k \\ i \end{bmatrix}}_{=(-1)^{m-k}\sigma_{m-k}\left(X_1,X_2,...,\widehat{X_i},...,X_m\right)} \cdot \underbrace{V^T\begin{bmatrix} j \\ k \end{bmatrix}}_{=V\begin{bmatrix} k \\ j \end{bmatrix}=X_j^{k-1}}$$

$$= \sum_{k=1}^{m} (-1)^{m-k} \sigma_{m-k}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right) X_j^{k-1}$$

$$= \sum_{k=0}^{m-1} (-1)^{k} \sigma_{k}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right) X_j^{m-1-k} \qquad \text{(here, we substituted } k \text{ for } m-k \text{ in the sum)}$$

$$= \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_j - X_\ell) \qquad \text{(by (4))}. \qquad (5)$$

Thus, if $j \neq i$, then $\left(WV^T\right)\begin{bmatrix} j \\ i \end{bmatrix} = 0$ (since $\left(WV^T\right)\begin{bmatrix} j \\ i \end{bmatrix} = \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_j - X_\ell)$, but the product $\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_j - X_\ell)$ contains the factor $X_j - X_j = 0$ and thus equals 0). Hence, the matrix WV^T is diagonal. Therefore,

$$\det\left(WV^T\right) = \prod_{i=1}^{m} \left(WV^T\right)\begin{bmatrix} i \\ i \end{bmatrix} = \prod_{i=1}^{m} \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_i - X_\ell)$$

$$\left(\text{since (5), applied to } j = i\text{, yields } \left(WV^T\right)\begin{bmatrix} i \\ i \end{bmatrix} = \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_i - X_\ell)\right)$$

$$= \prod_{\substack{(i,\ell)\in\{1,2,...,m\}^2;\\ \ell\neq i}} (X_i - X_j) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j\neq i}} (X_i - X_j) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$$

$$\left(\begin{array}{c}\text{since the set } \left\{(i,j)\in\{1,2,...,m\}^2 \mid j\neq i\right\} \text{ is the union of the two disjoint sets} \\ \left\{(i,j)\in\{1,2,...,m\}^2 \mid j>i\right\} \text{ and } \left\{(i,j)\in\{1,2,...,m\}^2 \mid i>j\right\}\end{array}\right).$$

But on the other hand,

$$\det\left(WV^T\right) = \det W \cdot \det\left(V^T\right) = \det W \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$$

(since $\det\left(V^T\right) = \det V = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$). Hence,

$$\det W \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j) = \det\left(WV^T\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j).$$

But since the element $\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$ of the ring $R[X_1, X_2, ..., X_m]$ is not a zero divisor (according to Lemma 5), this yields

$$\det W = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j).$$

Since $W = \left((-1)^{m-j} \sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}$, this becomes

$$\det\left(\left((-1)^{m-j} \sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j).$$

Thus, Theorem 6 is proven.

Next, we show:

Theorem 7. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of R. Let b_1, b_2, ..., b_m be m elements of R. Then,

$$\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} ((a_i - a_j)(b_j - b_i)).$$

Proof of Theorem 7. Consider the ring $R[X_1, X_2, ..., X_m]$ (the polynomial ring over R in m indeterminates X_1, X_2, ..., X_m).

Let $\widetilde{V} = \left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $\widetilde{V}\begin{bmatrix} j \\ i \end{bmatrix} = a_i^{j-1}$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

Let $W = \left((-1)^{m-j} \sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $W\begin{bmatrix} j \\ i \end{bmatrix} = (-1)^{m-j} \sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

For every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we can apply (1) to $x = a_j$ and $a_k = X_k$, and obtain

$$\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - X_\ell) = \sum_{k=0}^{m-1} (-1)^k \sigma_k\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right) a_j^{m-1-k}. \qquad (6)$$

Now, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$\left(W\widetilde{V}^T\right)\left[\begin{array}{c} j \\ i \end{array}\right] = \sum_{k=1}^{m} \underbrace{W\left[\begin{array}{c} k \\ i \end{array}\right]}_{=(-1)^{m-k}\sigma_{m-k}\left(X_1,X_2,...,\widehat{X_i},...,X_m\right)} \cdot \underbrace{\widetilde{V}^T\left[\begin{array}{c} j \\ k \end{array}\right]}_{=\widetilde{V}\left[\begin{array}{c} k \\ j \end{array}\right]=a_j^{k-1}}$$

$$= \sum_{k=1}^{m} (-1)^{m-k} \sigma_{m-k}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right) a_j^{k-1}$$

$$= \sum_{k=0}^{m-1} (-1)^{k} \sigma_{k}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right) a_j^{m-1-k} \qquad \text{(here, we substituted } k \text{ for } m-k \text{ in the sum)}$$

$$= \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - X_\ell) \qquad \text{(by (6))}.$$

Hence,

$$W\widetilde{V}^T = \left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - X_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}.$$

Thus,

$$\det\left(\underbrace{\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - X_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}}_{=W\widetilde{V}^T}\right) = \det\left(W\widetilde{V}^T\right) = \det W \cdot \det\left(\widetilde{V}^T\right)$$

$$= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j)$$

$$\left(\begin{array}{c}\text{since } \det W = \det\left(\left((-1)^{m-j}\sigma_{m-j}\left(X_1, X_2, ..., \widehat{X_i}, ..., X_m\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j) \\ \text{by Theorem 6 and } \det\left(\widetilde{V}^T\right) = \det\widetilde{V} = \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j) \\ \text{by Theorem 3}\end{array}\right)$$

$$= \prod_{\substack{(j,i)\in\{1,2,...,m\}^2;\\ i>j}} (X_j - X_i) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j)$$

(here, we renamed i and j as j and i in the first product)

$$= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_j - X_i) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j)$$

$$= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} ((X_j - X_i)(a_i - a_j)) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} ((a_i - a_j)(X_j - X_i)).$$

Both sides of this identity are polynomials over the ring R in m indeterminates X_1, X_2, ..., X_m. Evaluating these polynomials at $X_1 = b_1$, $X_2 = b_2$, ..., $X_m = b_m$, we obtain

$$\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} ((a_i - a_j)(b_j - b_i)).$$

This proves Theorem 7.

Proof of Theorem 2. For every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$\frac{1}{a_j - b_i} = \frac{\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)}{\prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell)}.$$

Hence, the matrix $\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}$ is what we obtain if we take the matrix $\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}$ and divide its j-th column by $\prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell)$ for every $j \in \{1, 2, ..., m\}$. Therefore,

$$\det\left(\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \frac{\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}{\prod_{j\in\{1,2,...,m\}} \prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell)}$$

$$= \frac{\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}{\prod_{(\ell,j)\in\{1,2,...,m\}^2} (a_j - b_\ell)} = \frac{\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} ((a_i - a_j)(b_j - b_i))}{\prod_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)}$$

(by Theorem 7). Thus, Theorem 2 is proven.

Theorem 8. Let $n \in \mathbb{N}$. Let a_1, a_2, ..., a_n be n pairwise distinct reals. Let c be a real such that $a_i + a_j + c > 0$ for every $(i, j) \in \{1, 2, ..., n\}^2$. Then, the matrix $\left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} \in \mathbb{R}^{n\times n}$ is positive definite.

Proof of Theorem 8. Let $A = \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n}$. Then, $A\begin{bmatrix} j \\ i \end{bmatrix} = \frac{1}{a_i + a_j + c}$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

Thus, $A \in \mathbb{R}^{n\times n}$ is a symmetric $n \times n$ matrix.

Define n reals b_1, b_2, ..., b_n by $b_i = -a_i - c$ for every $i \in \{1, 2, ..., n\}$. Then, $a_j \neq b_i$ for every $(i, j) \in \{1, 2, ..., n\}^2$ (since $a_j - b_i = a_j - (-a_i - c) = a_i + a_j + c > 0$).

Now, every $m \in \{1, 2, ..., n\}$ satisfies

$$\det\left(\left(A\begin{bmatrix} j \\ i \end{bmatrix}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \det\left(\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)$$

$$\left(\text{since } A\begin{bmatrix} j \\ i \end{bmatrix} = \frac{1}{a_i + a_j + c} = \frac{1}{a_j - (-a_i - c)} = \frac{1}{a_j - b_i}\right)$$

$$= \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2; \\ i>j}} ((a_i - a_j)(b_j - b_i))}{\prod\limits_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)} \qquad \left(\text{by Theorem 2, since } a_j \neq b_i \text{ for every } (i,j) \in \{1, 2, ..., m\}^2\right)$$

$$= \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2; \\ i>j}} (a_i - a_j)^2}{\prod\limits_{(i,j)\in\{1,2,...,m\}^2} (a_i + a_j + c)}$$

$$\left(\begin{array}{c} \text{since } (a_i - a_j)(b_j - b_i) = (a_i - a_j)((-a_j - c) - (-a_i - c)) = (a_i - a_j)^2 \\ \text{and } a_j - b_i = a_j - (-a_i - c) = a_i + a_j + c \end{array}\right)$$

$$> 0$$

(since $(a_i - a_j)^2 > 0$ for every $(i,j) \in \{1, 2, ..., m\}^2$ satisfying $i > j$ (because a_1, a_2, ..., a_n are pairwise distinct, so that $a_i \neq a_j$, thus $a_i - a_j \neq 0$), and $a_i + a_j + c > 0$ for every $(i,j) \in \{1, 2, ..., m\}^2$).

Hence, according to Theorem 1, the symmetric matrix A is positive definite. Since $A = \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n}$, this means that the matrix $\left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n}$ is positive definite. Thus, Theorem 8 is proven.

Corollary 9. Let $n \in \mathbb{N}$. Let a_1, a_2, ..., a_n be n pairwise distinct reals. Let c be a real such that $a_i + a_j + c > 0$ for every $(i,j) \in \{1, 2, ..., n\}^2$. Let v_1, v_2, ..., v_n be n reals. Then, the inequality $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \frac{v_i v_j}{a_i + a_j + c} \geq 0$ holds, with equality if and only if $v_1 = v_2 = ... = v_n = 0$.

Proof of Corollary 9. Define a vector $v \in \mathbb{R}^n$ by $v = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix}$. Then,

$$v^T \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} v = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{a_i + a_j + c} v_i v_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{v_i v_j}{a_i + a_j + c}. \qquad (7)$$

Also, obviously,

$$v = 0 \text{ holds if and only if } v_1 = v_2 = ... = v_n = 0. \qquad (8)$$

Now, since the matrix $\left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} \in \mathbb{R}^{n\times n}$ is positive definite (by Theorem 8), we have $v^T \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} v \geq 0$, with equality if and only if $v = 0$. According to (7) and (8), this means that $\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{v_i v_j}{a_i+a_j+c} \geq 0$, with equality if and only if $v_1 = v_2 = ... = v_n = 0$. Thus, Corollary 9 is proven.

Corollary 10. Let $n \in \mathbb{N}$. Let a_1, a_2, ..., a_n be n pairwise distinct reals. Let c be a real such that $a_i + a_j + c > 0$ for every $(i,j) \in \{1,2,...,n\}^2$. Let w_1, w_2, ..., w_n be n reals. Then, the inequality $\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j w_i w_j}{a_i+a_j+c} \geq -c\left(\sum_{i=1}^{n} w_i\right)^2$ holds, with equality if and only if $(c+a_1)w_1 = (c+a_2)w_2 = ... = (c+a_n)w_n = 0$.

Proof of Corollary 10. Define n reals v_1, v_2, ..., v_n by $v_i = (c+a_i)w_i$ for every $i \in \{1,2,...,n\}$.

Then,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j w_i w_j}{a_i+a_j+c} - \left(-c\left(\sum_{i=1}^{n} w_i\right)^2\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j w_i w_j}{a_i+a_j+c} + c\left(\sum_{i=1}^{n} w_i\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j w_i w_j}{a_i+a_j+c} + c\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j$$

$$\left(\text{since } \left(\sum_{i=1}^{n} w_i\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{a_i a_j w_i w_j}{a_i+a_j+c} + c w_i w_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{a_i a_j}{a_i+a_j+c} + c\right) w_i w_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(c+a_i)(c+a_j)}{a_i+a_j+c} w_i w_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(c+a_i)w_i(c+a_j)w_j}{a_i+a_j+c} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{v_i v_j}{a_i+a_j+c}$$

(since $(c+a_i)w_i = v_i$ and $(c+a_j)w_j = v_j$). Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j w_i w_j}{a_i+a_j+c} \geq -c\left(\sum_{i=1}^{n} w_i\right)^2 \text{ holds if and only if } \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{v_i v_j}{a_i+a_j+c} \geq 0. \tag{9}$$

Also, clearly,

$$v_1 = v_2 = ... = v_n = 0 \text{ holds if and only if } (c+a_1)w_1 = (c+a_2)w_2 = ... = (c+a_n)w_n = 0. \tag{10}$$

By Corollary 9, the inequality $\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{v_i v_j}{a_i+a_j+c} \geq 0$ holds, with equality if and only if $v_1 = v_2 = ... = v_n = 0$. According to (9) and (10), this means that $\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j w_i w_j}{a_i+a_j+c} \geq -c\left(\sum_{i=1}^{n} w_i\right)^2$, with equality if and only if $(c+a_1)w_1 = (c+a_2)w_2 = ... = (c+a_n)w_n = 0$. Thus, Corollary 10 is proven.

The problem follows from Corollary 10 (applied to $c = -1$ and $a_i = i$).